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A SPECTRAL ANALYSIS TO STABILIZATION OF A FLEXIBLE BEAM WITH STRUCTURAL DAMPING

Xuezhang Hou[1 §], Shenghan Lai[2], Hong Lai[3]

[1]Department of Mathematics
Towson University
Baltimore, MD 21252-0001, USA
e-mail:xhou@towson.edu

[2]Department of Pathology
Johns Hopkins Medical Institutions
Baltimore, MD 21287, USA

[3]Department of Ophthalmology
Johns Hopkins Medical Institutions
Baltimore, MD 21287, USA

Abstract: A flexible beam with structural damping described by partial differential equations with initial and boundary conditions is investigated in this paper. First, the system is transferred to an abstract evolution equation in an appropriate Hilbert space, and then spectral properties and semigroup generation of system operator are discussed and presented. Finally, the exponential stability of the system is discussed and explored.

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1. Introduction

A great attention has been paid to dynamics and control of flexible beams and robot arms ([3], [4], [15], [9], [6], [10], [18], [5]) in the past thirty years due to the requirements of high-speed performance and low energy consumption. In

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§Correspondence author

studying stabilization problems of engineering, as we know, the time domain criteria, frequency domain criteria, energy multiplier method and spectral analysis method play very important roles. However, spectral analysis method is very difficult to be used because the spectrum-determined growth condition is very hard to be found and shown. In this paper, as a continuation of our work [11], [12], and [13] we shall apply the spectral analysis method to investigate a hyperbolic system formulated by partial differential equations with initial-boundary value conditions. First, we describe this system as an evolution equation in an appropriate Hilbert space, and then discuss spectral properties and semigroup generation of the system operator corresponding to the evolution equation. Finally, we obtain well-posedness and exponential stability of the system.

We shall be concerned with the following system of partial differential equations with the initial and boundary conditions

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} + \eta \frac{\partial^5 u(x,t)}{\partial t \partial x^4} + EI \frac{\partial^4 u(x,t)}{\partial x^4} = 0, \\ \frac{\partial^2 u(x,t)}{\partial x^2}|_{x=0,l} = \frac{\partial^3 u(x,t)}{\partial x^3}|_{x=0,l} = 0, \\ u(x,0) = \varphi_0(x), \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = \psi_0(x), \end{cases} \tag{1.1}$$

which represents a flexible beam with two free ends. Here $u(x,t)$ is the transverse displacement of the beam at the point x and at the time t, l is the length of the beam, η is the structural damping coefficient, EI is the uniform flexural rigidity. As we know, the system (1.1) can be used to describe the motions of beams with two free ends such as slender flying vehicle and space shuttles, etc.

Now, we take $L^2[0,l]$ as a state space, with the inner product and norm as follows:

$$< f, g >= \int_0^l f(x)\overline{g(x)}dx, \quad f, g \in L^2[0,l]\,,$$

$$\|f\|^2 = \int_0^l |f(x)|^2 dx, \quad f \in L^2[0,l]\,.$$

Let H_1 =span$\{1, x\}$, then $L^2[0,l] = H_1 \oplus H_2$, where H_2 is the orthogonal complement of H_1 in $L^2[0,l]$. Suppose P_1 is the projection operator on H_1 and $I - P_1$ is the projection operator on H_2, and so the system (1.1) can be rewritten in H_1 as follows

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} = 0\,, \\ u(x,0) = P_1\varphi_0, \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = P_1\psi_0\,. \end{cases} \tag{1.2}$$

It is clear that the solution of (1.2) can be described as

$$u^{(1)}(x,t) = a_1 + a_2 x + a_3 t + a_4 tx\,, \tag{1.3}$$

where a_1, a_2, a_3 and a_4 are determined uniquely by $P_1\varphi_0$ and $P_1\psi_0$.

Consider the system (1.1) in H_2, we have

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} + \eta \frac{\partial^5 u(x,t)}{\partial t \partial x^4} + EI \frac{\partial^4 u(x,t)}{\partial x^4} = 0\,, \\ \frac{\partial^2 u(x,t)}{\partial x^2}|_{x=0,l} = \frac{\partial^3 u(x,t)}{\partial x^3}|_{x=0,l} = 0\,, \\ u(x,0) = (I - P_1)\varphi_0(x), \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = (I-P_1)\psi_0(x)\,. \end{cases} \tag{1.4}$$

If we denote the solution of (1.4) by $u^{(2)}(x,t)$, then the solution of system (1.1) can be described as

$$u(x,t) = u^{(1)}(x,t) \oplus u^{(2)}(x,t)\,. \tag{1.5}$$

It should be noted that the form of $u^{(1)}(x,t)$ is ready from (1.3), and $u^{(2)}(x,t)$ will play a key role in order to investigate the solution of the system (1.1).

