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NON-HOMOGENEOUS QUASI-LINEAR SYMMETRIC HYPERBOLIC SYSTEMS WITH CHARACTERISTIC BOUNDARY

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Abstract: We consider the initial-boundary value problem for quasi-linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity. We show the existence of regular solutions in suitable functions spaces which take into account the loss of regularity in the normal direction to the characteristic boundary. The paper extends known results for problems with homogeneous boundary conditions to the nonhomogeneous case, under conditions of sharp regularity for the boundary data.

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1. Introduction

Let Ω be the half-space $\mathbb{R}^n_+ = \{(x_1, x') \in \mathbb{R}^n : x_1 > 0,\ x' \in \mathbb{R}^{n-1}\}$. Let Γ be the boundary of Ω and set $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$, with $T > 0$. We study the following initial-boundary value problem

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$$\begin{cases} L(u)u = F & \text{in} \quad Q_T, \\ Mu = G & \text{on} \quad \Sigma_T, \\ u(x,0) = f & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where

$$L(u) = A_0(u)\partial_t + \sum_{j=1}^{n} A_j(u)\partial_j + B(u),$$

$\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$ and $A_j(u), j = 0, \ldots, n$, $B(u)$ are $N \times N$ matrix-valued functions of the unknown function $u(x,t)$. We suppose the matrices $A_j(u), j = 0, \ldots, n$ are symmetric, $A_0(u)$ is also positive-definite for suitable u. The functions u, $F = F(x,t)$, $f = f(x)$ are vector-valued functions with N components, F and f are defined on $\overline{Q}_T$ and $\overline{\Omega}$, respectively. We suppose M is a given $d \times N$ matrix defined on Γ with constant rank d at Γ. $G = G(x,t)$ is a d-vector defined on Σ_T.

Denote by ν the unit outnormal to Γ. The boundary matrix is

$$A_\nu(u) := \sum_{j=1}^{n} A_j(u)\nu_j = -A_1(u).$$

We assume that $A_\nu(u)$ is not invertible but it has a constant rank only on Γ. In this case the boundary is said to be characteristic of constant multiplicity. We also assume that the boundary condition is strictly dissipative in the following sense:

$$\exists\, \delta > 0 : \; (A_\nu(v)u, u) \geq \delta |Pu|^2 - \frac{1}{\delta}|Mu|^2, \quad \forall u \in \mathbb{R}^N, \forall\, (x,t) \in \Sigma_T, \tag{2}$$

where P is the sum of eigenprojections corresponding to the non-zero eigenvalues of $A_\nu(v)$ on Σ_T (see Secchi [14] for more details); here $(\cdot,\cdot)$ and $|\cdot|$ denotes the inner product and the Euclidean norm in $\mathbb{R}^N$.

The initial boundary value problem (1) has been considered by many authors, in particular by Kreiss [4] under quite general boundary conditions in the case of noncharacteristic boundaries, see also Majda [5] and Métivier [7]. The analysis has been extended by Majda et al [6] to characteristic boundaries in the case when the boundary matrix has constant rank in a neighborhood of the boundary. Unfortunately, this is not the case in many physical applications, such as the compressible Euler equations and the equations of ideal compressible magneto-hydrodynamics. The relevant case of characteristic boundary when the boundary matrix has constant rank *only* at the boundary has been studied in the linear case for maximally non-negative boundary conditions by Rauch

[9], Ohno et al [8], Secchi [11, 13]; for the applications to quasilinear problems see Secchi [12, 14, 15, 18]. In the previous papers the boundary conditions are always linear and homogeneous.

Motivated by the interest in relativity theory, see Friedrich et al [3] and Winicour [19], in the present paper we study the characteristic problem with non-homogeneous boundary conditions. We introduce condition (2) at the boundary which provide an optimal regularity for the trace at the boundary of the noncharacteristic part of the solution; this seems to be a fundamental point in view of possible extensions to nonlinear boundary conditions. Condition (2) is a natural extension to characteristic boundaries of the strict positivity condition

$$\exists\, \delta > 0 : \ \ (A_\nu u, u) \geq \delta |u|^2 - \frac{1}{\delta}|Mu|^2, \quad \forall u \in \mathbb{R}^N. \tag{3}$$

Problem (1) was studied by Secchi [16] in the noncharacteristic case under the previous assumption (3), when $G \neq 0$.

In the present paper we extend Secchi's results [14], [16] by considering the case of quasilinear system with linear non-homogeneous and characteristic boundary conditions. We extend to the quasilinear problem the result of Casella et al [2] for the linear case, and we prove the existence and the uniqueness of local in time solution of (1) which is continuous in times in $H^m_*(\Omega)$. Similar property holds for its derivatives in time. We prove the existence of solutions in anisotropic weighted Sobolev spaces $H^m_*(\Omega)$, which take account of the loss of regularity in the normal direction to the characteristic boundary.

Recall that for characteristic mixed problems the full regularity cannot be expected, in the sense that the regularity theory cannot be stated in terms of usual Sobolev spaces $H^m(\Omega)$. In this case the space $H^m_*(\Omega)$ seems to be more suitable. This function space was introduced by Yanagisawa et al [20]. The definition of $H^m_*(\Omega)$ is motivated by the observation that the normal differentiation of order one of the solutions results from the tangential differentiation of order two.

The content of this paper is the following. In Section 2 we fix the notation, the structural assumption on the matrices of the operator $L(u)$ and we state the main result. Section 3 is devoted to the proof of the existence of the solution of problem (1). In Section 4 we prove the energy-type estimate satisfied by the solution of problem (1). Finally, in the Appendix A we give several lemmas about the properties of the Sobolev spaces H^m_* and H^m_{**}, while in Appendix B we recall the existence result for the linear case obtained by Casella et al [2].

2. Notations and Main Results

2.1. Function Spaces

We denote by H^m the usual Sobolev space $H^m(\Omega)$ and by $||\cdot||_m$ its norm. For simplicity, we denote the norm of $L^2 = L^2(\Omega)$ by $||\cdot||$.
Let $\sigma = \sigma(x_1)$ be a smooth and positive function of $x_1 > 0$ such that $\sigma(x_1) = x_1$ in a neighbourhood of the origin and $\sigma(x_1) = 1$ for x_1 large enough. The differential operator in the tangential direction is $\partial_*^\alpha = (\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Given an integer $m \geq 1$, we denote by $H_*^m = H_*^m(\Omega)$ the space of distributions $u \in L^2$ such that $\partial_*^\alpha \partial_1^k u \in L^2$ for any multi-index α and non negative integer k such that $|\alpha| + 2k \leq m$. The norm in H_*^m is

$$||u||_{m,*}^2 = \sum_{|\alpha|+2k\leq m} ||\partial_*^\alpha \partial_1^k u||^2.$$

The space $H_{**}^m = H_{**}^m(\Omega), m \geq 1$, consists of the distributions $u \in L^2$ such that $\partial_*^\alpha \partial_1^k \in L^2$ if $|\alpha| + 2k \leq m + 1$, $|\alpha| \leq m$. H_{**}^m is normed by

$$\|u\|_{m,**}^2 = \sum_{\alpha,k} \|\partial_*^\alpha \partial_1^k u\|^2,$$

where the sum is taken over all the multi-indices α and indices k such that $|\alpha| + 2k \leq m + 1$, $|\alpha| \leq m$. We have

$$H^m \hookrightarrow H_{**}^m \hookrightarrow H_*^m \hookrightarrow H_{\mathrm{loc}}^m, \quad H_*^m \hookrightarrow H^{[m/2]}, \quad H_{**}^m \hookrightarrow H^{[(m+1)/2]}.$$

Note that $H_*^0 = H_{**}^0 = L^2$. In the sequel, we set

$$\partial_\star^\alpha = \partial_t^{\alpha_0}(\sigma(x_1)\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n},$$

for any multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n$.

