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ON THE EQUALITY IN DISTRIBUTION OF THE RANDOM VARIABLES X AND $g(X)$

Dan Kucerovsky[1 §], Eric Marchand[2], Robert D. Small[3]
[1,3]Department of Mathematics and Statistics
University of New Brunswick
P.0. Box 4400, Fredericton, N.B., E3B 5A3, CANADA
[1]e-mail: dan@math.unb.ca
[3]e-mail: don@math.unb.ca
[2]Department of Mathematics
Faculty of Science
University of Sherbrooke
2500 Boul. de l'Université
Sherbrooke, Québec, J1K 2R1, CANADA
e-mail: eric.marchand@usherbrooke.ca

Abstract: We study the equidistributional identity

$$X =^d g(X)$$

in $g(X)$ for continuous random variables X in the case of one-to-one and onto functions $g : \Re \longrightarrow \Re$ with at most one singularity. Such equidistributional identities and the related characterizations are not only of intrinsic interest, but also offer potential usefulness in the identification of probability models. We obtain a number of very general characterization results, one of which specializes to give a new characterization of the Cauchy distribution (Theorem 24).

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[§]Correspondence author

1. Introduction

Consider the family of Cauchy$(a,b), a \in \Re, b > 0$; distributions with probability density functions $f(x) = [\frac{\pi}{b}(b^2 + (x-a)^2)]^{-1}$. A well-known property of a standard Cauchy distributed random variable ($a = 0, b = 1$) is

$$X =^d \frac{1}{X}, \tag{1.1}$$

in other words X and its reciprocal $\frac{1}{X}$ are equidistributed. In fact, it is not too difficult to verify that among the members of the family of Cauchy(a,b) random variables, identity (1.1) holds if and only if $a^2+b^2 = 1$ (see Menon [7]). Identity (1.1) however is hardly a characterization in general of this Cauchy subfamily, say among continuous distributions, as (1.1) holds as soon as $X =^d \frac{Y_1}{Y_2}$ with Y_1, Y_2 independent and identically distributed continuous random variables. In particular, a standard Cauchy distribution arises when Y_1 and Y_2 are normally distributed with mean 0 and standard deviation σ ($\sigma > 0$) but, as shown by Lukacs and Laha (see [6], p. 56-58), the normality of Y_1 and Y_2 is not necessary for the ratio $\frac{Y_1}{Y_2}$ to be distributed as Cauchy$(0,1)$. On the other hand as shown by Menon, if F is restricted to being the cdf of a stable distribution, then property (1.1) implies that X is Cauchy distributed.

Even though the above construction of random variables X such that equation (1.1) holds is straightforward, it is still of interest to study which properties are implied for the cumulative distribution function (cdf) F. Similarly, it would be useful to understand the implications of an equivalence in distribution of X and $g(X)$. In this regard, a first non-trivial solution $g(X)$, where X is a continuous random variable, is given by $g(X) = F^{-1}(1-F(X))$, and arises immediately once one observes that $F(X) =^d 1 - F(X) =^d U$, with $U \sim$ Uniform$(0,1)$. Indeed this pivotal solution not only appears in Proposition 2, but is shown to be unique among globally decreasing functions g. Also, a key result which we will expand upon below and which is due to Williams [9], characterizes the standard Cauchy distribution as the only absolutely continuous distribution for which $X =^d \frac{1+mX}{m-X}$ whenever $\frac{1}{\pi}\arctan(m)$ is irrational. We shall, for example, prove the following result.

Proposition 1. *Given cdfs F_1 and F_2, and a number $m \in \Re$ that does not belong to a certain countable set S, we have that F_1 is equal to F_2 if and only if $g_{4,(F_2,m)}$ is equal to $g_{4,(F_1,m)}$, where the function g_4 is defined by*

$$g_{4,(F,m)}(x) = F^{-1}((F(x) - F(m)) \bmod 1), \tag{1.2}$$

The significance of the function g_4 in the above will be made clear in the body of the paper.

We will be generally concerned here with the equidistributional identity

$$X =^d g(X) \tag{1.3}$$

in $g(X)$ for continuous random variables X. We shall restrict ourselves in this paper to the case of one-to-one and onto functions $g : \Re \longrightarrow \Re$ with at most one singularity[1]. In short, our results below describe properties and characterizations related to pairs (g, F) satisfying equation (1.3). Our study is motivated by the fact that such equidistributional identities and related characterizations are not only of intrinsic interest, but also offer potential usefulness in the identification of probability models.

This paper is organized as follows. Section 2 gives some preliminary definitions, follows with determination and properties of g when F is given, and concludes with various remarks and lemmas. In Section 3, we consider pairs (g, F) where g is fixed and we obtain various properties and characterizations for such F's, concluding with several examples.

2. Determination of and Properties of *g* for a Given *F*

Our results pertain to:

(a) cumulative distribution functions F that are continuous, and strictly increasing on a given support set S_F (thus the support $S_F = \{y : 0 < F(y) < 1\}$ is necessarily of the form $S_F = (s_1, s_2)$, where, either s_1 or s_2 can have infinite magnitude);

(b) the class (or classes) of functions $C(S_F, m)$; $m \in \Re$; composed of one-to-one continuous and onto functions g from $S_F - \{m\}$ to $S_F - \{m\}$.

It will be useful to have some terminology for the classes of monotone functions with at most one discontinuity. We label the everywhere decreasing monotone functions with domain S_F by $\mathcal{C}_1(S_F)$, the everywhere increasing monotone functions by $\mathcal{C}_2(S_F)$, the locally decreasing monotone functions with a discontinuity at m by $\mathcal{C}_3(S_F, m)$, etc. as follows, where S_F will in the applications be an interval, possibly infinite or semi-infinite:

(i) $\mathcal{C}_1(S_F) = \{g \in \mathcal{C}(S_F, m) : g \text{ is globally decreasing}\}$;

(ii) $\mathcal{C}_2(S_F) = \{g \in \mathcal{C}(S_F, m) : g \text{ is globally increasing}\}$;

[1]Strictly speaking, such a function is only one-to-one if we use an extended real number system.

(iii) $\mathcal{C}_3(S_F, m) = \{g \in \mathcal{C}(S_F, m),\ g \notin \mathcal{C}_1(S_F) :\ g$ is locally decreasing$\}$;

(iv) $\mathcal{C}_4(S_F, m) = \{g \in \mathcal{C}(S_F, m),\ g \notin \mathcal{C}_2(S_F) :\ g$ is locally increasing$\}$.