We now define a differential operators A as follows:

$$A\varphi = \phi''''(x),\ \varphi \in D(A),$$
$$D(A) = \{\varphi | \varphi \in H^4(0,l),\ \varphi''(0) = \varphi''(l) = 0, \varphi'''(0) = \varphi'''(l) = 0, \varphi'''' \in H_2\}.$$

Then in terms of the operator A the system (1.4) can be written as follows:

$$\begin{cases} \ddot{u}(t) + \eta A \dot{u}(t) + EIAu(t) = 0\,, \\ u(0) = (I-P_1)\varphi_0, \quad \dot{u}(x,0) = (I-P_1)\varphi_0\,, \end{cases} \tag{1.6}$$

where the prime " $'$ " and the over dot " $\cdot$ " denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ respectively.

To establish an abstract evolution equation for (1.6), we now introduce a Hilbert space $\mathcal{H} = H_2 \times H_2$ equipped with the inner product

$$\langle \vec{u}, \vec{v} \rangle_H = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle\,,$$

where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are any elements in $\mathcal{H}$, $u_i \in H_2$, $v_i \in H_2$, i = 1, 2.

Let $u_1 = u$, $u_2 = du/dt$, and

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -EIA & -\eta A \end{bmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A)\,,$$

$$\vec{u}_0 = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} (I - P_1)\varphi_0 \\ (I - P_1)\psi_0 \end{bmatrix}.$$

Then the system (1.6) or (1.1) is equivalent to the following first order evolution equation

$$\begin{cases} \frac{d\vec{u}(t)}{dt} = \mathcal{A}\vec{u}(t)\,, \\ \vec{u}(0) = \vec{u}_0\,. \end{cases} \tag{1.7}$$

2. The Spectral Properties of A and $\mathcal{A}$

We shall discuss spectral properties of operators A in (1.6) and $\mathcal{A}$ in (1.7) in this section. Let us start with investigation of spectrum of A.

Theorem 2.1. *The operator A is a positive self-adjoint operator in Hilbert space H_2, and A^{-1} exist and is a compact operator. Therefore, the spectrum $\sigma(A)$ of A consists entirely of isolated eigenvalues $\{\lambda\}_{n=1}^{\infty}$ with finite multiplicity so that*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \text{and} \quad \lambda_n \to \infty \quad (n \to \infty)$$

and the set of all normalized eigenvectors $\{\phi_{k_1}, \phi_{k_2}, \cdots, \phi_{k_{j_k}}\}_{k=1}^{\infty}$ constitutes an orthonormal basis of H_2.

Proof. Apply integration by parts with the definition of A and boundary conditions included in $D(A)$ to find

$$\begin{aligned} \langle A\varphi, \varphi\rangle &= \int_0^l \varphi''''(x)\overline{\varphi(x)}dx \\ &= \int_0^l \varphi'''(x)\overline{\varphi'(x)}dx = \int_0^l \varphi''(x)\overline{\varphi''(x)}dx = \|\varphi''\|^2 \geq 0. \end{aligned} \tag{2.1}$$

Hence A is a symmetric operator. In order to show A to be self-adjoint, it suffices to show that there is a constant c such that $\|A\varphi\| \geq c\|\varphi\|$ for every $\varphi \in D(A)$, see [2].

Actually, we can apply the boundary conditions of φ in $D(A)$ and the result of [7] to obtain

$$\int_0^l |\varphi(x)|^2 dx \leq \frac{l^4}{12}\int_0^l |\varphi''|^2 dx,$$

namely,

$$\|\varphi''\|^2 \geq \frac{12}{l^4}\|\varphi\|^2. \tag{2.2}$$

Combining (2.1) and (2.2) leads to the following inequality

$$\|A\varphi\|\|\varphi\| \geq \langle A\varphi, \varphi\rangle = \|\varphi''\|^2 \geq \frac{12}{l^4}\|\varphi\|^2,$$

and so $\|A\varphi\| \geq \frac{12}{l^4}\|\varphi\|$. Hence,

$$\|A\varphi\| \geq c\|\varphi\| \quad \text{for every} \quad \varphi \in D(A)\,, \tag{2.3}$$

where $c = 12/l^2 > 0$. Thus, A is positive self-adjoint.

It can also be seem from (2.3) that A^{-1} exists. Now, let $A\varphi = \psi$, then $\varphi = A^{-1}\psi$, and the inequality (2.3) is equivalent to

$$\|A^{-1}\psi\| \leq \frac{1}{c}\|\psi\|.$$

This implies that the mapping $A^{-1} : H^4(0,l) \to H^4(0,l)$ is bounded, and

$$\|A^{-1}\| \leq \frac{1}{c}. \tag{2.4}$$

Thus, we can conclude by Sobolev embedding Theorem, see [1], that A^{-1} is a compact operator. The proof is complete.

By applying the result of [1], we can arrive at the fact that the spectrum $\sigma(A)$ of A consists entirely of isolated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with finite multiplicity, so that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\,, \quad \text{and} \quad \lambda_n \to \infty\,.$$

The proof is complete. □

A significant result on spectrum of $\mathcal{A}$ is stated and proved in the next theorem.