As usual, for any given $T > 0$ and for any normed space X, the symbol $L_T^p(X) = L^p(0,T;X)$, with $1 \leq p < +\infty$, denotes the set of all measurable functions $u(t)$ with values in X such that

$$\|u\|_{L_T^p(X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}} < +\infty,$$

where $\|\cdot\|_X$ is the norm in X. The set of all essentially bounded (with respect to the norm of X) measurable functions of t with values in X is denoted by

$L^\infty(0,T;X)$. We equip this space with the usual norm

$$\|u\|_{L^\infty(X)} = \sup_{t\in(0,T)} \|u(t)\|_X.$$

Let $C^m([0,T];X)$ denote the set of all X-valued m-times continuously differentiable functions of t, for $0 \le t \le T$. The norm in $C([0,T];L^2)$ is denoted by $||\cdot||_{0,T}$.

For $t \in [0,T]$, let $u(t)$ belong to H^m_* and be such that $\partial_t^k u(t) \in H^{m-k}_*$ for $k = 1,\dots,m$. Then, we set

$$|||u(t)|||^2_{m,*} = \sum_{k=0}^{m} ||\partial_t^k u(t)||^2_{m-k,*}.$$

Put $\mathcal{L}^2_T(H^m_*) = \bigcap_{k=0}^{m} H^k(0,T;H^{m-k}_*)$, $\mathcal{C}_T(H^m_*) = \bigcap_{k=0}^{m} C^k([0,T];H^{m-k}_*)$ and $\mathcal{L}^\infty_T(H^m_*) = \bigcap_{k=0}^{m} W^{k,\infty}(0,T;H^{m-k}_*)$, equipped, respectively, with the norms

$$[u]^2_{m,*,T} = \int_0^T |||u(t)|||^2_{m,*}\,dt\,, \quad |||u|||_{m,*,T} = \sup_{[0,T]} |||u(t)|||_{m,*}.$$

When H^m_* is replaced by the usual Sobolev space H^m we have $\mathcal{L}^2_T(H^m) = H^m(Q_T)$.

Given system (1), we recursively define $f^{(k)}, k \ge 1$, by formally taking $k-1$ time derivatives of $L(u)u = F$, solving for $\partial_t^k u$ and evaluating it at initial time $t = 0$. For $k = 0$, let $f^{(0)} = f$. We set $|||f|||^2_{m,*} = \sum_{k=0}^m ||f^{(k)}||^2_{m-k,*}$. The compatibility condition of order $m \ge 0$ for problem (1) (see Rauch et al [10]) is

$$M\, f^{(k)} = \partial_t^k G|_{t=0} \quad \text{on } \Gamma,\ k = 0,\dots,m. \tag{4}$$

In this paper, we shall denote by $c, c_i, C, C_i \dots$ different positive constants, which may vary in the same line and from line to line. The symbols $c(\cdot)$, $C(\cdot)$ mean that c and C depend at most on the quantities inside the brackets.

2.2. Main Results

The main result of the paper is given by the following theorem.

Theorem 2.1. *Let $m \ge 4[\frac{n}{2}] + 12$ be an integer. Fix $T_0 > 0$. Assume that:*

(i) $F \in \mathcal{L}^2_{T_0}(H^m_*)$, $f^{(k)} \in H^{m-k}_*$ *for* $k = 0, \dots, m$ *and* $G \in H^m(\Sigma_{T_0})$. *The data* (f, F, G) *satisfy the compatibility conditions of order* $m-1$.

(ii) for some constant $\alpha_0 > 0$ *the matrices* $A_0, \dots, A_n, B$ *are in* $C^m(N_0)$, *where* $N_0 = \{v \in \mathbb{R}^N : |v - f|_\infty \le \alpha_0\}$. *The matrices* $A_0(v), \dots, A_n(v)$ *are symmetric for all* $v \in N_0$. *There exists a constant* $a_0 > 0$ *such that* $A_0(v) \ge a_0$ *for all* $v \in N_0$.

(iii) The rank of the boundary matrix $A_\nu(v(x,t))$ *is constant,* $0 < \mathrm{rank} A_\nu(v(x,t)) < N$ *for* $(x,t) \in \Sigma_{T_0}$ *and* $v \in N_1 = \{v \in C(\overline{Q}_{T_0}; N_0) : \mathrm{M}v = G \,\mathrm{on}\, \Sigma_{T_0}\}$. *We write the boundary matrix* $A_\nu(v)$ *in the following block-form*

$$A_\nu(v) = \begin{pmatrix} A_\nu^{I,I}(v) & A_\nu^{I,II}(v) \\ \\ A_\nu^{II,I}(v) & A_\nu^{II,II}(v) \end{pmatrix}, \tag{5}$$

where $A_\nu^{I,I}(v), A_\nu^{I,II}(v), A_\nu^{II,I}(v), A_\nu^{II,II}(v)$ *are respectively* $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ *submatrices. We suppose that* $A_\nu^{I,II}(v)$, $A_\nu^{II,I}(v)$, $A_\nu^{II,II}(v)$ *vanish on* Σ_T *for every* $v \in N_1$. *We also suppose that* $A_\nu^{I,I}(v)$ *is invertible on* Σ_T *and there exists* $a_1 > 0$ *such that satisfies*

$$|(A_\nu^{I,I})^{-1}(v)| \le a_1 \qquad \forall v \in N_1. \tag{6}$$

Furthermore, we suppose that there exists $\delta > 0$ *such that*

$$(A_\nu(v)u, u) \ge \delta |Pu|^2 - \frac{1}{\delta}|Mu|^2 \ \ \forall u \in \mathbb{R}^N, \forall (x,t) \in \Sigma_{T_0}, \ \forall v \in N_1, \tag{7}$$

where P *is the orthogonal projection onto* $(\ker A_\nu(v))^\perp$.