The introduction of the subclasses of the type $\mathcal{C}_3(S_F, m)$ and $\mathcal{C}_4(S_F, m)$ will permit us to enrich considerably the number of solutions and characterizations in (g, F) associated with equation (1.3). In brief, we shall find that certain one-to-one functions (in classes $\mathcal{C}_3(S_F, m)$ and $\mathcal{C}_4(S_F, m)$) encapsulate a great deal of information about the symmetry properties of a given cdf. Furthermore by a closer study of the functions introduced, we will recover and considerably extend Williams's characterization result, as well as obtain a new proof of Williams result. As will be shown in Proposition 2, the restriction of functions g to one of the defined subclasses $\mathcal{C}(S_F, m)$ for a given (S_F, m) permits an essentially unique specification of g via an equidistribution condition. In what follows and throughout this paper, we define the fractional part of $x \in \Re$ to be $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the largest integer which is smaller than or equal to x.

Proposition 2. *If, for a given cdf F and real number m, a function $g \in \mathcal{C}(S_F, m)$ satisfies $X =^d g(X)$, then g is uniquely determined within each one of the subclasses: $\mathcal{C}_1(S_F)$, $\mathcal{C}_2(S_F)$, $\mathcal{C}_3(S_F, m)$, and $\mathcal{C}_4(S_F, m)$ by:*

(i) $g_{1,F}(x) = F^{-1}(1 - F(x))$ if $g \in \mathcal{C}_1(S_F)$;

(ii) $g_{2,F}(x) = x$ if $g \in \mathcal{C}_2(S_F)$;

(iii) $g_{3,(F,m)}(x) = F^{-1}(\{F(m) - F(x)\})$ if $g \in \mathcal{C}_3(S_F, m)$;

(iv) $g_{4,(F,m)}(x) = F^{-1}(\{F(x) - F(m)\})$ if $g \in \mathcal{C}_4(S_F, m)$.

Corollary 3. *If F_1 and F_2 are two continuous cdf's, and if m_1, m_2 are such that $F_1(m_1) = F_2(m_2)$ then the functions $g_{i,(F_k,m_k)}$ defined in the above proposition satisfy $F_1 \circ g_{i,F_1,m_k} \circ F_1^{-1} = F_2 \circ g_{i,F_2,m_k} \circ F_2^{-1}$, for $i = 3, 4$. Similarly, for any two continuous cdf's, $F_1 \circ g_{i,F_1} \circ F_1^{-1} = F_2 \circ g_{i,F_2} \circ F_2^{-1}$, for $i = 1, 2$.*

Remark 4. By breaking up g_3 into the sum of its restrictions on $(-\infty, m)$ and (m, ∞) respectively, it follows from Proposition 2 that:

$$g_{3,(F,m)}(x) = g_{1,F_{(-\infty,m)}}(x)[x < m] + g_{1,F_{(m,\infty)}}(x)[x > m]\,,$$

where F_A is defined as the cumulative distribution function of the conditional distribution $X|X \in A$ under F. In this sense, part (iii) of Proposition 2 follows from its part (i).

Proposition 5. *If, for a given (F, m), functions g_i in $\mathcal{C}_i(S_F, m)$ or in $\mathcal{C}_i(S_F)$ all satisfy $X =^d g(X)$, then they also satisfy the following identities:*

$$g_{4,(F,m)} = g_{1,F} \circ g_{3,(F,m)} \tag{2.1}$$

and

$$g_{4,(F,m)} = g_{3,(F,g_1(m))} \circ g_{1,F}\,. \tag{2.2}$$

Moreover, an equidistribution function g that is of class $\mathcal{C}_1(S_F, m)$, $\mathcal{C}_2(S_F, m)$, or $\mathcal{C}_3(S_F, m)$ has the following unusual self-inverse property:

$$g(g(x)) = x. \tag{2.3}$$

Remark 6. Proposition 5 can be regarded as a useful constructive scheme and characterization of g_4 in terms of the other g_i, since

$$g_4(F, m) = g_1(F, m) \circ g_3(F, m). \tag{2.4}$$

In this sense, Proposition 2, part (iv), follows from the above proposition and the remainder of Proposition 2. Note that $g_{4,(F,m)}$ functions are generally not self-inverses, except in the case that where m is a fixed point of g_1, as may be verified by working with equations (2.1) and (2.2). It is easy to see from these equations that $g_1 \circ g_4$ must have the self-inverse property (of equation (2.3)).

From Proposition 2 we see that a function $g_{1,F}$ has exactly one fixed point at $F^{-1}(1/2)$, in other words at the median under F. Similarly, a function $g_{3,(F,m)}$ has exactly two fixed points which are $F^{-1}(\frac{1}{2}F(m))$ and $F^{-1}(\frac{1+F(m)}{2})$. These points are in fact the medians of the conditional distributions $X|X \leq m$ and $X|X > m$ under F.

Finally, one may be curious as to the generation of functions that are self-inverses, and hence possibly admissible functions of class $\mathcal{C}_1(S_F, m)$, or $\mathcal{C}_3(S_F, m)$ for the identity $X =^d g(X)$ to hold. Starting with self-inverses s, such as $s(x) = \frac{1}{x}$ or $s(x) = -x$, and invertible but otherwise arbitrary functions H, self-inverses g can be generated as $g(x) = H^{-1}(s(H(x)))$. Functions g_1 and g_3 given in Proposition 2 may be viewed as special cases of the above self-inverse generation scheme with $H = F$, and $s(x) = 1-x$ or $s(x) = \{F(m)-x\}$. Furthermore, a huge class of self-inverses may be generated by starting with a strictly monotone function h having fixed point r, and defining a self-inverse function s by

$$s(x) := \begin{cases} h(x) & \text{if} \quad x \leq r, \\ h^{-1}(x) & \text{if} \quad x > r. \end{cases}$$

Proof of the Proposition 2 and Proposition 5. 1. Here, each one of the four results is derived by establishing the following equivalent form of equation (1.3):

$$P(g(X) \leq x) = P(X \leq x) \text{ for all } x \in S_F. \tag{2.5}$$

(a) If $g \in \mathcal{C}_1(S_F)$, then

$$\{x : g(x) \leq t\} = \{x : x \geq g^{-1}(t)\}\,,$$

which implies $P(g(X) \leq t) = 1 - F(g^{-1}(t))$. Hence, (2.5) now becomes with the substitution $x = g^{-1}(t)$,

$$F(x) + F(g(x)) = 1, \tag{2.6}$$

which leads to $g(x) = F^{-1}(1 - F(x))$ as claimed.

(b) If $g \in \mathcal{C}_2(S_F)$, then

$$\{x : g(x) \leq t\} = \{x : x \leq g^{-1}(t)\}\,,$$

which implies $P(g(X) \leq t) = F(g^{-1}(t))$. Hence, (2.5) now becomes with the substitution $x = g^{-1}(t)$,

$$F(x) - F(g(x)) = 0, \tag{2.7}$$

which leads to $g(x) = x$ as stated.

