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A NOTE ON OPERATORS CONJUGATE TO d-HOMOMORPHISMS

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Abstract: Let X, Y be two Banach f-modules over the same Banach f-algebra A and suppose that T is an A-orthomorphism continuous linear operator from X into Y. It is shown that $T' \in dh(Y', X')$, where T' is the continuous adjoint of T.

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1. Introduction

Let X be a Banach space and A be a Banach f-algebra with unit. By $L(X)$ we denote the set of all continuous linear operators from X into X. We say that X is a Banach f-module if there exists a bilinear mapping $p : A \times X \to X, (a, x) \to a.x$ satisfying the following conditions:

(i) $1.x = x$ for all $x \in X, 1 \in A$;

(ii) $(ab).x = a.(b.x)$ for all $a, b \in A, x \in X$;

(iii) $\|a.x\| \leq \|a\|\|x\|$ for all $a \in A, x \in X$.

Bilinear mapping p induces $m : A \to L(X), (a, x) \to m(a)x = a.x$ is a unital norm $\|.\|$ to SOT (the strong operator topology) continuous algebra homomorphism. We let X' denote the dual of a Banach X. We can establish

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the following bilinear mappings:

(A) $$X' \times X \to A', (f,x) \to (f.x)(a) = f(a.x), \quad f \in X', x \in X;$$

(B) $$A'' \times X' \to X', (a,f) \to (a.f)(x) = a(f.x), \quad a \in A'', f \in X', x \in X;$$

(C) $$A'' \times X'' \to X'', (a,z) \to (a.z)(f) = z(a.f), \quad f \in X', a \in A'', z \in X''.$$

When X is taken as A, then (C) becomes the Arens product on A''. The bilinear mapping (B) defines a Banach A''-module structure on X' that gives a homomorphism $m^\star : A'' \to L(X')$ defined by $m^\star(a) = a.f$. The bilinear mapping (C) defines a Banach A''-module structure on X''. These are called the Arens extensions of the module multiplication on X, Arens [3]. Note that for each $a \in A, m^\star(a)$ is the adjoint in $L(X')$ of the operator $m(a)$ in $L(X)$, and $m^\star$ is $(weak^\star, weak^\star$ -operator) continuous, and also for each $f \in X'$ the linear mapping from A'' into X' that sends a to $a.f$ is $\sigma(A'', A'), \sigma(X', X)$ continuous, Hadwin et al [4]. For terminology and unproved properties of Banach modules, Banach lattices,and f-algebra we refer to the reader to Abramovich et al [1],Aliprantis et al [2], Meyer [5]. We need the following result from Abramovich et al [1, Theorem 3.3].

Theorem 1. *Let X and Y be vector lattices, and $T : X \to Y$ a linear operator. The following conditions are equivalent:*

(1) T is an order bounded disjoint homomorphism.

(2) $|x_1| \leq |x_2|$ implies that $|Tx_1| \leq |Tx_2|$.

Definition. If X is a Banach f-module over a Banach f-algebra A with unit and $x \in X$, then

$$\Delta(x) = \mathrm{Cl}_X\{a.x : a \in A, \|a\| \leq 1\},$$

where Cl_X denotes the norm closure in X. A linear subspace Y of a Banach f-module X over a Banach f-algebra A is said to be an order ideal if for each $x \in Y$ the whole interval $\Delta(x)$ belongs to Y. Two elements x, y of a Banach f-module X over a Banach f-algebra A are called disjoint $(x \perp y)$ if $\Delta(x+y) = \Delta(x) + \Delta(y)$ and $\Delta(x) \cap \Delta(y) = \{0\}$. Let X be a Banach f-module over a Banach f-algebra A and let $f \in X'$, then

$$\Delta(f) = \mathrm{Cl}_{X'} = \{a.f : a \in A'', \|a\| \leq 1\},$$

where Cl_X denotes the closure in X'.

Lemma 2. *Let X be a Banach f-module over a Banach f-algebra A and $x \in X$. Let*

$$X(x)_+ = \mathrm{Cl}_X\{a.x : 0 \leq a \in A\}.$$

Then, the space $X(x)$ ordered by the cone $X(x)_+$ is a Banach lattice.

We need the following lemma and we refer to Abramovich et al [1, Lemma 7.11] for a Banach $C(K)$-module case.