(iv) $M = (I_d, 0)$ *with* $d \le r$, *where* I_d *is the* $d \times d$ *unit matrix.*

Then there exists $0 < T \le T_0$ *such that in* $(0,T)$ *the mixed problem (1) has a unique solution* $u \in \mathcal{C}_T(H^m_*)$ *with* $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$. *A lower bound for* T *is given by inequalities of the form*

$$T\Phi_1(|||f|||_{[m/2],*}) \le 1, \quad [F]_{[m/2],*,T}\Phi_2(|||f|||_{[m/2],*}) \le 1. \tag{8}$$

The solution satisfies the estimate

$$|||u|||_{m,*,T} + \|Pu\|_{H^m(\Sigma_T)} \le \Phi_3(|||f|||_{m,*}, [F]_{m,*,T}, ||G||_{H^m(\Sigma_T)}), \tag{9}$$

where the functions Φ_i, $i = 1,2,3$ *are increasing in their arguments.*

Remark 2.2. As in Secchi [14] it can be proved that if the initial data f are in the usual Sobolev space H^m we can weaken the assumption on m by taking $m \ge 2[\frac{n}{2}] + 6$ (see Theorem 3.2). We underline that the solution still lives in $\mathcal{C}_T(H^m_*)$.

Observe that conditions (iii) and (iv) yield $\ker A_\nu(v) \subseteq \ker M$ for suitable v.

In the sequel, for $j = 0, \dots, n$, we write the matrices $A_j(u)$ in the block-form as in assumption (iii), i.e.

$$A_j(u) = \begin{pmatrix} A_j^{I,I}(u) & A_j^{I,II}(u) \\ A_j^{II,I}(u) & A_j^{II,II}(u) \end{pmatrix}$$

and we decompose $u = (u^I, u^{II})$, $F = (F^I, F^{II})$, and so on. Observe that $Pu = (u^I, 0)$. We now introduce the space

$$\mathcal{H}^m = \{u \in H^m_* : \ \partial_1 u^I \in H^{m-1}_*\} = \{u \in H^m_* : u^I \in H^m_{**}\},$$

with the norm

$$||u||^2_{\mathcal{H}^m} = ||u||^2_{m,*} + ||\partial_1 u^I||^2_{m-1,*}.$$

We set $\mathcal{H}^0 = L^2$. Let also

$$\mathcal{C}_T(\mathcal{H}^m) = \bigcap_{k=0}^{m} C^k([0,T]; \mathcal{H}^{m-k}),$$

with the norm

$$|||u|||_{m,T} = \sup_{[0,T]} |||u(t)|||_m,$$

where $|||u(t)|||^2_m = |||u(t)|||^2_{m,*} + |||\partial_1 u^I|||^2_{m-1,*}$. Define $\mathcal{C}_T(H^m)$ similarly by using H^{m-k} instead of $\mathcal{H}^{m-k}$.

We can prove an additional regularity for the solution of problem (1). More precisely we have the following results.

Theorem 2.3. *Assume that conditions of Theorem 2.1 hold.*

(i) Then $f^{(k)} \in \mathcal{H}^{m-k}$ and $Mf^{(k)} \in H^{m-k-1/2}(\Gamma)$, $k = 0, 1, \dots, m-1$.

(ii) Let $u \in \mathcal{C}_T(H^m_*)$ be a solution of problem (1). Then $u \in \mathcal{C}_T(\mathcal{H}^m)$ and, for each $t \in [0,T]$, $M\partial_t^k u(t) \in H^{m-k-1/2}(\Gamma)$, $k = 0, 1, \dots, m-1$.

In particular, the compatibility conditions $Mf^{(k)} = \partial_t^k G_{|t=0}$, $k = 0, \dots, m-1$, hold in the sense of $H^{m-k-1/2}(\Gamma)$. The proof of Theorem 2.3 follows by adapting to the half-plane case the proofs of Lemma 6.2 and Corollary 6.3 in Secchi [14], obtained in the case of bounded domain. We omit the details.

Remark 2.4. From Theorem 2.3, the solution $u \in \mathcal{C}_T(H^m_*)$ of (1) satisfies $M\partial_t^k u(t) = \partial_t^k G(t)$ on $[0,T] \times \Gamma$ for $k = 0, \dots, m-1$. Then $Mf^{(k)} = \partial_t^k G_{|t=0}$

on Γ, for $k = 0, \dots, m-1$. It follows that the compatibility conditions of order $m-1$ are not only sufficient, as shown by Theorem 2.1, but also necessary for the existence of a solution $u \in \mathcal{C}_T(H^m_*)$.

3. Proof of Theorem 2.1: The Existence

We prove the existence of the solution in two steps: first we prove that the solution belongs to $\mathcal{C}_T(H^{[\frac{m}{2}]}_*)$, then we pass from $[\frac{m}{2}]$ to m.

3.1. Existence in $\mathcal{C}_T(H^{[\frac{m}{2}]}_*)$

Lemma 3.1. *Let μ be an integer such that $2[\frac{n}{2}] + 6 \le \mu \le m$. Assume that the hypotheses (i)–(iv) of Theorem 2.1 hold, with μ instead of m. Assume also that there exists a function $w \in \mathcal{C}_{T_0}(H^\mu_*)$ such that $\partial_t^k w(0) = f^{(k)}$ for $k = 0, \dots, \mu$ and that $Mw = G$ on Σ_{T_0}. Then there exists $T \le T_0$ such that in $(0,T)$ the mixed problem (1) has a unique solution $u \in \mathcal{C}_T(H^\mu_*)$. T and u satisfy estimates of the form (8) and (9) with μ instead of m.*

Proof. The proof is quite the same as that of Lemma 3.1 in Secchi [14] with small modifications due to the non-homogeneous boundary conditions. For convenience of the reader we sketch the fundamental steps of the proof. We prove existence using a fix-point argument.

For $T \in (0, T_0]$ we define

$$K = \{w \in \mathcal{L}^\infty_T(H^\mu_*) \,:\, |||w|||_{\mu,*,T} \le b_1,\ |||w|||_{\mu-2,*,T} \le b_2,$$
$$\partial_t^k w(0) = f^{(k)}, k = 0, \dots, \mu,\ Mw = G \,\text{on}\, \Sigma_T\},$$

with $b_1 > |||f|||_{\mu,*}$ and $b_2 > |||f|||_{\mu-2,*}$ positive constants to be chosen later. By the assumptions in the lemma, K is not empty and it is a closed subset of $\mathcal{L}^\infty_T(H^{\mu-2}_*)$. Moreover, for $w \in K$, we have

$$|w - f|_{\infty,T} \le |\partial_t w|_{\infty,T} T \le c b_2 T.$$

Choose T such that $cb_2T \le \alpha_0$. Hence $w \in N_1$, the matrices $A_j(w)$ for $j = 0, \dots, n$, $A_\nu(w)$ satisfy the hypotheses (ii), (iii) of Theorem 2.1. Furthermore, by Lemma A.7, we get $A_0(w), \dots, A_n(w), B(w) \in \mathcal{L}^\infty_T(H^\mu_*)$. Consider the map Φ defined by $\Phi(w) = u_w$ where u_w is the solution of the linearized problem

$$\begin{cases} L(w)u = F & \text{in} \quad Q_T, \\ Mu = G & \text{on} \quad \Sigma_T, \\ u(x,0) = f & \text{in} \quad \Omega. \end{cases} \tag{10}$$