(c) If $g \in \mathcal{C}_3(S_F, m)$, then

$$\{x : g(x) \leq t\} = \{x : g^{-1}(t) \leq x \leq m\} \text{ if } g^{-1}(t) < m\,,$$

$$\text{and } \{x : g(x) \leq t\} = (-\infty, m] \cup \{x : x \geq g^{-1}(t)\} \text{ if } g^{-1}(t) > m.$$

Consequently, for $g^{-1}(t) \neq m$,

$$\begin{aligned} &P(g(X) \leq t) \\ &= [g^{-1}(t) < m]\{F(m) - F(g^{-1}(t))\} + [g^{-1}(t) > m]\{F(m) + 1 - F(g^{-1}(t))\}, \end{aligned}$$

in which case (2.5) becomes, for $g^{-1}(t) \neq m$, $F(g^{-1}(t)) + F(t) = F(m) + [g^{-1}(t) > m]$, or again with the substitution $g^{-1}(t) = x$,

$$F(x) + F(g(x)) = F(m) + [x > m];\quad x \neq m; \tag{2.8}$$

which leads to the unique stated solution in g.

(d) The proof is similar to that of (iii).

3. Observe first that $g_{1,F} \circ g_{3,(F,m)} \in \mathcal{C}_4(S_F, m)$. Now, since $g_{1,F} \circ g_{3,(F,m)}(X) =^d X$ by applying part (a) twice, identity (2.1) follows from the uniqueness of $g_{4,(F,m)}$ within $\mathcal{C}_4(S_F, m)$. The same reasoning applied to $g_{3,(F,g_1(m))} \circ g_{1,(F,m)}$ yields identity (2.2).

4. The self-inverse property of $g_{2,F}$ is obvious, while the result for $g_{1,F}$ and $g_{3,(F,m)}$ follows from the already proven uniqueness in Proposition 2. and the previously observed (Remark 6) self-inverse property that they possess.

The self-inverse property for $g_{2,F}$ is obvious, while the result for $g_{1,F}$, and $g_{3,(F,m)}$ follows by applying the formulas from Proposition 2 part (i) and (iii) to the quantities $x = z$ and $x = g(z)$. In each case, we are able to infer that $g(g(x)) = x$ as claimed in (2.3). For instance, we have from (iii) that:

$$F(x) + F(g(x)) = F(m) + [x > m]$$

and

$$F(g(x)) + F(g(g(x))) = F(m) + [g(x) > m] ,$$

which implies that $|F(x) - F(g(g(x)))|$ equals 0 or 1. But $|F(x) - F(g(g(x)))|$ cannot equal 1 by virtue of our conditions on F; whence $F(x) = F(g(g(x)))$, establishing equation (2.3) as claimed.

We now expand somewhat on the above general results. First, as shown in Proposition 2, the specification of g within a class $\mathcal{C}_2(S)$ leads to the unique trivial solution $g(x) = x$. Similarly as we now show, if F is symmetric about a, then the specification of g within $\mathcal{C}_1(S_F)$ leads to the unique solution $g(x) = 2a - x$, hence suggesting that the g_i functions do indeed reflect (sometimes subtle) symmetry properties of a cdf. □

Corollary 7. *If a cdf F possesses a probability density function $f = F'$ symmetric about a, then we must have $g_{1,F}(x) = 2a - x$.*

Proof. Whenever the probability density function $f = F'$ is symmetric about a, we may represent F and F^{-1} as $F(x) = \frac{1}{2} + Z(x - a)$ and $F^{-1}(x) = a + Z^{-1}(x - \frac{1}{2})$ where $-Z(x) = Z(-x)$. Now, from the first part of Proposition 2,

$$\begin{aligned} g_{1,F}(x) &= a + Z^{-1}(1 - F(x) - \frac{1}{2}) = a + Z^{-1}(-Z(x - a)) \\ &= a + Z^{-1}(Z(a - x)) = 2a - x , \end{aligned}$$

establishing the result. □

Example 8. Take X to be Cauchy(a, b) distributed with cdf $\Phi_{a,b}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(\frac{x-a}{b})$. Corollary 7 tells us that $g_{1,\Phi_{a,b}}(x) = 2a - x$, while Proposition 2 gives us, with a little bit of computation

$$g_{3,(\Phi_{a,b},m)}(x) = \frac{mx + a^2 + b^2 - 2am}{x - m} ,$$

and

$$g_{4,(\Phi_{a,b},m)}(x) = \frac{mx + a^2 + b^2 - 2ax}{m - x}\,.$$

Remark 9. Observe that the Cauchy g_4 given above may be derived as $2a - g_3(x)$ (since $g_1(x) = 2a - x$ by virtue of Corollary 7).

It is quite interesting that in the above example if we set $m = 0$ and/or $(a, b) = (0, 1)$, we recover the following known results:

(i) $\frac{a^2+b^2}{X}$ has the same distribution as X (for any Cauchy(a,b) X);

(ii) $\frac{1}{X}$ has the same distribution as X among Cauchy (a,b) if and only if $a^2 + b^2 = 1$, recovering Menon's [7] result (see Introduction, using uniqueness of g in $\mathcal{C}_3(S_F, m)$);

As well, both equidistributional identities:

(a) $\frac{1+mX}{X-m} =^d X$, and

(b) $\frac{1+mX}{m-X} =^d X$,

hold for a standard Cauchy cdf $\Phi_{0,1}$. As mentioned earlier, Williams [9] established that identity (b) characterizes $\Phi_{0,1}$ as long as $\frac{1}{\pi}\arctan(m)$ is an irrational number. We will see later how this result can be extended, and why the irrationality requirement is necessary. Also, we will see that identity (a) does not come even close to characterizing $\Phi_{0,1}$ among all cdfs, but it does so among cdfs F with symmetric pdfs as long as $\frac{1}{\pi}\arctan(m)$ is an irrational number (see Corollary 23).

Remark 10. Members of a same location-scale family with cdfs $F_{a,b}(x) = F_{0,1}(\frac{x-a}{b}); a \in \Re, b > 0$; have related equidistributional g_i's. Moreover, for a given location-scale family $F_{0,1}$, one can construct $g_{1,F_{a,b}}$, $g_{3,(F_{a,b},m)}$, and $g_{4,(F_{a,b},m)}$; $m \in \Re$; from standardized versions with $(a, b) = (0, 1)$. Indeed, since:

(i) the equidistributional result $X =^d \psi_{0,1}(X)$ under $F_{0,1}$ implies the equidistributional result $X =^d \psi_{a,b}(X)$ under $F_{a,b}$ with $\psi_{a,b}(x) = b\psi_{0,1}(\frac{x-a}{b}) + a$;

(ii) $\psi_{0,1} \in \mathcal{C}_1(S_{F_{0,1}})$ implies $\psi_{a,b} \in \mathcal{C}_1(S_{F_{a,b}})$, and $\psi_{0,1} \in \mathcal{C}_i(F_{0,1}, m)$ implies $\psi_{a,b} \in \mathcal{C}_i(S_{F_{a,b}}, a + mb)$; $i = 3, 4$;

it follows that:

$$g_{1,F_{a,b}}(x) = b g_{1,F_{0,1}}(\frac{x-a}{b}) + a\,, \tag{2.9}$$

$$g_{i,(F_{a,b},m)}\,(x) = b g_{i,(F_{0,1},(m-a)/b)}(\frac{x-a}{b}) + a;\; i = 3, 4. \tag{2.10}$$

As an illustration, refer to the Cauchy case of Example 8, where $g_{3,(F_{0,1},m)}(x) = \frac{mx+1}{x-m}$, and $g_{3,(F_{a,b},m)}\,(x)$ is necessarily given by $bg_{3,(F_{0,1},(m-a)/b)}\,(\frac{x-a}{b}) + a = \frac{b^2+a^2+mx-2am}{x-m}$.