Theorem 2.2. *Let $\sigma(\mathcal{A})$ and $\sigma_p(\mathcal{A})$ be the spectrum and the point spectrum of $\mathcal{A}$ respectively, then*

$$(i)\sigma(\mathcal{A}) = \sigma_E I(\mathcal{A}) \cup \{-EI/\eta\}, \quad \sigma_E I(\mathcal{A}) = \{\xi_k, \mu_k\}_{k=1}^{\infty}\,,$$

where

$$\xi_k = (-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})/2 \text{ and } \mu_k = (-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})/2$$

are eigenvalues of $\mathcal{A}$, and the eigenvectors of $\mathcal{A}$ for ξ_k and μ_k are

$$\vec{\phi}_{k_j} = \frac{1}{\sqrt{\lambda_k^2 + |\xi_k|^2}} \begin{bmatrix} \phi_{k_j} \\ \xi_k \phi_{k_j} \end{bmatrix}, \quad \vec{\psi}_{k_j} = \frac{1}{\sqrt{\lambda_k^2 + |\mu_k|^2}} \begin{bmatrix} \phi_{k_j} \\ \mu_k \phi_{k_j} \end{bmatrix},$$
$$\|\vec{\phi}_{k_j}\| = \|\vec{\psi}_{k_j}\| = 1, j = 1, 2, \cdots, j_k;\ k = 1, 2, \cdots,$$

respectively.

(ii) *If $\alpha \in \rho(\mathcal{A})$, then*

$$(\alpha I - \mathcal{A})^{-1} = \begin{bmatrix} (\alpha^2 + \eta\alpha A + EIA)^{-1}(\alpha + \eta A) & (\alpha^2 + \eta\alpha A + EIA)^{-1} \\ -I + (\alpha^2 + \eta\alpha A + EIA)^{-1}(\alpha^2 + \eta\alpha A) & \alpha(\alpha^2 + \eta\alpha A + EIA)^{-1} \end{bmatrix}. \tag{2.5}$$

Proof. By verifying directly, we can see that $\{\xi_k, \mu_k\}_{k=1}^{\infty} \subset \sigma_p(\mathcal{A})$, and $\vec{\phi}_{k_j}$, $\vec{\psi}_{k_j}$ are eigenvectors for ξ_k and μ_k, respectively.

Since

$$\begin{aligned}
\lim_{k\to\infty} \xi_k &= \lim_{k\to\infty} \frac{-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k}}{2} \\
&= \lim_{k\to\infty} \frac{(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2} - 4EI\lambda_k)}{2(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})} \\
&= \lim_{k\to\infty} \frac{4EI\lambda_k}{-2(\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})} = \lim_{k\to\infty} (-2)\frac{EI}{\eta + \sqrt{\eta^2 - \frac{4EI}{\lambda_k}}} \\
&= (-2)\frac{EI}{\eta + \eta} = -\frac{EI}{\eta} < 0,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\lim_{k\to\infty} \mu_k &= \lim_{k\to\infty} \frac{(-\eta\lambda_k - \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2} - 4EI\lambda_k)}{2(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})} \\
&= \lim_{k\to\infty} \frac{4EI\lambda_k}{2(-\eta\lambda_k + \sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k})} = \lim_{k\to\infty} \frac{2EI}{-\eta + \sqrt{\eta^2 - \frac{4EI}{\lambda_k}}} = -\infty,
\end{aligned}$$

we have

$$-\frac{EI}{\eta} \in \sigma(\mathcal{A}) \quad \text{and} \quad \{\xi_k, \mu_k\}_{k=1}^{\infty} \cup \{-\frac{EI}{\eta}\} \subseteq \sigma(\mathcal{A}).$$

On the other hand, let $\alpha \neq \xi_k, \mu_k, -\frac{EI}{\eta}$, and $f(\lambda) = \alpha^2\lambda^{-1} + \eta\alpha + EI$, then $f(A) = \alpha^2 A^{-1} + \eta\alpha + EI$ by functional calculus. Since the extended spectrum,

$\sigma_e(A)$ of A, is equal to $\sigma(A)\cup\{\infty\} = \{\lambda_k\}_{k=1}^{\infty}\cup\{\infty\}$, and $\alpha \neq \xi_k$, μ_u and $-\frac{EI}{\eta}$, we have

$$\begin{aligned} f(\lambda_k) &= \alpha^2\lambda_k^{-1} + \eta\alpha + EI \neq 0, \quad \text{otherwise } \ \alpha = \xi_k \text{ or } \mu_k; \\ f(\infty) &= \lim_{k\to\infty} f(\lambda_k) = \eta\alpha + EI \neq 0, \quad \text{otherwise } \alpha = -\frac{EI}{\eta}. \end{aligned}$$