We apply Theorem B.1 for $\mu = s = m$ and find a unique solution $u_w \in \mathcal{C}_T(H^\mu_*)$ of (10) which satisfies the estimate

$$|||u_w(t)|||^2_{\mu,*} \leq \{C_1|||f|||^2_{\mu,*} + C_2[F]^2_{\mu,*,t} + \frac{C_1}{\delta}||G||^2_{H^\mu(\Sigma_t)}\}e^{C_2 t}, \qquad (11)$$

where C_1 is an increasing function of $|||w|||_{\mu-2,*,T}$ and C_2 is an increasing function of $|||w|||_{\mu,*,T}$. Hence

$$C_1(|||w|||_{\mu-2,*,T}) \leq C_1(b_2), \quad C_2(|||w|||_{\mu,*,T}) \leq C_2(b_1).$$

As in Lemma 3.1 of Secchi [14] we get

$$\begin{aligned} &|||u_w|||_{\mu-2,*,T} \\ &\leq |||f|||_{\mu-2,*} + cTb_2\Big(c'(|||f|||_{\mu,*}) + T[F]_{\mu,*,T} + b_2|||u_w|||_{\mu,*,T}\Big). \end{aligned} \qquad (12)$$

Now, by choosing $b_2 = 2|||f|||_{\mu-2,*}$ in (12), and $C_1(b_2) = \frac{b_1^2}{4|||f|||^2_{\mu,*}}$ in (11), for T sufficiently small, we get $\Phi(K) \subset K$. Moreover, by writing the linear problem satisfied by $u_{w_1} - u_{w_2}$, for $w_1, w_2 \in K$ and applying again (28) we find that, for T sufficiently small, Φ is a contraction in $\mathcal{L}^\infty_T(H^{\mu-2}_*)$ (see Lemma 3.1 of Secchi [14]). Then there exists a unique fixed point $w = \Phi(w) = u_w \in K \cap \mathcal{C}_T(H^\mu_*)$. All the restrictions on T take the form (8) and from $|||u_w|||_{\mu,*,T} \leq b_1$ we get (9) with μ instead of m. □

In order to obtain the existence of the solution of problem (1) it remains to construct the function w of Lemma 3.1. We construct w as the solution of a suitable linear mixed-boundary problem with data in the usual Sobolev spaces. We follow the same idea of Secchi [14]. More precisely, let $m \geq 2[\frac{n}{2}] + 6$ and consider the initial-boundary-value problem

$$\begin{cases} L(f)w = H & \text{in} \quad Q_T, \\ Mw = G & \text{on} \quad \Sigma_T, \\ w(x,0) = f & \text{in} \quad \Omega, \end{cases} \qquad (13)$$

where (f, F, G) are the data of problem (1), and the function H is such that

$$\begin{aligned} &H(0) = F(0), \\ &\partial_t^k H(0) = \partial_t^k F(0) - [\partial_t^k, A_0(w)]\partial_t w_{|t=0} - \sum_{j=1}^n [\partial_t^k, A_j(w)]\partial_j w_{|t=0} \\ &\qquad -[\partial_t^k, B(w)]w_{|t=0}, \qquad k = 1, \dots, m-1. \end{aligned} \qquad (14)$$

If $f \in H^m$ and $F \in \mathcal{L}^2_{T_0}(H^m_*)$ is such that $\partial_t^k F(0) \in H^{m-k-1/2}(\Omega)$ for $k = 0, \dots, m-1$, then

$$\partial_t^k H(0) \in H^{m-k-1/2}(\Omega), \quad k = 0, \dots, m-1.$$

Then there exists $H \in H^m(Q_{T_0})$ which satisfies (14). Moreover, if the data (f, F, G) of the original problem (1) satisfy the compatibility condition of order $m-1$, then from (13) and (14) we formally obtain $\partial_t^k w(0) = f^{(k)}$ for $k = 0, \dots, m$. In particular, the compatibility conditions of order $m-1$ hold for problem (13), hence, by applying Theorem B.1 for $s = m$, there exists a unique solution $w \in \mathcal{C}_{T_0}(H^m_*)$ of (13). If we choose $\mu = m$, this function w satisfies all the conditions of Lemma 3.1. Note that from Lemma 3.1 we get that the original problem (1) has a unique solution $u \in \mathcal{C}_T(H^m_*)$. We have proved the following result.

Theorem 3.2. *Let $m \geq 2[\frac{n}{2}] + 6$ be an integer. Assume that hypotheses (ii)–(iv) of Theorem 2.1 hold. Let $f \in H^m$, $F \in \mathcal{L}^2_{T_0}(H^m_*)$ such that $\partial_t^k F(0) \in H^{m-k-1/2}(\Omega)$ for $k = 0, \dots, m-1$ and $G \in H^m(\Sigma_{T_0})$. Assume the data (f, F, G) satisfy the compatibility conditions of order $m-1$. Then there exist T and a solution $u \in \mathcal{C}_T(H^m_*)$ of problem (1) as in Theorem 2.1. A lower bound for T is given by estimates of the form (8) with m instead of $[\frac{m}{2}]$.*

We are now in the position to find the existence in $\mathcal{C}_T(H_*^{[\frac{m}{2}]})$.

Proposition 3.3. *Under the assumptions of Theorem 2.1, there exists T such that in $(0, T)$ problem (1) has a unique solution $u \in \mathcal{C}_T(H_*^{[\frac{m}{2}]})$. T and u satisfy estimates of the form (8) and (9) with $[\frac{m}{2}]$ instead of m.*

Proof. By Sobolev embeddings we reduce the problem to one which has the data in the usual Sobolev spaces. More precisely, we observe that $[\frac{m}{2}] \geq 2[\frac{n}{2}] + 6$. Moreover, $f \in H^m_* \hookrightarrow H^{[m/2]}$, $F \in \mathcal{L}^2_{T_0}(H^m_*) \hookrightarrow H^{[m/2]}(Q_{T_0})$, which gives $\partial_t^k F(0) \in H^{[m/2]-k-1/2}(\Omega)$, for $k = 0, \dots, [\frac{m}{2}] - 1$ and the compatibility conditions hold up to order $m-1$. We apply Theorem 3.2 which gives the conclusion. □

3.2. Passage from $[\frac{m}{2}]$ to m

Assume that we have already proved $u \in \mathcal{C}_T(H^\mu_*)$ for $[\frac{m}{2}] \leq \mu \leq m-1$. We prove that $u \in \mathcal{C}_T(H^{\mu+1}_*)$. By finite induction we get that $u \in \mathcal{C}_T(H^m_*)$.
In order to show that $u \in \mathcal{C}_T(H^{\mu+1}_*)$, we observe that it is sufficient to prove:

(i) $\partial_\star^\alpha \partial_1^k u \in \mathcal{C}_T(H^1_*)$ for every α, k such that $|\alpha| + 2k = \mu$;

(ii) if μ is odd, $\partial_\star^\alpha \partial_1^{k+1} u \in \mathcal{C}_T(L^2)$ for every α, k such that $|\alpha| + 2k = \mu - 1$.