3. Characterizations of CDFs F for Given g's

3.1. Characterizations Based on g_4 Type Functions

This section deals with the reciprocal question, namely: given an admissible g, what can be stated about cdfs that satisfy an equidistributional identity

$$X =^d g(X) \quad ? \tag{3.1}$$

Below, we show not only how to find such $F's$ but also establish various characterizations, some of which follow along the lines of Williams Cauchy(0,1) characterization which we alluded to in Example 8. We show that, in general, admissible g_4 functions lead to cdf characterizations.

Lemma 11. *For a given (S, m), cdfs F_1 and F_2 have the same g_4 functions if and only if the function $\psi = F_2 \circ F_1^{-1}$ satisfies:*

$$\psi(\{\lambda - \lambda_0\}) = \{\psi(\lambda) - \psi(\lambda_0)\}, \tag{3.2}$$

for $\lambda \in (0, 1)$, with $\lambda_0 = F_1(m)$.

Proof. First, suppose the cdfs F_1 and F_2 share the same g_4 function. With the representation of g_4 given in Proposition 2, the definition of ψ gives us

$$\psi(\{F_1(x) - F_1(m)\}) = F_2(g_4(x)) = \{F_2(x) - F_2(m)\},$$

or, equivalently by setting $\lambda = F_1(x)$,

$$\psi(\{\lambda - \lambda_0\}) = \{\psi(\lambda) - \psi(\lambda_0)\},$$

for all $\lambda \in (0, 1)$; whence equation (3.2) is established.

Conversely, if the function $\psi = F_2 \circ F_1^{-1}$ satisfies equation (3.2), we have

$$\begin{aligned} g_{4,(F_2,m)}(x) &= F_2^{-1}(\{F_2(x) - F_2(m)\}) = F_1^{-1}(\psi^{-1}(\{\psi(F_1(x)) - \psi(F_1(m))\})) \\ &= F_1^{-1}(\psi^{-1}(\psi(\{F_1(x) - F_1(m)\})) = F_1^{-1}(\{F_1(x) - F_1(m)\}) = g_{4,(F_1,m)}(x), \end{aligned}$$

as was to be proven. ◻

The proof of the main result of this section will make use of the following lemma which is a direct consequence of a dynamical systems theorem given by Cornfeld, Fomin, and Sinai [2], Theorem 5.4.1, p. 133.

Lemma 12. *For a transformation $T_{\lambda_0} : [0,1] \to [0,1]$ of the form: $t \to \{t - \lambda_0\}$ with $\lambda_0 \in (0,1)$, the orbit of $(1-\lambda_0)$ under the transformation: $\{T^{[n]}_{\lambda_0}(1-\lambda_0) \mid n \in Z\}$, is either a dense subset of $[0,1]$, or finite (in which case the transformation is periodic).*

We now give the main result of this section, in the process proving Proposition 1:

Proposition 13. *Given cdfs F_1 and F_2, and an number $m \in \Re$ such that either $F_1(m)$ or $F_2(m)$ is irrational, we have that F_1 is equal to F_2 if and only if $g_{4,(F_2,m)}$ is equal to $g_{4,(F_1,m)}$.*

Proof. We shall prove that given a cdf F_1 and a $m \in \Re$, any cdf F_2 with $g_{4,(F_2,m)} = g_{4,(F_1,m)}$ is neccessarily equal to F_1, provided that the technical condition that $F_1(m)$ is irrational holds.

First, notice that $\psi_0 = F_1 \circ F_2^{-1}$ is a solution of the equation $\psi(\{\lambda - \lambda_0\}) = \{\psi(\lambda) - \psi(\lambda_0)\}$ when λ_0 is of course irrational.

It follows that ψ_0 commutes with R, where R is the mapping given by $R_{\lambda_0}(x) = \{x - F_2(m)\}$. Moreover, R is invertible, and, for $n \in Z$, $R^{[n]}_{\lambda_0}(x) = \{x - n\lambda_0\}$. Now the above Lemma 12 tells us the set $\{R^{[n]}_{\lambda_0}(1-\lambda_0) : n \in Z\}$ is, as a subset of $[0,1]$, either dense or finite. But, if it were to be finite, there would exist some $m \in Z$ such that $R^{[m]}_{\lambda_0}(1-\lambda_0) = 1 - \lambda_0$ which then gives $\{m\lambda_0\} = 0$, implying the contradictory result that λ_0 is a rational number. Since ψ_0 and R commute, we see that since $R^{[n]} \circ \psi_0$ and $\psi_0 \circ R^{[n]}$ must be equal, the unique discontinuity of $R^{[n]}$ must be at a fixed point of ψ_0. Therefore the fixed points of ψ_0 are a dense set, and thus $\psi_0(x) = x$ proving the result. □

We shall later obtain characterization results as a corollary of the above. For the moment, we content ourselves with the observation that Williams result is obtained as a corollary of the above.

Corollary 14. *The Cauchy distribution is characterized among continuous distributions by the property*

$$\frac{1 + m'X}{m' - X} \sim X$$

for a number m' that is not a rational multiple of π.

Proof. In Example 8 we saw that the g_4 function for a Cauchy(0,1) distribution is $g_4(x) = \frac{mx+1}{m-x}$, and hence we have the claimed result. $\square$

We can in fact reverse part of the argument, using Corollary 14 to prove Proposition 13. This is done in an Appendix.

Now, despite being perhaps peculiar, the irrationality of $F_1(m)$ requirement in Proposition 13 is indeed necessary. Counterexamples are immediately given with $g_4(x) = -\frac{1}{x}$ by random variables symmetric about 0 which satisfy the equidistributional result $X =^d \frac{1}{X}$ (see Introduction). Of course, in such situations, Proposition 13 does not apply since $m = 0$ and $F_1(m) = \frac{1}{2}$ is rational. To better visualize, however, why the irrationality of $F_1(m)$ is needed to obtain characterizations, and to indicate the existence of a family of counterexamples, we give, in the following example with $F_1(m) \in Q$, other cdfs with a g_4 function matching that of F_1. It is moreover clear that the argument is completely general, being based on the construction of a function ψ having fixed points on a given finite set. In this construction, we must choose an arbitrary periodic function (which may be conveniently represented by a Fourier series, as is done below).