Lemma 3. *Let X be a Banach f-module and let $f, g \in X'$. The following statements are true.*

(i) An element $x \in X$(as a functional on X') is orthogonal to $X'(f)$ if and only if $f.x = 0$.

(ii) An element $x \in X$ (as a functional on X') is positive on $X'(f)_+$ if and only if $0 \leq f.x$.

(iii) $f \in \Delta(g)$ if and only if $\|f.x\| \leq g.x\|$ for all $x \in X$ if and only if $|f.x| \leq |g.x|$ for all $x \in X$.

(iv) $f \perp g$ if and only if the elements f, g belong to the lattice $X'(f+g)$ and are disjoint in it if and only if $(f.x) \perp (g.x)$ for all $x \in X$.

Definition. Let X, Y be two Banach f-modules over a Banach f-algebra A. A linear operator $T : X \to Y$ is called an A-orthomorphism if $T(a.x) = a.Tx$ for all $a \in A, x \in X$.

Definition. Let $T : X \to Y$ be a linear operator from a Banach f-module X into a Banach f-module Y over the Banach f-algebra A. We call operator T a (disjoint) d-homomorphism if $x \perp z$ in X implies $Tx \perp Tz$ in Y. By $dh(X, Y)$ we denote d-homomorphisms from X into Y.

Theorem 4. *Let T be a continuous linear operator from a Banach f-module X over a Banach f-algebra A into a Banach f-module Y over the same Banach f-algebra A. The following are equivalent:*

(i) $T' \in dh(Y', X')$;

(ii) $f \in \Delta(g) \Rightarrow T'f \in \Delta(T'g)$ for $f, g \in Y'$.

Proof. (i)$\Rightarrow$(ii). Let $f \in \Delta(g)$. Fix $x \in X$ and consider an operator $S : Y'(g) \to A'$ defined by $Sh = T'h.x$. By Lemma 3 (iv) the operator S is a continuous d-homomorphism from $Y'(g)$ into A'. But each continuous d-homomorphism between Banach lattices is regular. Since $f \in \Delta(g)$, Theorem 1 implies that $Sf \in \Delta(Sg)$ or $|T'f.x| \leq |T'g.x|$. By Lemma 3(iii), we conclude that $T'f \in \Delta(T'g)$.

(ii)$\Rightarrow$(i). Let $f, g \in Y'$ and $f \perp g$. Then, by Lemma 3 (iv) $f \perp g$ in the vector lattice $Y'(f+g)$, and (ii) implies that the restriction of T' to $Y'(f+g)$ takes values in the vector lattice $X'(T'(f+g))$ and satisfies the condition (2) of Theorem 1. By this theorem, T' is a d-homomorphism between the vector lattices $Y'(f+g)$ and $X'(T'(f+g))$, and hence $T'f \perp T'g$. $\square$

Theorem 5. *Let T be an A-orthomorphism continuous linear operator from a Banach f-module X into a Banach f-module Y over the same Banach f-algebra A. Then, $T' \in dh(Y', X')$, where T' is the continuous adjoint of T.*

Proof. It suffices to show that if $f \in \Delta(g)$ then $T'f \in \Delta(T'g)$. Suppose that $f \in \Delta(g)$. Then, there exists a net (a_α) in A'', $\|a_\alpha\| \leq 1$ such that $a_\alpha.g \to f$ in X'. Since T' is continuous, it implies that $T'(a_\alpha.g) \to T'(f)$. T is an A-orthomorphism operator, i.e., for each $a \in A$, $Ta = aT$. Therefore, $T'(a.f) = a.T'f$ for all $a \in A''$. In fact, for an arbitrary $x \in X$ and $f \in X'$ and $a \in A, T'(a.f)x = T'(m^\star(a)f)x = T'((m(a))^\star f)x = f(m(a)T)x = f(Tm(a))x = (m^\star(a)T'f)x = (a.T'f)x$, i.e., $T'(a.f) = a.T'f$. It is well known that A is $\sigma(A'', A')$ dense in A''. We obtain that for each $a \in A'', a.T'f = T'(a.f)$. Since $T'(a_\alpha.g) = a_\alpha.T'g$ and $\Delta(T'g)$ is closed, it follows that $T'f \in \Delta(T'g)$. Hence, $T' \in dh(Y', X')$. $\square$

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