It follows from the spectral mapping theorem (see [17]) that $0 \notin f(\sigma_e(A)) = \sigma(f(A))$, namely $0 \in \rho(f(A))$. This implies that the inverse of $\alpha^2+\eta\alpha A+EIA$ exists, and

$$(\alpha^2 + \eta\alpha A + EIA)^{-1} = A^{-1}(\alpha^2 A^{-1} + \eta\alpha + EI)^{-1} = A^{-1}[f(A)]^{-1}$$

is a bounded linear operator on H_2. Therefore, the operator defined by

$$T = \begin{bmatrix} (\alpha^2 + \eta\alpha A + EIA)^{-1}(\alpha + \eta A) & (\alpha^2 + \eta\alpha A + EIA)^{-1} \\ -I + (\alpha^2 + \eta\alpha A + EIA)^{-1}(\alpha^2 + \eta\alpha A) & \alpha(\alpha^2 + \eta\alpha A + EIA)^{-1} \end{bmatrix}$$

is a bounded linear operator on H. A simple computation shows that

$$(\alpha I - \mathcal{A})T = I_{\mathcal{H}}, \quad T(\alpha I - A) = I_{D(\mathcal{A})},$$

and $\alpha \in \rho(\mathcal{A})$. This implies that $\sigma(\mathcal{A}) \subseteq \{\xi_k, \mu_k\}_{k=1}^{\infty} \cup \{-\frac{EI}{\eta}\}$. Thus, $\sigma(\mathcal{A}) = \{\xi_k, \mu_k\}_{k=1}^{\infty}\cup\{-\frac{EI}{\eta}\}$, and $(\alpha I-\mathcal{A})^{-1} = T$. The proof of the theorem is complete. □

Corollary 2.1. *$\mathcal{A}$ is a closed linear operator.*

Proof. From Theorem 2.1, we know that $\rho(\mathcal{A}) \neq \emptyset$. Since for any $\alpha \in \rho(\mathcal{A}), (\alpha I - \mathcal{A})^{-1}$ is a bounded linear operator, it is closed. It follows that $\lambda I - \mathcal{A} = [(\lambda I - \mathcal{A})^{-1}]^{-1}$ is closed, and hence $\mathcal{A}$ is a closed operator. □

Corollary 2.2. $\sup\{\text{Re }\lambda | \lambda \in \sigma(\mathcal{A})\} = -\omega < 0$.

Proof. It can be seen from the expressions of ξ_k and μ_k that $\text{Re}\xi_k < 0$ and $\text{Re}\mu_k < 0 \quad (k = 1, 2, \cdots)$, and

$$\lim_{k\to\infty} = \xi_k = -\frac{EI}{\eta} \quad \text{and} \quad \lim_{k\to\infty} \mu_k = -\infty$$

and therefore,

$$\lim_{k\to\infty} \text{Re } \xi_k = -\frac{EI}{\eta} \quad \text{and} \quad \lim_{k\to\infty} \text{Re } \mu_k = -\infty.$$

It follows that $\sup\{\text{Re}\, \sigma_n | \sigma_n \in \sigma_p(\mathcal{A})\} \stackrel{\text{def}}{=} -\omega_1 < 0$. Thus, we conclude from (i) of Theorem 2.1 that $\sup\{\text{Re } \mu | \mu \in \sigma(\mathcal{A})\} = \max\{-\omega_1, -\frac{EI}{\eta}\} \stackrel{\text{def}}{=} -\omega < 0$. □

Lemma 2.1. *The family of the following vectors*

$$\left\{\begin{pmatrix}\phi_{k_1}\\0\end{pmatrix},\begin{pmatrix}0\\\phi_{k_1}\end{pmatrix},\cdots,\begin{pmatrix}\phi_{k_{j_k}}\\0\end{pmatrix},\begin{pmatrix}0\\\phi_{k_{j_k}}\end{pmatrix}\right\}_{k=1}^{\infty} \tag{2.6}$$

constitute an orthonormal basis of $\mathcal{H}$.

Proof. Since $\{\phi_{k_1},\cdots,\phi_{k_{j_k}}\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_2$, we see from Theorem 2.1 that form any $\binom{v}{w}\in\mathcal{H}$, where $v,w\in H_2$,

$$v=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}v_{k_j}\phi_{k_j},\quad \text{where}\quad v_{k_j}=\langle v,\,\phi_{k_j}\rangle\,,$$

$$w=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}w_{k_j}\phi_{k_j},\quad \text{where}\quad w_{k_j}=\langle w,\,\phi_{k_j}\rangle.$$

Hence,

$$\begin{pmatrix}v\\w\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\begin{pmatrix}v_{k_j}\phi_{k_j}\\w_{k_j}\phi_{k_j}\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\left[v_{k_j}\begin{pmatrix}\phi_{k_j}\\0\end{pmatrix}+w_{k_j}\begin{pmatrix}0\\\phi_{k_j}\end{pmatrix}\right],$$

and

$$\left\|\begin{pmatrix}v\\w\end{pmatrix}\right\|^2=\sum_{k=1}^{\infty}\sum_{j=1}^{j_k}\Big(\lambda_k^2|v_{k_j}|^2+|w_{k_j}|^2\Big).$$

This implies that the family (2.6) is complete, and it constitutes an orthonormal basis of H. □