Example 15. In the context of Proposition 13, let $\lambda_0 = F_1(m)$ be a rational number. By constructing nontrivial functions ψ that satisfy

$$\psi(\{\lambda - \lambda_0\}) = \{\psi(\lambda) - \psi(\lambda_0)\}\,, \tag{3.3}$$

we obtain other cdfs $F_2 = \psi \circ F_1$ with a g_4 function matching that of F_1. Indeed, consider the class of functions

$$\psi(x) = x + \frac{\sum_{i=1}^{\infty} a_i \sin(2c_i \pi x)}{2\pi \sum_{i=1}^{\infty} a_i c_i}\,, \tag{3.4}$$

with the a_i's and c_i's being nonnegative, such that all the c_i's and $c_i\lambda_0$'s are integers, and such that $\sum_{i=1}^{\infty} a_i c_i < \infty$; but otherwise arbitrary (observe that the rationality of λ_0 renders possible such a choice of c_i's, while, in contrast, the irrationality of λ_0 would render impossible such a selection of c_i's). Now it is straightforward to check that such a ψ satisfies equation (3.3). Hence, any of the above ψ yields a cdf $F_2 = \psi \circ F_1$ with the same g_4 function as that of F_1.

In particular, for the case of a Cauchy distribution, if $X \sim$ Cauchy $(0, 1)$ with $F_1(x) = \Phi_{0,1}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$, and $\frac{1}{\pi}\arctan(m) \in \overline{Q}$, then $\frac{1+mX}{m-X} =^d X$ not only under F_1, but also under any F_2 such that

$$F_2(x) = F_1(x) + \frac{\sum_{i=1}^{\infty} a_i \sin(2c_i \pi F_2(x))}{2\pi \sum_{i=1}^{\infty} a_i c_i}.$$

$$= \frac{1}{2} + \frac{1}{\pi}\arctan(x) + \frac{\sum_{i=1}^{\infty} a_i \sin(2c_i \arctan(x))}{2\pi \sum_{i=1}^{\infty} a_i c_i};$$

which also can be expressed as

$$\frac{1}{2} + \frac{1}{\pi}\arctan(x) + \frac{\sum_{i=1}^{\infty} a_i \mathrm{Im}((\frac{i-x}{i+x})^c)}{2\pi \sum_{i=1}^{\infty} a_i c_i}.$$

For instance, if $m = 1$ (and $\lambda_0 = 3/4$), $a_i = 0$ for $i \geq 2$, $c_1 = 2$, we have that $\frac{1+X}{1-X} =^d X$ for $X \sim$ Cauchy$(0,1)$, as well as for any cdf $F_2(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x) + \frac{a_1 \sin(4\arctan(x))}{4a_1\pi}$, with $a_1 > 0$.

3.2. CDF Characterizations Based on Pairs (g_1, g_3)

We now turn to properties and characterizations of cdfs such that

$$X =^d g(X) \tag{3.5}$$

for functions g of classes $\mathcal{C}_1$ and $\mathcal{C}_3$. In contrast to the characterizations related to single g_4 type functions, a single g_1 or g_3 type function may be associated with infinitely many distinct cdfs. This will be shown and illustrated with Lemma 16 and Lemma 17.

However, as presented in Theorem 19 and Corollary 21, pairs (g_1, g_3) will lead to characterizations given that they lead, by part (b) of Proposition 2, to unique g_4's, and that g_4's in turn lead to cdf characterizations as seen in Proposition 13. Related implications and examples are presented in the latter part of this section.

Lemma 16. *(a) For a given support S, cdfs F_1 and F_2 have the same g_1 function if and only if $\psi = F_2 \circ F_1^{-1}$ is symmetric about $\frac{1}{2}$, i.e.,*

$$\psi(1-\lambda) = 1 - \psi(\lambda) \tag{3.6}$$

for $\lambda \in (0,1)$.

(b) For a given (S, m), cdfs F_1 and F_2 have the same g_3 functions if and only if the function $\psi = F_2 \circ F_1^{-1}$ satisfies:

$$\psi(\{\lambda_0 - \lambda\}) = \{\psi(\lambda_0) - \psi(\lambda)\}, \tag{3.7}$$

for $\lambda \in (0,1)$, with $\lambda_0 = F_1(m)$; (in other words, ψ is symmetric about $\lambda_0/2$ and $(1+\lambda_0)/2$ respectively on the intervals $(0,\lambda_0)$ and $(\lambda_0, 1)$).

Proof. (a) First, suppose the cdfs F_1 and F_2 share the same g_1 function. The definition of ψ and representation of g_1, give us $\psi(1-F_1(x)) = F_2 \circ F_1^{-1}(1-F_1(x)) = F_2(g_1(x)) = 1 - F_2(x) = 1 - \psi(F_1(x))$, or, equivalently by setting $\lambda = F_1(x)$, $\psi(1-\lambda) = 1 - \psi(\lambda)$, for all $\lambda \in (0,1)$; and equation (3.6) follows.

Conversely, if the function $\psi = F_2 \circ F_1^{-1}$ satisfies (3.6), we have

$$\begin{aligned} g_{1,F_2}(x) = F_2^{-1}(1-F_2(x)) &= F_1^{-1} \circ \psi^{-1}(1-\psi(F_1(x))) \\ &= F_1^{-1} \circ \psi^{-1}(\psi(1-F_1(x))) = g_{1,F_1}(x), \end{aligned}$$

as was to be proven.

(b) A proof of part (b) follows along the lines of Lemma 11, but also follows from part (a) in view of Remark 4. ☐

Lemma 17. *(a) For a given S, and a self-inverse $g \in \mathcal{C}_1(S)$, a cumulative distribution function F satisfies the equidistribution condition* (3.5) *if and only if there exists a cdf Q with support S_F such that*

$$F(x) = \frac{1}{2}(\ Q(x) - Q(g(x)) + 1\ ). \tag{3.8}$$

(b) For a given (S, m), and a self-inverse $g \in \mathcal{C}_3(S, m)$, a cumulative distribution function F satisfies the equidistribution condition (3.5) *if and only if there exists a cdf Q with support S_F such that*

$$F(x) = \frac{1}{2}(\ Q(x) - Q(g(x)) + Q(m) + [x > m]\ ). \tag{3.9}$$

Proof. We prove part (a) only, as part (b) is perfectly analogous. Necessity follows by observing that the choice $Q = F$ in (3.8) implies $F(x) + F(g(x)) = 1$ and thus equation (3.5) holds, as was to be shown. Sufficiency follows as whenever (3.8) holds for some Q, it follows that

$$F(x) + F(g(x)) = \frac{1}{2}[Q(x) - Q(g(x)) + 1 + Q(g(x)) - Q(g(g(x))) + 1] = 1\ ,$$

given g is (must be) a self-inverse. Thus, we have the claimed equidistribution property. ☐

Remark 18. (a) Lemma 17 provides a universal and simple mechanism to find at least one cdf F satisfying the equidistribution condition (3.5) for a given admissible g, in a class $\mathcal{C}_1(S)$ or $\mathcal{C}_3(S, m)$ (see Remark 6). Interestingly, it may be interpreted as giving solutions F which are the result of onto mappings

$Q \to F$, with fixed point $Q = F$, so that for a chosen Q at most one solution can be generated by iteration of identities (3.8) or (3.9).