Note that (i) means that u has one more tangential derivative, while (ii) that u has one more normal derivative than in $\mathcal{C}_T(H_*^\mu)$. This is enough to conclude that $u \in \mathcal{C}_T(H_*^{\mu+1})$. In order to prove (i) and (ii) we proceed by induction on k.

Lemma 3.4. *Let $u \in \mathcal{C}_T(H_*^\mu)$ be a solution of (1), $[\frac{m}{2}] \leq \mu \leq m-1$. Then $\partial_\star^\alpha u \in \mathcal{C}_T(H_*^1)$, with $|\alpha| = \mu$.*

Proof. Since $A_1^{I,I}(u)$ is invertible on Γ we write $\partial_1 u^I$ as the sum of tangential derivative

$$\partial_1 u^I = -(A_1(u)^{I,I})^{-1} \left[\left(A_0(u)\partial_t u + \sum_{j=2}^n A_j(u)\partial_j u \right)^I + A_1^{I,II}(u)\partial_1 u^{II} \right] + R(u) \,, \quad (15)$$

where

$$R(u) = (A_1^{I,I}(u))^{-1}(F - B(u)u)^I.$$

Since $A_1^{I,II}(u) = 0$ for $x_1 = 0$, we can write the term $A_1^{I,II}(u)\partial_1 u^{II} = H(u)\sigma(x_1)\partial_1 u$, where $H(u) = (0, H^{II}(u))$ is a $r \times N$ matrix, $H^{II}(u)$ is a $r \times (N-r)$ submatrix which satisfies the estimates of Lemma A.5. Let $\Lambda(u)$ be the quasilinear operator defined by

$$\begin{aligned} \Lambda(u)\partial_\star u &= \Lambda(u)\big(\partial_t u, \sigma(x_1)\partial_1 u, \partial_2 u, \ldots, \partial_n u\big) \\ &= -(A_1(u)^{I,I})^{-1} \left[\left(A_0(u)\partial_t u + \sum_{j=2}^n A_j(u)\partial_j u \right)^I + H(u)\sigma(x_1)\partial_1 u \right]. \end{aligned} \quad (16)$$

Then, by (15) and (16), we obtain

$$\partial_1 u^I = \Lambda(u)\partial_\star u + R(u). \quad (17)$$

We have $\Lambda(u) \in \mathcal{L}_T^\infty(H_*^{\mu-2})$. We now derive the equation for $\partial_\star^\alpha u$, $|\alpha| = \mu$. We apply the operator $\partial_\star^\alpha$ to $(1)_1$ and, proceeding as in Secchi [14], we get

$$\big(L(u) + \overline{B}(u)\big)\, \partial_\star^\alpha u = \overline{F}_\alpha,$$

for a suitable linear operator $\overline{B}(u) \in L^\infty(0,T;L^2)$. By estimating the products of functions in spaces H_*^m by means of Lemma A.2 – Lemma A.7, we obtain $\overline{F}_\alpha \in \mathcal{L}_T^2(H_*^1)$ (see Secchi [14] for details).

We consider now the problem satisfied by the vector of all the tangential derivatives of order α, by abuse of notation still denoted by $\partial_\star^\alpha u$. It takes the form

$$\begin{cases} (\mathcal{L}+\mathcal{B})\partial_\star^\alpha u = \mathcal{F} & \text{in } Q_T, \\ \mathcal{M}\partial_\star^\alpha u = \partial_\star^\alpha G & \text{on } \Sigma_T, \\ \partial_\star^\alpha u|_{t=0} = \tilde{f} & \text{in } \Omega, \end{cases} \tag{18}$$

where

$$\mathcal{L} = \begin{pmatrix} L(u) & & \\ & \ddots & \\ & & L(u) \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix},$$

$\mathcal{B} \in L^\infty(0,T;L^2)$ is a suitable linear operator and $\mathcal{F}$ is the vector of all right-hand sides $\overline{F}_\alpha$. The initial datum $\tilde{f}$ is the vector of functions $\partial_*^{\alpha'} f^{(\alpha_0)}$ if $\alpha = (\alpha_0, \alpha')$, $\alpha' = (\alpha_1, \dots, \alpha_n)$. From the hypotheses, we get $\mathcal{F} \in \mathcal{L}_T^2(H_*^1)$, $\tilde{f} \in H_*^1$ such that $\tilde{f}^{(k)} \in H_*^{1-k}$ for $k = 0,1$, and $\partial_\star^\alpha G \in H^1(\Sigma_T)$. By applying Theorem B.1 for $m = 1$, we see that problem (18) has a unique solution $\partial_\star^\alpha u \in \mathcal{C}_T(H_*^1)$, with $|\alpha| = \mu$. □

Lemma 3.5. *Let $u \in \mathcal{C}_T(H_*^\mu)$ be a solution of (1), $[\frac{m}{2}] \le \mu \le m-1$. Then $\partial_\star^\alpha \partial_1 u^I \in C_T(L^2)$ for every $|\alpha| = \mu - 1$ and*

$$|||\partial_1 u^I(t)|||_{\mu-1,*} \le C_3 \left\{|||F(t)|||_{\mu-1,*} + |||u(t)|||_{\mu,*}\right\},$$

where C_3 is a constant depending on C_σ, $|||A_j(u)|||_{\mu,*,T}$, $j = 0,\dots,n$ and $|||B(u)|||_{\mu-1,*,T}$.

Proof. We apply the operator $\partial_\star^\alpha \partial_1^k$, $|\alpha| + 2k \le \mu - 1$ to both sides of (15). We use (16) and Lemma 3.4 and we get $\partial_\star^\alpha \partial_1^{k+1} u^I \in C_T(L^2)$. Hence, we obtain the estimate

$$\begin{aligned} |||\partial_1 u^I(t)|||_{\mu-1,*} \le c|||(A_1^{I,I}(u))^{-1}|||_{\mu-1,*,T}\Big\{ |||F(t)|||_{\mu-1,*} \\ + \left(\sum_{j=0}^n |||A_j(u)|||_{\mu,*,T} + |||B(u)|||_{\mu-1,*,T}\right) |||u(t)|||_{\mu,*}\Big\}, \end{aligned} \tag{19}$$

from which we get the result. □

Lemma 3.6. *Let $u \in \mathcal{C}_T(H_*^\mu)$ be a solution of (1), $[\frac{m}{2}] \le \mu \le m-1$. Then $\partial_\star^\gamma \partial_1 u^{II} \in C_T(L^2)$ for every $|\gamma| = \mu - 1$.*