Lemma 2.2. *(i) Let $\eta\neq 2\sqrt{EI}\lambda_k^{-\frac{1}{2}}$, $(k=1,2,\cdots)$ then the set of the pairs eigenvectors of $\mathcal{A}$, $\{\vec{\phi}_{k_1},\vec{\psi}_{k_1},\cdots,\vec{\phi}_{k_{j_k}},\vec{\psi}_{k_{j_k}}\}_{k=1}^{\infty}$, constitutes a Riesz basis of $\mathcal{H}$.*

(ii) If $\eta=2\sqrt{EI}\lambda_{k^*}^{-\frac{1}{2}}$ for some k^* (there exist at most one k^*), then the set of eigenvectors of $\mathcal{A}$, $\{\vec{\phi}_{k_1},\vec{\psi}_{k_1},\cdots,\vec{\phi}_{k_{j_k}},\vec{\psi}_{k_{j_k}}\}_{k_i\neq k^*}\cup\{\vec{\phi}_{k_1^*},\cdots,\vec{\phi}_{k^*_{j_{k^*}}}\}$ together with $\{\binom{0}{\phi_{k_1^*}},\cdots,\binom{0}{\phi_{k^*_{j_{k^*}}}}\}$, constitute a basis of $\mathcal{H}$.

Proof. (i) Since for arbitrary $\binom{v}{w}\in\mathcal{H}$, we have from Lemma 2.1 and the definitions of $\vec{\phi}_k$ and $\vec{\psi}_k$ that

$$\begin{pmatrix}v\\w\end{pmatrix}=\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}[v_{k_j}\begin{pmatrix}\phi_{k_j}\\0\end{pmatrix}+w_{k_j}\begin{pmatrix}0\\\phi_{k_j}\end{pmatrix}]=\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}(a_{k_j}^{(1)}\vec{\phi}_{k_j}+a_{k_j}^{(2)}\vec{\psi}_{k_j})\,,$$

where

$$a_{k_j}^{(1)} = (-\mu_k v_{k_j} - w_{k_j})\frac{\sqrt{\lambda_k^2 + |\xi_k|^2}}{\sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k}},$$

$$a_{k_j}^{(2)} = (\xi_k v_{k_j} - w_{k_j})\frac{\sqrt{\lambda_k^2 + |\mu_k|^2}}{\sqrt{(\eta\lambda_k)^2 - 4EI\lambda_k}}$$

and thus it is an Riesz basis of $\mathcal{H}$, see [6], [10], [18].

(ii) If $\eta = 2\sqrt{EI}\lambda_{k^*}^{-\frac{1}{2}}$, we can obtain from Theorem 2.1 that $\xi_{k^*} = \mu_{k^*} = -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}$, and the eigenvectors corresponding to ξ_{k^*} and μ_{k^*} are as follows

$$\vec{\phi}_{k_j^*} = \vec{\psi}_{k_j^*} = \frac{1}{\sqrt{\lambda_{k^*}^2 + EI\lambda_{k^*}}}\begin{pmatrix} \phi_{k_j^*} \\ -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix} \quad (j = 1, 2, \cdots, j_{k^*})\,.$$

Then

$$v_{k_j^*}\begin{pmatrix} \phi_{k_j^*} \\ 0 \end{pmatrix} + w_{k_j^*}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} = (\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}} v_{k_j^*} + w_{k_j^*})\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + v_{k_j^*}\begin{pmatrix} \phi_{k^*} \\ -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix}$$
$$= b_{k_j^*}^{(1)}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)}\vec{\phi}_{k_j^*} \quad (j = 1, 2, \cdots, j_{k^*})\,,$$

where

$$b_{k_j^*}^{(1)} = \sqrt{EI}\lambda_{k^*}^{\frac{1}{2}} v_{k_j^*} + w_{k_j^*}, \quad b_{k_j^*}^{(2)} = v_{k_j^*}\sqrt{\lambda_{k^*}^2 + EI\lambda_{k^*}}\,.$$

Thus, for every $\binom{v}{w} \in \mathcal{H}$, we have

$$\begin{pmatrix} v \\ w \end{pmatrix} = \sum_{k=1}^{\infty}\sum_{j=1}^{j_k}[v_{k_j}\begin{pmatrix} \phi_{k_j} \\ 0 \end{pmatrix} + w_{k_j}\begin{pmatrix} 0 \\ \phi_{k_j} \end{pmatrix}]$$
$$= \sum_{\substack{k=1 \\ k\neq k^*}}^{\infty}\sum_{j=1}^{j_k}(a_{k_j}^{(1)}\vec{\phi}_{k_j} + a_{k_j}^{(2)}\vec{\psi}_{k_j}) + \sum_{j=1}^{j_{k^*}}[b_{k_j^*}^{(1)}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)}\vec{\phi}_{k_j^*}]\,,$$

and so the conclusion of (ii) of Lemma 2.2 is obtained. □

3. Semigroup Generation

In this section, we shall study the semigroup generation of $\mathcal{A}$. The main results are shown in the following theorems.