As an example, consider $g(x) = \frac{1}{x}[x > 0]$. Then equation (3.8) tells us a cdf solution of equation (3.5) may be generated as $F(x) = \frac{1}{2}(Q(x) - Q(\frac{1}{x}) + 1)[x > 0]$, for some arbitrary chosen Q. Equivalently, pdfs giving rise to identity (3.5) may be generated from arbitrary pdfs q on $(0, \infty)$ by setting $f(x) = \frac{1}{2}(q(x) + \frac{1}{x^2} q(\frac{1}{x}))$.

(b) Lemma 16 provides a universal and simple mechanism to generate cdfs F satisfying equation (3.5), for a given admissible g, in a set $\mathcal{C}_1(S)$ or $\mathcal{C}_3(S, m)$. In contrast to Lemma 17 however:

(i) we need to start with an initial solution $F = F_1$, and

(ii) the generation of solutions via the properly chosen maps $F_2 = \psi \circ F_1$ seems more straightforward.

Now, in contrast to these above results which provide much freedom in the the choice of cdfs satisfying the equidistribution condition (3.5) for a given g_1 or g_3, we now turn to situations, where F is uniquely determined if restricted to an appropriately chosen class.

Theorem 19. *Given a cdf F_1 and $m \in \Re$, a cdf F_2 with the same (g_1, g_3) pair as that of F_1 is necessarily F_1 whenever $F_1(m) \in \overline{Q}$.*

Proof. The result follows directly from Proposition 5 and Proposition 13. Indeed, matching pairs $(g_1, g_{3,(F_i,m)})$; $i = 1, 2$; implies matching $g_{4,(F_i,m)}$; $i = 1, 2$; by equation (2.1), which in turn implies matching cdfs whenever $F_1(m)$ is irrational. □

Remark 20. 1. To produce counterexamples to the claim of Theorem 19 whenever $F_1(m) \in Q$, we use the principle of Example 15. We start with the pair $(g_{1,F_1}, g_{3,(F_1,m)})$ and generate $g_{4,(F_1,m)}$ using the identity (2.1). Now using the mapping ψ given in equation (3.2), we produce other cdfs F_2 with the same g_4 as that of F_1. Now observe directly that these mappings ψ satisfy equation (3.6), hence indicating given Lemma 16 that the common g_{1,F_1} has been preserved by these F_2's. Furthermore, a common (g_1, g_4) imply $g_{3,(F_2,m)} = g_{3,(F_1,m)}$ given equation (2.1) again. Hence, we have shown that $(g_{1,F_1}, g_{3,(F_1,m)}) = (g_{1,F_2}, g_{3,(F_2,m)})$ for any $F_2 = \psi \circ F_1$ with ψ any mapping of the class satisfying equation (3.2).

2. It is possible to numerically determine the graph of a cdf F given the $g_4 \in \mathcal{C}_4(\Re)$ (and the value of $F(m)$) for the cdf. The key observation is that if (x_n, y_n) is a point on the graph of the cdf, then $(g_4(x_n), \{y_n - F(m)\})$ is another point on the graph, as can be verified from the identities in Proposition

2. Lemma 12 implies that the sequence of points generated by this iteration will be dense in the graph if and only if $F(m)$ is irrational. Observing that g_4 will have a horizontal asymptote as $x \longrightarrow \infty$, we take $y_0 := 1$ and $x_0 := \infty$. It can be verified, incidently, that then $x_1 = g_4(x_0) := \lim_{x \longrightarrow \infty} g_4(x) = g_1(m)$, and of course $y_1 := 1 - F(m)$. In this way, given g_4 and $F(m)$ we obtain a sequence of points that are theoretically dense in the graph of the cdf. We have tested this algorithm on a computer, and have not found any evidence of numerical instability.

Now, another way to state Theorem 19 is as follows.

Corollary 21. *Given a pair* (F_0, m),

(a) If $X =^d g_{3,(F_0,m)}(X)$ *under a cdf* F*, then* F *is uniquely determined as* F_0 *among the class of cdfs* $\{F : g_{1,F} = g_{1,F_0}\}$ *as long as* $F_0(m)$ *is irrational (note that the cardinality of this class of cdfs is not 1(!) given part (a) of Lemma 17).*

(b) If $X =^d g_{1,F_0}(X)$ *under a cdf* F*, then* F *is uniquely determined as* F_0 *among the class of cdfs* $\{F : g_{3,(F,m)} = g_{3,(F_0,m)}\}$ *as long as* $F_0(m)$ *is irrational (note that the cardinality of this class of cdfs is not 1(!) given part (b) of Lemma 17).*

As was the case for Proposition 13, Corollary 21 contains an automatic mechanism to produce characterizations of cdfs. Indeed, it suffices to initially select a pair (F_0, m) such that $F_0(m)$ is irrational, and use Proposition 2 to obtain the corresponding g_{1,F_0} and $g_{3,(F_0,m)}$. Then we have, for instance, the characterization: $X =^d g_{3,(F_0,m)}(X)$ under F where $F \in \{F : g_{1,F} = g_{1,F_0}\}$ implies $F = F_0$. We can also start with a self-inverse $g \in \mathcal{C}_3(S, m)$ (or $g \in \mathcal{C}_1(S)$) then find an associated cdf F_0 using Lemma 17 which satisfies equation (3.5) for which $F_0(m)$ is irrational, and proceed as above. The potential advantage of the latter scheme is that we can obtain characterizations for explicit g_3's or g_1's (while the previous scheme permits us to select the cdf F_0).

Notwithstanding the above generality, specific examples remain of interest; and we thus pursue with various, and believed to be novel, applications of Corollary 21 and our other results. Note that we could easily extend the following list, as we have in effect constructed a machine for producing characterization theorems.

Example 22. We here give a selection of the corollaries of our above results, and also derive g_1, g_3, and g_4 for various cdfs F.