Proof. Applying to the part II of $(1)_1$ the operator $\partial_\star^\gamma \partial_1$, with $|\gamma| = \mu - 1$, shows that the vector of all derivatives $\partial_\star^\gamma \partial_1 u^{II}$ solves a system of the form

$$(\tilde{\mathcal{L}}(u) + \tilde{\mathcal{C}}(u))\partial_\star^\gamma \partial_1 u^{II} = \mathcal{G}, \tag{20}$$

where

$$\tilde{\mathcal{L}}(u) = \begin{pmatrix} \tilde{L}(u) & & \\ & \ddots & \\ & & \tilde{L}(u) \end{pmatrix},$$

and $\tilde{\mathcal{C}}(u) \in L^\infty(0, T; L^2)$ is a suitable linear operator (see Secchi [14] for details). Since $\tilde{L}(u) = A_0^{II,II}(u)\partial_t + \sum_j A_j^{II,II}(u)\partial_j$ has the boundary matrix vanishing on Γ, no boundary condition is required for (20). We observe that $\mathcal{G}$ contains only tangential derivatives of order at most $\mu + 1$. Furthermore, by using Lemma 3.4 and Lemma A.2 – Lemma A.7 we obtain $\mathcal{G} \in L^2(Q_T)$. By Theorem 2.2 of Beirão Da Veiga [1], we get that there exists a unique solution of (20) $\partial_\star^\gamma \partial_1 u^{II} \in C_T(L^2)$, with $|\gamma| = \mu - 1$. □

Lemma 3.7. *Let $u \in \mathcal{C}_T(H_*^\mu)$ be a solution of (1), $[\frac{m}{2}] \leq \mu \leq m-1$. Then $\partial_\star^\alpha \partial_1 u \in C_T(L^2)$ for every $|\alpha| = \mu - 1$.*

Proof. The result follows by Lemma 3.5 and Lemma 3.6. □

Lemma 3.8. *Let $u \in \mathcal{C}_T(H_*^\mu)$ be a solution of (1), $[\frac{m}{2}] \leq \mu \leq m - 1$. Let $1 < k_0 \leq [\frac{\mu+1}{2}]$ and assume that $\partial_\star^\alpha \partial_1^k u \in C_T(L^2)$, for every α, k such that $k = 0, \dots, k_0 - 1$, $|\alpha| + 2k \leq \mu + 1$. Then $\partial_\star^\beta \partial_1^{k_0} u \in C_T(L^2)$ for every β such that $|\beta| + 2k_0 = \mu + 1$.*

Proof. From Lemma 3.5 we already know that $\partial_1 u^I \in \mathcal{C}_T(H_*^{\mu-1})$ which implies that $\partial_\star^\beta \partial_1^{k_0} u^I \in C_T(L^2)$ for every β such that $|\beta| + 2k_0 = \mu + 1$.

We apply the operator $\partial_\star^\beta \partial_1^{k_0}$, $|\beta| + 2k_0 = \mu + 1$ to the part II of $(1)_1$. As in Secchi [14] we obtain the system

$$(\mathcal{L}'(u) + C'(u))\partial_\star^\beta \partial_1^{k_0} u^{II} = \mathcal{G}_{k_0},$$

where $\mathcal{C}'(u)$ is a linear operator such that $\mathcal{C}'(u) \in L^\infty(0, T; L^2)$ and $\mathcal{L}'(u)$ is a linear operator with boundary matrix equal to zero on Γ. We show $\mathcal{G}_{k_0} \in L_T^2(L^2)$. Again by Theorem 2.2 of Beirão Da Veiga [1], the solution $\partial_\star^\beta \partial_1^{k_0} u^{II} \in C_T(L^2)$ for any β, k_0 such that $|\beta| + 2k_0 = \mu + 1$ which concludes the proof. □

We apply Lemma 3.8 recursively and we obtain that $\partial_\star^\beta \partial_1^k u^{II} \in C_T(L^2)$ for every β, k such that $|\beta| + 2k = \mu + 1$. By Lemma 3.5 we know that $\partial_1 u^I \in \mathcal{C}_T(H_*^{\mu-1})$, thus $\partial_\star^\beta \partial_1^k u^I \in C_T(L^2)$ for every $|\beta| + 2k \le \mu + 1$. It follows that $\partial_\star^\beta \partial_1^k u \in C_T(L^2)$ for every β, k such that $|\beta| + 2k = \mu + 1$. This gives $u \in \mathcal{C}_T(H_*^{\mu+1})$. By finite induction we conclude that $u \in \mathcal{C}_T(H_*^m)$.

4. Proof of Theorem 2.1: Energy-Type Estimate

In this section we derive estimate (9). Recall that the boundary matrix $A_\nu(u)$ satisfies the strictly dissipative condition (7).

Lemma 4.1. *Let $u \in \mathcal{C}_T(H_*^m)$ be the solution of system* (1) *and let α be a multi-index with $|\alpha| \le m$. Then, for each $t \in [0,T]$, the estimate*

$$\frac{d}{dt}\|A_0^{1/2}(u)\partial_\star^\alpha u(t)\|^2 + \delta\|P\partial_\star^\alpha u(t)\|^2_{L^2(\Gamma)} \le c(t)(|||u(t)|||^2_{m,*} + |||F(t)|||^2_{m,*}) + \frac{c}{\delta}\|\partial_\star^\alpha G(t)\|^2_{L^2(\Gamma)},$$

holds, with $c(t) = c(|||A_j(u)(t)|||_{m,*}, |||B(u)(t)|||_{m,*})$ for $j = 0, \dots, n$.

Proof. We apply $\partial_\star^\alpha$ to $(1)_1$ and obtain

$$L(u)(\partial_\star^\alpha u) = F_\alpha, \tag{21}$$

where

$$F_\alpha = \partial_\star^\alpha F - [\partial_\star^\alpha, A_0(u)\partial_t]\,u - \sum_{j=1}^n [\partial_\star^\alpha, A_j(u)\partial_j]\,u - [\partial_\star^\alpha, B(u)]\,u.$$

We multiply (21) by $\partial_\star^\alpha u$ and we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A_0^{1/2}(u)\partial_\star^\alpha u|^2 + \frac{1}{2}\int_\Gamma (A_\nu(u)\partial_\star^\alpha u, \partial_\star^\alpha u) \le \int_\Omega (F_\alpha, \partial_\star^\alpha u) + c(t)\|\partial_\star^\alpha u(t)\|^2,$$

where $c(t)$ is a time function depending on the L^∞-norm of the matrices $\partial_t A_0(u)$, $\partial_j A_j(u)$, $j = 1, \dots, n$ and $B(u)$, hence by the norm of $|||A_j(u)(t)|||_{m,*}$, $j = 0, \dots, n$ and $|||B(u)(t)|||_{m,*}$ (see Lemma A.7 and use Sobolev embeddings). By estimating $\|F_\alpha\|$, it follows that

$$\int_\Omega (F_\alpha, \partial_\star^\alpha u) \le c(t)(|||u(t)|||^2_{m,*} + |||F(t)|||^2_{m,*}),$$

where $c(t)$ is as above. The result now follows from (2). □

Lemma 4.2. *Let $u \in \mathcal{C}_T(H^m_*)$ be the solution of system* (1)*. Then $\partial_1 u^I$ satisfies*

$$|||\partial_1 u^I(t)|||_{m-2,*} \leq C_4 \Big\{ |||u(t)|||_{m-1,*} + |||F(t)|||_{m-2,*} \Big\},$$

for each $t \in [0,T]$ and $C_4 = c\,(C_\sigma, |||A_j(u)|||_{m-1,*,T}, |||B(u)|||_{m-2,*,T})$, $j = 0,\dots,n$, a real positive constant

Proof. We proceed as in Lemma 3.5 with $m-2$ instead of $\mu - 1$. □

As in Secchi [14] we prove the following result.