Theorem 3.1. *The operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{H}$, and there is constant $M > 0$ such that*

$$\|T(t)\| \le Me^{-\omega t},$$

where $-\omega = \sup\{\mathrm{Re}\,\mu | \mu \in \sigma(\mathcal{A})\} < 0$.

Proof. We shall prove the theorem in two different cases,

Case 1. $\eta \neq 2\sqrt{EI}\lambda_k^{-\frac{1}{2}}(k = 1, 2, \cdots)$. For the sake of simplicity, we denote the eigenpairs of $\mathcal{A}$ by $\{\sigma_n, \vec{e}_n\}, n = 1, 2, \cdots$. For every real λ, $\lambda > -\omega = \sup\{\mathrm{Re}\ \mu|\mu \in \sigma(\mathcal{A})\}$, we see that $\lambda \in \rho(\mathcal{A})$. For any $\vec{u} \in \mathcal{H}$, since $\{\vec{e}_n\}$ constitutes a Riesz basis of $\mathcal{H}$ (see (i) of Lemma 2.2), $\vec{u} = \sum_{n=1}^{\infty} a_n\vec{e}_n$. A simple computation shows that

$$(\lambda I - \mathcal{A})^{-1}\vec{u} = \sum_{n=1}^{\infty} a_n \frac{1}{\lambda - \sigma_n}\vec{e}_n$$

and

$$\begin{aligned}\|[\lambda I - \mathcal{A})^{-1}]^m\vec{u}\| &= \|\sum_{n=1}^{\infty} a_n \frac{1}{(\lambda - \sigma_n)^m}\vec{e}_n\| \\ &= \|\sum_{n=1}^{\infty} a_n \frac{(\lambda+\omega)^m}{(\lambda - \sigma_n)^m}\frac{1}{(\lambda+\omega)^m}\vec{e}_n\|.\end{aligned}$$

It should be noted that $|(\lambda+\omega)^m/(\lambda-\sigma_n)^m| \le 1$ because $-\omega = \sup\{\mathrm{Re}\ \mu|\mu \in \sigma(\mathcal{A})\}$ and $\lambda > -\omega$. Therefore,

$$\|[\lambda I - \mathcal{A})^{-1}]^m\vec{u}\| \le \frac{1}{(\lambda+\omega)^m}\|\sum_{n=1}^{\infty} a_n\vec{e}_n\| = \frac{1}{(\lambda+\omega)^m}\|\vec{u}\|.$$

We thus arrive at the following result:

$$\|[\lambda I - \mathcal{A})^{-1}]^m\| \le \frac{1}{\lambda+\omega)^m}, \quad \lambda > -\omega, \quad m = 1, 2, \cdots.$$

It follows from the Theorem 1.5.3 of [16] that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{H}$, and $\|T(t)\| \le Me^{-\omega t}$, where $M \ge 1$.

Case 2. $\eta = 2\sqrt{EI}\lambda_{k^*}^{-\frac{1}{2}}$ for some positive integer k^*. We see from (ii) of Lemma 2.2 that for any $\vec{u} \in \mathcal{H}$,

$$\vec{u} = \sum_{\substack{k=1\\k\neq k^*}}^{\infty} \sum_{j=1}^{j_k} (a_{k_j}^{(1)} \vec{\phi}_{k_j} + a_{k_j}^{(2)} \vec{\phi}_{k_j}) + \sum_{j=1}^{j_{k^*}} [b_{k_j^*}^{(1)} \begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} + b_{k_j^*}^{(2)} \vec{\phi}_{k_j^*}] .$$

Since

$$\mathcal{A}\vec{\psi}_{k_j^*} = \mu_{k^*}\vec{\psi}_{k_j^*}, \quad j = 1, 2, \cdots, j_{k^*}$$

we have

$$\mathcal{A}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -\eta A \end{pmatrix}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} \phi_{k_j^*} \\ -\eta A\phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} \phi_{k_j^*} \\ -2\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}} A\phi_{k_j^*} \end{pmatrix}$$

$$= \begin{pmatrix} \phi_{k_j^*} \\ -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix} + \begin{pmatrix} 0 \\ -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}\phi_{k_j^*} \end{pmatrix}.$$

Since $\eta = 2\sqrt{EI}\lambda_{k^*}^{-\frac{1}{2}}$, we refer to (i) of Theorem 2.2 to find

$$\begin{aligned}\mu_{k^*} &= (-\eta\lambda_{k^*} - \sqrt{(\eta\lambda_{k^*})^2 - 4EI\lambda_{k^*}})/2 \\ &= (-2\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}} - \sqrt{(2\sqrt{EI}\lambda_{k^*}^{-\frac{1}{2}}\lambda_{k^*})^2 - 4EI\lambda_{k^*}})/2 \\ &= (-2\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}} - \sqrt{4EI\lambda_{k^*} - 4EI\lambda_{k^*}})/2 = -2\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}}/2 = -\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}},\end{aligned}$$

and so

$$\mathcal{A}\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix} = \begin{pmatrix} \phi_{k_j^*} \\ \mu_{k_j^*}\phi_{k_j^*} \end{pmatrix} + (-\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}})\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix}$$

$$= \sqrt{\lambda_{k^*}^2 + EI\lambda_{k^*}}\ \vec{\psi}_{k_j^*} + (-\sqrt{EI}\lambda_{k^*}^{\frac{1}{2}})\begin{pmatrix} 0 \\ \phi_{k_j^*} \end{pmatrix}.$$