3.2.1. Cauchy

Suppose F_0 is the cdf of a Cauchy(a_0, b_0).

As shown in Example 8, we have $g_{3,(F_0,m)}(x) = \frac{mx+a_0^2+b_0^2-2a_0m}{x-m}$ and $g_{1,F_0}(x) = 2a_0 - x$.

Corollary 23. *The cdf F_0 corresponds to the unique symmetric distribution about a_0 such that $X =^d \frac{mX+a_0^2+b_0^2-2a_0m}{X-m}$, as long as $F_0(m) = \frac{1}{\pi}\arctan(\frac{m-a_0}{b_0})$ is irrational (note that the case $a_0 = m$ is ruled out).*

The next quite interesting corollary seems important enough to be a theorem.

Theorem 24. *The identity $X =^d \frac{a_0^2+b_0^2}{X}$ characterizes the Cauchy(a_0, b_0) distribution among distributions symmetric about a_0, as long as $\frac{1}{\pi}\arctan(\frac{-a_0}{b_0})$ is irrational.*

For the particular case where $a_0 = 0$, we obtain from Corollary 23 that the identity $X =^d \frac{mX+b_0^2}{X-m}$ characterizes the Cauchy$(0, b_0)$ distribution among symmetric distributions about 0, as long as $\frac{1}{\pi}\arctan(\frac{m}{b_0})$ is irrational.

3.2.2. Uniform

For $X \sim U(\alpha, \beta)$ with $F(x) = (\frac{x-\alpha}{\beta-\alpha})[\alpha < x < \beta] + [x \geq \beta]$, we have $g_{1,F}(x) = \alpha + \beta - x$, and, for $m \in (\alpha, \beta)$, we obtain as one may anticipate the piecewise linear solutions:

$$g_{3,(F,m)}(x) = (\alpha + m - x)[x < m] + (\beta + m - x)[x > m],$$

and

$$g_{4,(F,m)}(x) = (x - m + \beta)[x < m] + (x - m + \alpha)[x > m].$$

Corollary 25. *(a) The equidistributional identity $X =^d (X - m + \beta)[X < m] + (X - m + \alpha)[X > m]$ characterizes the $U(\alpha, \beta)$ distribution among symmetric distributions (about $\frac{\alpha+\beta}{2}$) as long as $\frac{m-\alpha}{\beta-\alpha}$ is an irrational number.*

(b) Any continuous cdf F satisfying $X =^d (X - m + \beta)[X < m] + (X - m + \beta)[X > m]$ for some number m in (α, β) such that $\frac{m-\alpha}{\beta-\alpha}$ is an irrational number is necessarily the cdf of a $U(\alpha, \beta)$ distribution.

3.2.3. Pareto

For Pareto distributions with cdfs $F(x) = (1-(\frac{\alpha}{x})^{\beta})[x > \alpha]$; where $\alpha > 0, \beta > 0$; we obtain

$$g_{1,F}(x) = \frac{\alpha x}{(x^{\beta} - \alpha^{\beta})^{1/\beta}}; \qquad g_{3,(F,m)}(x) = \alpha\{(\frac{\alpha}{m})^{\beta} - (\frac{\alpha}{x})^{\beta}\}^{-1/\beta};$$

and

$$g_{4,(F,m)}(x) = \alpha\{(\frac{\alpha}{x})^{\beta} - (\frac{\alpha}{m})^{\beta}\}^{-1/\beta};$$

for $m > \alpha$.

Note that the case $\beta = 1$ leads to a simple rational function for g_1.

Corollary 26. *(a) The equidistributional identity* $X =^d \alpha(\{(\frac{\alpha}{X})^{\beta} - (\frac{\alpha}{m})^{\beta}\}^{-1/\beta})$ *characterizes the Pareto distribution with cdf* $F(x) = (1-(\frac{\alpha}{x})^{\beta})[x > \alpha]$ *among continuous cdfs, as long as* $(\frac{\alpha}{m})^{\beta}$ *is irrational.*

(b) The equidistributional identities $X =^d \alpha(\{(\frac{\alpha}{m})^{\beta} - (\frac{\alpha}{X})^{\beta}\}^{-1/\beta})$ *and* $X =^d \frac{\alpha X}{(X^{\beta}-\alpha^{\beta})^{1/\beta}}$ *characterize the Pareto distribution; with cdf* $F(x) = (1-(\frac{\alpha}{x})^{\beta})[x > \alpha]$ *among continuous cdfs, provided* $(\frac{\alpha}{m})^{\beta}$ *is irrational.*

3.2.4. Hyperbolic Secant

Consider the hyperbolic Secant distribution with cdf $F(x) = \frac{2}{\pi}\arctan(e^x)$; (and with pdf $F'(x) = \frac{1}{\pi}\text{sech}(x)$, whence its hyperbolic secant designation). Given that F' is symmetric about 0, Corollary 7 tells us that $g_{1,F}(x) = -x$, while expression (2.1) tells us that $g_{4,(F,m)} = -g_{3,(F,m)}$. Now, calculations using Proposition 2 yield

$$g_{3,(F,m)}(x) = 2\text{sgn}(m-x)\log\left|\frac{e^x - e^m}{1 + e^{x+m}}\right|.$$

Notice that the case $m = 0$ simplifies to $g_{3,(F,0)}(x) = -2\text{sgn}(x)\ \text{arctanh}(e^{|x|})$.

Corollary 27. *(a) The equidistributional identity* $X =^d 2\text{sgn}(m-X)\log|\frac{e^X-e^m}{1+e^{X+m}}|$ *characterizes the hyperbolic secant distribution (with cdf* $F(x) = \frac{2}{\pi}\arctan(e^x)$*); among distributions symmetric about* 0*, as long as* $\frac{2}{\pi}\arctan(e^m)$ *is irrational.*

(b) The equidistributional identity $X =^d -2\text{sgn}(m-X)\log|\frac{e^X-e^m}{1+e^{X+m}}|$ *characterizes the hyperbolic secant distribution (with cdf* $F(x) = \frac{2}{\pi}\arctan(e^x)$*); among distributions symmetric about* 0*, as long as* $\frac{2}{\pi}\arctan(e^m)$ *is irrational.*

3.2.5. Logistic

For the logistic distribution having cdf $F(x) = (1+e^{-x})^{-1}$, the symmetry of F' again implies $g_{1,F}(x) = -x$, and $g_{4,(F,m)}(x) = -g_{3,(F,m)}(x)$. Then computations using Proposition 2 lead to:

$$g_{3,(F,m)}(x) = \log \frac{(e^m - e^x) + ((e^m+1)(e^x+1))[x > m]}{(e^x - e^m) + ((e^m+1)(e^x+1))[x < m]}.$$

Corollary 28. *(a) Any cdf F having the equidistributional identity*

$$X =^d -\log \frac{(e^m - e^X) + ((e^m+1)(e^X+1))[X > m]}{(e^X - e^m) + ((e^m+1)(e^X+1))[X < m]}$$

for m such that e^{-m} is irrational is necessarily a logistic cdf (with $F(x) = (1+e^{-x})^{-1}$).