Lemma 4.3. *Let $u \in \mathcal{C}_T(H^m_*)$ be the solution of system* (1)*. Let $k \geq 1$ be an integer and α be a multi-index such that $|\alpha| + 2k \leq m$. Then, for each $t \in [0,T]$, $\partial^\alpha_\star \partial^k_1 u^{II}$ satisfies*

$$\frac{d}{dt} \|(A^{II,II}_0)^{1/2} \partial^\alpha_\star \partial^k_1 u^{II}(t)\|^2 \leq c(t) \left(|||u(t)|||^2_{m,*} + |||F(t)|||^2_{m,*} \right),$$

where $c(t) = c\,(|||A_j(u)(t)|||_{m,*}, |||B(u)(t)|||_{m,*})$ for $j = 0,\dots,n$.

We conclude the proof. Let us first observe that

$$|||u(t)|||_{m,*} = \sum_{|\alpha| \leq m} ||\partial^\alpha_\star u(t)|| + |||\partial_1 u^I|||_{m-2,*} + \sum_{|\alpha|+2k \leq m} ||\partial^\alpha_\star \partial^k_1 u^{II}||. \tag{22}$$

We add the inequalities appearing in Lemma 4.1, Lemma 4.3 and substitute (22). This gives

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{|\alpha| \leq m} ||A^{1/2}_0 \partial^\alpha_\star u(t)||^2 + \sum_{|\beta|+2k \leq m} ||(A^{II,II}_0)^{1/2} \partial^\beta_\star \partial^k_1 u^{II}(t)||^2 \right) \\ &\quad + \delta \sum_{|\alpha| \leq m} ||\partial^\alpha_\star P u(t)||^2_{L^2(\Gamma)} \leq c(t) \left(\sum_{|\alpha| \leq m} ||\partial^\alpha_\star u(t)||^2 + |||\partial_1 u^I(t)|||^2_{m-2,*} \right. \\ &\qquad \left. + \sum_{|\alpha|+2k \leq m} ||\partial^\alpha_\star \partial^k_1 u^{II}||^2 + |||F(t)|||^2_{m,*} \right) + \frac{c}{\delta} \sum_{|\alpha| \leq m} ||\partial^\alpha_\star G(t)||^2_{L^2(\Gamma)}, \end{aligned}$$

where $c(t) = c(\sigma^{-1}, |||A_j(t)|||_{m,*}, |||B(t)|||_{m,*})$ for $j = 0,\dots,n$ is a suitable function of time. We use Lemma 4.2 and we get

$$\frac{d}{dt}\left(\sum_{|\alpha|\le m}||A_0^{1/2}\partial_\star^\alpha u(t)||^2+\sum_{|\beta|+2k\le m}||(A_0^{II,II})^{1/2}\partial_\star^\beta\partial_1^k u^{II}(t)||^2\right)$$
$$+\,\delta\sum_{|\alpha|\le m}||\partial_\star^\alpha Pu(t)||^2_{L^2(\Gamma)}$$
$$\le c(t)\left(\sum_{|\alpha|\le m}||\partial_\star^\alpha u(t)||^2+|||u(t)|||^2_{m-1,*}+\sum_{|\alpha|+2k\le m}||\partial_\star^\alpha\partial_1^k u^{II}||^2\right.$$
$$\left.+|||F(t)|||^2_{m,*}\right)+\frac{c}{\delta}\sum_{|\alpha|\le m}||\partial_\star^\alpha G(t)||^2_{L^2(\Gamma)}.$$

We integrate in time, use $A_0\ge a_0I$ and get

$$\sum_{|\alpha|\le m}||\partial_\star^\alpha u(t)||^2+\sum_{|\alpha|+2k\le m}||\partial_\star^\alpha\partial_1 u^{II}(t)||^2$$
$$\le C\left\{|||f|||^2_{m,*}+[F]^2_{m,*,t}+[u]^2_{m,*,t}+\tfrac{1}{\delta}[G]^2_{H^m(\Sigma_t)}\right\}. \tag{23}$$

We add in the left-hand side of (23) $|||\partial_1 u^I|||^2_{m-2,*}$. By observing that

$$|||F(t)|||_{m-2,*}\le|||F(0)|||_{m-2,*}+t^{1/2}[F]_{m,*,t}$$
$$\le c|||f|||_{m,*}+t^{1/2}[F]_{m,*,t}, \tag{24}$$

we get

$$|||u(t)|||^2_{m,*}\le C\left\{|||f|||^2_{m,*}+[F]^2_{m,*,t}+[u]^2_{m,*,t}+||G||^2_{H^m(\Sigma_t)}\right\}. \tag{25}$$

An application of the Gronwall Lemma gives the estimate (9).

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Appendix A

In this appendix we recall some fundamental results about function spaces H^m_* and H^m_{**}. For the proof we refer to Secchi [17], [13], [14].

Lemma A.1. *Let $m \geq 1$ be an integer. Then:*

(i) $C^\infty(\overline{\Omega})$ is dense in H^m_*,in H^m_{**} and in $\mathcal{H}^m$.

(ii) $C^\infty([0,T] \times \overline{\Omega})$ is dense in $\mathcal{C}_T(H^m_*)$ and in $\mathcal{L}^2_T(H^m_*)$.

Lemma A.2. *Let $s \geq 2\left[\frac{n}{2}\right] + 3$, $m = 0, \ldots, s$. Consider functions $u \in H^m_*$ and $v \in H^s_*$. Then $uv \in H^m_*$ and*

$$\|uv\|_{m,*} \leq c\|u\|_{m,*}\|v\|_{s,*}.$$

Lemma A.3. *Let $r \geq 2\left[\dfrac{n}{2}\right] + 3$, and $1 \leq m \leq r$.*

(i) If $u \in \mathcal{L}^2_T(H^m_*)$ and $v \in \mathcal{L}^\infty_T(H^r_*)$, then $uv \in \mathcal{L}^2_T(H^m_*)$ and

$$[uv]_{m,*,T} \leq c[u]_{m,*,T}|||v|||_{r,*,T}.$$

(ii) If $u \in \mathcal{C}_T(H^m_*)$ and $v \in \mathcal{C}_T(H^r_*)$, then $uv \in \mathcal{C}_T(H^m_*)$ and

$$|||uv|||_{m,*,T} \leq c|||u|||_{m,*,T}||||v||||_{r,*,T}.$$

(iii) If $f^{(k)} \in H^{m-k}_*$, for $k = 0, \ldots, m$, and $v \in \mathcal{C}_T(H^r_*)$, then

$$|||v(0)f|||_{m,*} \leq c|||v(0)|||_{r,*}|||f|||_{m,*}.$$

Lemma A.4. *Let $A \in H^\sigma_*(\mathbb{R}^n_+)$, $\sigma = 2\left[\frac{n}{2}\right]+4$, be a matrix-valued function such that $A = 0$ if $x_1 = 0$. Then, for each regular enough vector-valued function u,*

$$\|A\partial_1 u\| \leq c\|A\|_{\sigma,*}\|\sigma(x_1)\partial_1 u\|. \tag{26}$$