Hence, the space spanned by

$$\left\{\vec{\psi}_{k_1^*}, \cdots, \vec{\psi}_{k_{j_{k^*}}^*}\right\} \cup \left\{\begin{pmatrix} 0 \\ \phi_{k_1^*} \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \phi_{k_{j_{k^*}}^*} \end{pmatrix}\right\}$$

is an invariant subspace of $2j_{k^*}$ dimensions of $\mathcal{A}$, denoted by $\mathfrak{M}_{k^*}$. From the theory of finite dimensional space, we assert that

$$\sigma(\mathcal{A}|\mathfrak{M}_{k^*}) = \sigma_p(\mathcal{A}|\mathfrak{M}_{k^*}) \subseteq \sigma_p(\mathcal{A}) \subseteq \sigma(\mathcal{A}),$$

and therefore $-\omega^* = \sup\{\text{Re }\mu|\mu \in \sigma(\mathcal{A}|\mathfrak{M}_{k^*})\} \leq \sup\{\text{Re }\mu|\mu \in \sigma(\mathcal{A})\} = -\omega$. Actually, we can arrange the vectors spanning $\mathfrak{M}_{k^*}$ as follows

$$\begin{pmatrix} 0 \\ \phi_{k_1^*} \end{pmatrix}, \vec{\psi}_{k_1^*}, \begin{pmatrix} 0 \\ \phi_{k_2^*} \end{pmatrix}, \vec{\psi}_{k_2^*}, \cdots, \begin{pmatrix} 0 \\ \phi_{k^*_{j_{k^*}}} \end{pmatrix}, \vec{\psi}_{k^*_{j_{k^*}}}.$$

Let

$$\mathbb{A} = \begin{pmatrix} -\lambda_{k^*}^{\frac{1}{2}} & \sqrt{\lambda_{k^*}^2 + \lambda_{k^*}} \\ 0 & \lambda_{k^*} \end{pmatrix},$$

then $\mathcal{A}|\mathfrak{M}_{k^*}$ has the form

$$\mathcal{A}|\mathfrak{M}_{k^*} = \begin{bmatrix} \mathbb{A} & \ddots & 0 \\ 0 & & \mathbb{A} \end{bmatrix} \quad (\text{there are} j_{k^*} \mathbb{A}'s \text{ in the diagonal}).$$

Apply the result of the Theorem 4.4.3 of [16] to conclude that $\mathcal{A}$ generates a C_0-semigroup $T_1(t)$ satisfying $\|T_1(t)\| \leq M_1 e^{-\omega^* t}$, and so

$$\|T_1(t)\| \leq M_1 e^{-\omega t}. \tag{3.1}$$

On the other hand, since the family $\{\vec{\phi}_{k_1}, \vec{\psi}_{k_1}, \cdots, \vec{\phi}_{k_{j_k}}, \vec{\psi}_{k_{j_k}}\}_{k \neq k^*}$ consists of the eigenvectors of $\mathcal{A}$, the subspace $\mathfrak{M}$ spanned by them is an invariant subspace of $\mathcal{A}$, and this family is just a Riesz basis of $\mathfrak{M}$ (see Lemma 2.2). Thus, form Case 1, it is aware of the fast that $\mathcal{A}$ generates a C_0-semigroup $T_2(t), (t \geq 0)$ in $\mathfrak{M}$. For $\mathfrak{M} \in \rho(\mathcal{A})$, we have

$$(\mu I - \mathcal{A})^{-1}\vec{\phi}_{k_j} = \frac{1}{\mu - \xi_k}\vec{\phi}_{k_j}, \quad (\mu I - \mathcal{A})^{-1}\vec{\varphi}_{k_j} = \frac{1}{\mu - \mu_k}\vec{\psi}_{k_j}$$
$$(j = 1, 2, \cdots, j_k; k \neq k^*)$$

and $(\mu I - \mathcal{A})^{-1}\mathfrak{M}_k \subset \mathfrak{M}_k$, it follows from [8] that $\sigma(\mathcal{A}|\mathfrak{M}_k) \subseteq \sigma(\mathcal{A})$ and $-\omega_2 = \sup\{\text{Re }\mu|\mu \in \sigma(\mathcal{A}|\mathfrak{M}_k)\} \leq \sup\{\text{Re }\mu|\mu \in \sigma(\mathcal{A})\} = -\omega$. Thus, there is $M_2 > 0$ such that

$$\|T_2(t)\| \leq M_2 e^{-\omega t}. \tag{3.2}$$

Since $\mathfrak{M}_{k^*}$ is finite dimensional, it is a closed subspace of $\mathcal{H}$, and so $\mathcal{H} = \mathfrak{M}_{k^*} \oplus \mathfrak{M}_k$, where $\oplus$ expresses orthogonal sum in Hilbert space $\mathcal{H}$. Based on the fact that $T_1(t)T_2(t) = T_2(t)T_1(t) = 0$), we now define $T(t) \overset{\text{def}}{=} T_1(t) \oplus T_2(t)$.