(b) The equidistributional identity

$$X =^d -\log \frac{(e^m - e^X) + ((e^m+1)(e^X+1))[X > m]}{(e^X - e^m) + ((e^m+1)(e^X+1))[X < m]}$$

characterizes the logistic distribution with cdf $F(x) = (1+e^{-x})^{-1}$ among distributions symmetric about 0, as long as e^{-m} is irrational.

3.2.6. Weibull

For Weibull distributions with cdfs $F(x) = (1 - e^{-x^\beta})[x > 0]$; $\beta > 0$; we obtain

$$g_{1,F}(x) = (-\log(1 - e^{-x^\beta}))^{1/\beta}; \quad g_{3,(F,m)}(x) = (-\log(\{e^{-m^\beta} - e^{-x^\beta}\}))^{1/\beta};$$

and

$$g_{4,(F,m)}(x) = (-\log(\{e^{-x^\beta} - e^{-m^\beta}\}))^{1/\beta}$$

for $m > 0$.

Corollary 29. *(a) The equidistributional identity $X =^d (-\log(\{e^{-X^\beta} - e^{-m^\beta}\}))^{1/\beta}$ characterizes a Weibull distribution (with $F(x) = 1 - e^{-x^\beta}$), provided that m is positive and e^{-m^β} is irrational.*

(b) The equidistributional identities $X =^d (-\log(\{e^{-m^\beta} - e^{-X^\beta}\}))^{1/\beta}$ and $X =^d (-\log(1 - e^{-X^\beta}))^{1/\beta}$ characterize the Weibull distribution (with $F(x) = 1 - e^{-x^\beta}$), provided that m is positive and e^{-m^β} is irrational.

As expanded upon in Remark 10, corresponding solutions g for members of the hyperbolic secant, logistic, or Weibull location-scale families are obtainable using the above solutions and expressions (2.9) and (2.10).

The next interesting theorem applies in situations where g_3 is known for two different values of m.

Theorem 30. *For a cdf F_0, and constants m_1, m_2 (with $m_1 \neq m_2$), the system $X =^d g_{3,(F_0,m_1)}(X) =^d g_{3,(F_0,m_2)}(X)$ under F admits the unique solution $F = F_0$ as long as $F_0(m^*)$ is irrational, where $m^* = g_{3,(F_0,m_2)}(m_1)$.*

Proof. Set $H = g_{3,(F_0,m_1)} \circ g_{3,(F_0,m_2)}$. Observe that (i) $H(X) =^d X$, and that (ii) $H \in \mathcal{C}_4(S_{F_0}, m^*)$. Also, from the uniqueness claimed in Proposition 2, we must have $H = g_{4,(F_0,m^*)}$. Finally, the result follows as an application of Proposition 5. □

3.3. Concluding Remarks

We have limited our study of the equidistributional identity $X =^d g(X)$ to one-to-one and onto functions g. Extensions to other types of functions g are also of interest. In this regard, Arnold (1979) showed that $X =^d g(X)$, with $g(x) = \frac{1}{2}(x - \frac{1}{x})$ leads to a characterization of the Cauchy(0,1) distribution.

Also, some past work has dealt with determining classes of cdfs such that their members F are necessarily solutions of $X =^d g(X)$ under F, for a class of $g's$. For instance Knight [4] (also see Letac [5] shows that if $g(x)$ is of the form $\frac{ax+b}{cx+d}$ (with $ad \neq bc$), then F must be in the class of Cauchy cdfs.

The results of this paper may well be useful in addressing extensions like these. For instance, we showed in Lemma 17.b that the functional equation $X =^d g_{3,(F_0,m)}(X)$ under F has many solutions in F. In contrast, the results of this paper can be used, as in Theorem 30 to show that a system of two such equations typically leads to unique solutions in F.

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Appendix

We can obtain Proposition 13 starting from Williams' characterization of the Cauchy distribution.

Proposition 31. *Given a cdf* F_1 *and a* $m \in \Re$, *any cdf* F_2 *with* $g_{4,(F_2,m)} = g_{4,(F_1,m)}$ *is necessarily* F_1 *whenever* $F_1(m)$ *or* $F_2(m)$ *is irrational. Moreover, we have*

$$g_{4,(F_1,m)}(y) = F_1^{-1}(\Phi_{0,1}(\frac{1 + m'\Phi_{0,1}^{-1}(F_1(y))}{m' - \Phi_{0,1}^{-1}(F_1(y))})),$$

where $\Phi_{0,1}$ *is the cdf of a Cauchy*$(0,1)$ *(i.e.,* $\Phi_{0,1}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$*), and* $m' = \Phi_{0,1}^{-1}(F_1(m))$.

Proof. From Williams [9],

$$X =^d \frac{1+m'X}{m'-X} \text{ under } F \Leftrightarrow F = \Phi_{0,1} \text{ whenever } \Phi_{0,1}(m') \in \overline{Q}.$$

Rewrite the above in terms of $X = \Phi_{0,1}(^{-1}F_1(Y))$ to establish that for a given continous random variable Y:

$$\Phi_{0,1}^{-1}(F_1(Y)) =^d \frac{1+m'\Phi_{0,1}^{-1}(F_1(Y))}{m'-\Phi_{0,1}^{-1}(F_1(Y))} \text{ under } F$$
$$\Leftrightarrow F = F_1 \text{ whenever } \Phi_{0,1}(m') \in \overline{Q},$$

or equivalently

$$Y =^d F_1^{-1}\Phi_{0,1}(\frac{1+m'\Phi_{0,1}^{-1}(F_1(Y))}{m'-\Phi_{0,1}^{-1}(F_1(Y))}) \text{ under } F$$
$$\Leftrightarrow F = F_1 \quad \text{whenever } \Phi_{0,1}(m') \in \overline{Q}.$$

Finally, observing that the function $F_1^{-1}\Phi_{0,1}(\frac{1+m'\Phi_{0,1}^{-1}(F_1(y))}{m'-\Phi_{0,1}^{-1}(F_1(y))})$ belongs to $\mathcal{C}_{4(S_{F_1},m)}$, with $m = F^{-1}(\Phi_{0,1}(m'))$, and therefore must coincide with $g_{4,(F_1,m)}$ by Proposition 2's uniqueness of $g_{4,(F_1,m)}$ within $\mathcal{C}_{4(S_{F_1},m)}$, [2] the result follows with the condition $\Phi_{0,1}(m') \in \overline{Q} \Leftrightarrow F_1(m) \in \overline{Q}$. □

[2]The equivalence may be verified directly as well.