Lemma A.5. *Let $s \geq 2\left[\frac{n}{2}\right]+6$. Let $A = A(u)$ be a $N \times N$ matrix-valued function of u such that $A(u) \in H^s_*(\mathbb{R}^n_+)$ and $A = 0$ if $x_1 = 0$. Let $H = H(u)$ be defined by*

$$H(u(x_1, x')) = \frac{1}{x_1}\int_0^{x_1} \partial_1 A(u(y, x'))dy.$$

Then,

$$\|H\|_{s-2,*} \leq c\|A\|_{s,*}.$$

Lemma A.6. *Let $\tau = 2\left[\frac{n}{2}\right]+6$ and let $A \in H^\tau_*(\mathbb{R}^n_+)$ be a matrix-valued function such that $A = 0$ if $x_1 = 0$. Then, for each smooth enough vector-valued function u,*

$$\|\partial^\alpha_\star A\partial_1 u\| \leq c\|A\|_{\tau,*}||\sigma(x_1)\partial_1 u\| \qquad \text{if } \|\alpha| \leq 2,$$

$$|||\partial^\alpha_\star A\partial_1 u|||_{1,*} \leq c\|A\|_{\tau,*}|||\sigma(x_1)\partial_1 u|||_{1,*} \quad \text{if } |\alpha| \leq 1.$$

Lemma A.7. *Let $r > n+2$ be an integer. Let u and v be in $\mathcal{C}_T(H^r_*)$ and let $A = A(u)$ be an $N \times N$ smooth matrix-valued function of u. Then $A(u) \in \mathcal{C}_T(H^r_*)$ and*

$$|||A(u)|||_{r,*,T} \leq C(M_{[n/2]+1})\{1 + |||u|||^r_{r,*,T}\}.$$

Moreover,

$$\begin{aligned}&|||A(u) - A(v)|||_{r,*,T}\\ &\quad \leq C(N_{[n/2]+1})|||u - v|||_{r,*,T}\{1 + |||u|||^r_{r,*,T} + |||v|||^r_{r,*,T}\}.\end{aligned}$$

The constants $M_{[n/2]+1}$ and $N_{[n/2]+1}$ are such that $||u||_{[n/2]+1,T} \leq M_{[n/2]+1}$, $||v||_{[n/2]+1,T} \leq N_{[n/2]+1}$. $C(\cdot)$ is an increasing function of its argument.

We now give the trace theorem in the space H^m_{**}.

Theorem A.8. *Let Ω be $\mathbb{R}^n_+$ or a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ with C^∞ boundary. Let $m \geq 1$ be an integer. Then the mapping*

$$u \longmapsto \{\partial^k_\nu u|_\Gamma \, : k = 0, \ldots, [1/2(m-1)]\}$$

of

$$C^\infty(\overline{\Omega}) \longmapsto C^\infty(\Gamma) \times \cdots \times C^\infty(\Gamma)$$

extends by continuity to a continuous linear mapping of

$$H^m_{**}(\Omega) \longmapsto H^{m-1/2(\Gamma)} \times \prod_{k=1}^{[(m-1)/2]} H^{m-2k}(\Gamma).$$

Appendix B: The Linear Case

Consider the linear differential operator of first order

$$L = A_0\partial_t + \sum_{j=1}^{n} A_j\partial_j + B,$$

where the matrices A_j for $j = 0, \dots, n$ and B are given real $N \times N$ matrix-valued functions of t and x, defined on $\overline{Q}_T$. Consider the initial-boundary-value problem

$$\begin{cases} Lu = F & \text{in} \quad Q_T, \\ Mu = G & \text{on} \quad \Sigma_T, \\ u(x,0) = f & \text{in} \quad \Omega. \end{cases} \tag{27}$$

In Casella et al [2] we prove the following theorem.

Theorem B.1. *Let $T > 0$, let s, m be integers such that $s \geq 2\left[\frac{n}{2}\right] + 6$ and $1 \leq m \leq s$. Assume that:*

(i) the matrices $A_0, \dots, A_n$ are symmetric in $\overline{Q}_T$; there exists a constant $a_0 > 0$ such that $A_0(t,x) \geq a_0$ for all $(t,x) \in \overline{Q}_T$. Furthermore

$$\begin{aligned} &A_j \in \mathcal{L}^\infty_T(H^s_*),\ j = 0, \dots, n \text{ and } B \in \mathcal{L}^\infty_T(H^{s-1}_*), \text{if } m \leq s-1, \\ &A_j, B \in \mathcal{L}^\infty_T(H^s_*),\ j = 0, \dots, n \text{ if } m = s. \end{aligned}$$

(ii) The boundary matrix $A_\nu(t,x)$ is singular on Σ_T, with constant rank r where $0 < r < N$. We suppose $A_\nu(t,x)$ has the block-form (5), the sub-matrices $A_\nu^{I,II}$, $A_\nu^{II,I}$, $A_\nu^{II,II}$ vanish on Σ_T and there exists a constant $\mu > 0$ such that $|det A_\nu^{I,I}| \geq \mu$ on Σ_T. Moreover, the boundary condition is strictly dissipative, i.e.:

$$\exists \delta > 0\ : (A_\nu u, u) \geq \delta |Pu|^2 - \frac{1}{\delta}|Mu|^2 \ \ \forall u \in \mathbb{R}^N,\ \forall (x,t) \in \Sigma_T,$$

where P is the orthogonal projection onto $(\ker A_\nu)^\perp$.

(iii) $M = (I_d, 0)$ with $d \leq r$, where I_d is the $d \times d$ unit matrix.

(iv) $(F,G) \in \mathcal{L}^2_T(H^m_*) \times H^m(\Sigma_T)$, $f^{(k)} \in H^{m-k}_*$, *for* $k = 0, \ldots, m$. *The data* (f, F, G) *satisfy the compatibility conditions of order* $m-1$.

Then problem (27) has a unique solution $u \in \mathcal{C}_T(H^m_*)$ *with* $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$ *and such that, for each* t *in* $[0,T]$,

$$|||u(t)|||^2_{m,*} + \delta ||Pu||^2_{H^m(\Sigma_t)} \leq \left\{ C_1 |||f|||^2_{m,*} + C_2 [F]^2_{m,*,t} + \frac{C_1}{\delta} ||G||^2_{H^m(\Sigma_t)} \right\} e^{C_2 t}, \quad (28)$$

where $C_1 = C(\mu^{-1}, a_0^{-1}, T, |||A_j|||_{s-2,*,T}, |||B|||_{s-2,*,T})$, *while* $C_2 = C(\mu^{-1}, a_0^{-1}, T, |||A_j|||_{s,*,T}, |||B|||_{s,*,T})$. *If* $m \leq s-1$, C_2 *depends on* $|||B|||_{s-1,*,T}$ *instead of* $|||B|||_{s,*,T}$.