We shall next prove an interesting result that $T(t)$ is exactly a C_0-semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. The semigroup properties of $T(t)$ can be easily presented as follows:

(i) $T(0) = T_1(0) \oplus T_2(0) = I_{\mathfrak{M}_{k^*}} \oplus I_{\mathfrak{M}_k} = I_{\mathcal{H}}$.

(ii) $T(t+s) = T_1(t+s) \oplus T_2(t+s) = [T_1(t)T_1(s) \oplus [T_2(t)T_2(s)]$
$= [T_1(t) \oplus T_2(t)][T_1(s) \oplus T_2(s)] = T(t)T(s) \qquad (t, s \geq 0).$

(iii) For every $x \in \mathcal{H}$, since $\mathcal{H} = \mathfrak{M}_{k^*} \oplus \mathfrak{M}_k$, $x = x_{k^*} \oplus x_k$, where $x_{k^*} \in \mathfrak{M}_{k^*}$, $x_k \in \mathfrak{M}_k$, and

$$\begin{aligned}
\lim_{t\to 0^+} T(t)x &= \lim_{t\to 0^+} [T_1(t) \oplus T_2(t)](x_{k^*} \oplus x_k) \\
&= \lim_{t\to 0^+} T(t)x \lim_{t\to 0^+} [T_1(t) \oplus T_2(t)]x_{k^*} \oplus [T_1(t) \oplus T_2(t)]x_k \\
= \lim_{t\to 0^+} [T_1(t)x_{k^*} \oplus T_2(t)x_k] &= \lim_{t\to 0^+} T_1(t)x_{k^*} \oplus \lim_{t\to 0^+} T_2(t)x_k = x_{k^*} \oplus x_k = x.
\end{aligned}$$

(iv) For any $x \in D(\mathcal{A})$, we have $x = x_{k^*} \oplus x_k$, $x_{k^*} \in \mathfrak{M}_{k^*}$ and $x_k \in \mathfrak{M}_k$, and

$$\begin{aligned}
\mathcal{A}x &= \mathcal{A}(x_{k^*} \oplus x_k) = \mathcal{A}x_{k^*} \oplus \mathcal{A}x_k \\
&= \left(\lim_{t\to 0+} \frac{T_1(t)x_{k^*} - x_{k^*}}{t}\right) \oplus \left(\lim_{t\to 0+} \frac{(T_1(t)x_{k^*} \oplus T_2(t)x_{k^*}) - (x_{k^*} \oplus x_k)}{t}\right) \\
&= \lim_{t\to 0+} \frac{Tx - x}{t}.
\end{aligned}$$

Thus, $T(t)$ defined by the orthogonal sum $T_1(t) \oplus T_2(t)$ is exactly C_0-semigroup on $\mathcal{H}$ generated by $\mathcal{A}$. Taking $M = \max\{M_1, M_2\}$ from (3.1) and (3.2), leads to the following result

$$\|T(t)\| \leq Me^{-\omega t} \quad (t \geq 0).$$

The proof of Theorem 3.1 is complete. □

4. Exponential Stability of the System

The significant result on the stability of the system (1.7) can be obtained and stated in the next theorem.

Theorem 4.1. *The system (1.7) or hyperbolic system (1.1) has unique solution $\vec{u}(t)$, and there is constant $\tilde{M} > 0$ such that*

$$\|\vec{u}(t) \leq \tilde{M}e^{-\omega t}.$$

That is, the solution of the system (1.7) or (1.1) is exponential stable.

Proof. From the Theorem 3.1, we see that $\mathcal{A}$ generates a C_0-semigroup $T(t)$ on $\mathcal{H}$, and therefore the system (1.7) has a unique solution $\vec{u}(t) = T(t)\vec{u}_0$ based on the theory of semigroup of linear operators [16].

From the inequality that $\|T(t)\| \leq Me^{-\omega t}$ we have

$$\|\vec{u}(t)\| = \|T(t)\vec{u}_0\| \leq \|T(t)\|\|\vec{u}_0\| \leq M\|\vec{u}_0\|e^{-\omega t}.$$

Denote $M\|\vec{u}_0\|$ by $\tilde{M}$, then it is obvious that $\tilde{M}$ is a constant and $\tilde{M} > 0$, moreover

$$\|\vec{u}(t)\| \leq \tilde{M}e^{-\omega t}$$

This implies that the solution of the system (1.7) or (1.1) is exponential stable. The proof is complete. □

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