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WEIGHTED HERMITE AND BIRKHOFF INTERPOLATION FOR POLYNOMIAL RINGS

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Abstract: Here we study some Hermite and (in one-variable) Birkhoff polynomial interpolation problems in which we insert "weights". In several variables, this is just Hermite interpolation for weighted polynomials, but in one variable we allow greater generality.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field. Fix integers $n \geq 1$ and a_i, $1 \leq i \leq n$, such that $1 \leq a_1 \leq \cdots \leq a_n$. For any integer $k \geq 0$ let $W(n;k)$ denote the vector space of all polynomials of degree at most k in the variables y_i, $1 \leq i \leq n$, over $\mathbb{K}$. Set $y_i := x_i^{a_i}$ and let $V(n;k;a_1,\dots,a_n)$ (or just $V(n;k)$) denote the vector space $W(n,k)$ seen as a vector space of polynomials in the variables $x_1,\dots,x_n$. Then we add a variable x_0 and see the vector space $V(n,k;a_1,\dots,a_n)$ as a linear subspace of the set of all homogeous degree a_nk polynomials in the variables $x_0,\dots,x_n$. The interpolation problems for these vector spaces may be tricky. For instance, take $n=2$, $a_0=1$, $a_1=2$ and set $P=(1;0)$. Then any element of $V(n;k;a_0,\dots,a_n)$ vanishes with even order at P. Hence when $p = \mathrm{char}(\mathbb{K}) > 0$ we will need to assume $a_nk < p$. Interpolation at $P \in \mathbf{P}^n$ of order 0 plus all first order derivatives at P corresponds to the evaluations associated to the double

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point $2P$ of $\mathbf{P}^n$.

Theorem 1. *Fix integers $n \geq 1$, $k \geq 5$, $s > 0$, and a_i, $1 \leq i \leq n$, such that $1 \leq a_1 \leq \cdots \leq a_n$. If $p = \mathrm{char}(\mathbb{K}) > 0$ assume $a_n k < p$. Let Z be a general union of s double points of $\mathbf{P}^n$. Then the restriction map $\rho_{n,k,Z} : V(n;k) \to H^0(Z, \mathcal{O}_Z(a_n k))$ has maximal rank, i.e. it is injective if $(n+1)s \geq \binom{n+k}{n}$ and it is surjective if $(n+1)s \leq \binom{n+k}{n}$.*

Remark 1. Here we consider the same topic for the Segre products of projective spaces, i.e. for the tensor product of the corresponding rings of polynomials. For any integer $s \geq 1$ and integers $n_i \geq 10$, $1 \leq i \leq s$, set $\Pi(s; n_1, \dots, n_s) := \prod_{i=1}^{s} \mathbf{P}^{n_i}$. Fix integers $k_i > 0$, $1 \leq i \leq s$ and $a_{i,j} > 0$, $1 \leq i \leq s$, $1 \leq j \leq n_i$. Let $W(s; n_1, \dots n_s; k_1; \dots; k_s)$ denote the vector space of all polynomials in the variables $y_{i,j}$, $1 \leq i \leq s$, $1 \leq j \leq n_i$, with degree at most k_i with respect to the variables $y_{i,j}$, $1 \leq j \leq n_i$. Set $y_{i,j} := x_{i,j}^{a_{i,j}}$. Then add the variables $x_{i,0}$, $1 \leq i \leq s$, and let $V(s; n_1, \dots n_s; k_1; \dots; k_s; a_{i,j})$ denote the vector space $W(s; n_1, \dots n_s; k_1; \dots; k_s)$ seen as a vector space of multihomogeneous polynomials of multidegree $(k_1, \dots, k_s)$ in the variables $x_{i,j}$.

Then we will look at the "weighted" Birkhoff interpolation problem on $\mathbb{A}^1_{\mathbb{K}}$. See [3] or [6] for the one-variable characteristic zero theory of Birkhoff interpolation. For the theory of Hasse derivatives, see [4], §3. For any multi-index α, let D^α denote the Hasse derivative of order α and δ^α the usual (partial) derivative of order α. In Section 3 we will prove a few results on this topic. Among them there is the following one.

Proposition 1. *Fix integers $n > 0$, $0 < a_1 < \cdots < a_n$ and $0 < b_1 < \cdots < b_n \leq a_n$. Let $E := (E_{1,k})_{1 \leq k \leq a_n}$ be the $1 \times (a_n + 1)$ Birkhoff interpolation matrix with one row and $E_{1,k} = 1$ if and only if $k \in \{0, b_1, \dots, b_n\}$. Let V the linear subspace of $\mathbb{K}[t]$ spanned by $1, t^{a_1}, \dots, t^{a_n}$. Then the Birkhoff interpolation problem for the matrix E with respect to the Hasse derivatives and the linear subspace V is almost-regular over $\mathbb{K}$ if either $\mathrm{char}(\mathbb{K}) = 0$ or $p := \mathrm{char}(\mathbb{K}) > 0$ and none of the integers $\binom{a_i}{b_i}$ is divisible by p.*

2. Proof of Theorem 1

Notation 1. We use the homogeneous coordinates $x_0, \dots, x_n$ on $\mathbf{P}^n$. Set $H := \{x_n = 0\}$.

Remark 2. Take any $f \in V(n;k)$ such that $f|H \equiv 0$. Then $x_n^{a_n} | f$ and hence there is $g \in V(n;k-1)$ such that $f = x_n^{a_n} g$.

Let Y be any projective variety, $Z \subseteq Y$ a closed subscheme and $D \subset Y$ an effective Cartier divisor of Y. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of Y with $\mathrm{Hom}(\mathcal{I}_{D,Y}, \mathcal{I}_{Z,Y})$ as ideal sheaf. Thus $\mathrm{Res}_D(Z) \subseteq Z$. For any $L \in \mathrm{Pic}(Y)$ we have an exact sequence on Y:

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)},Y} \otimes L(-D) \to \mathcal{I}_{Z,Y} \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0 \qquad (1)$$

From (1) we obtain the following very elementary form of Horace Lemma.

Lemma 1. *For any $L \in \mathrm{Pic}(Y)$ we have*

$$h^0(Y, \mathcal{I}_{Z,Y} \otimes L) \le h^0(Y, \mathcal{I}_{\mathrm{Res}_{D(Z)},Y} \otimes L(-D)) + h^0(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

and

$$h^1(Y, \mathcal{I}_{Z,Y} \otimes L) \le h^1(Y, \mathcal{I}_{\mathrm{Res}_{D(Z)},Y} \otimes L(-D)) + h^1(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D)).$$

Definition 1. Fix integers $n \ge 1$, $k \ge 0$, and a_i, $1 \le i \le n$, such that $1 \le a_1 \le \cdots \le a_n$. Let $Z \subset \mathbf{P}^n$ be a zero-dimensional scheme. We will say that the interpolation problem corresponding to $V(n;k)$ is well-behaved at Z if the restriction map $\rho_{n,k,Z} : V(n;k) \to H^0(Z, \mathcal{O}_Z(a_n k))$ has maximal rank, i.e. it is injective if $\binom{n+k}{n} \le \mathrm{length}(Z)$ and it is surjective if $\binom{n+k}{n} \ge \mathrm{length}(Z)$.

For all integers $n \ge 1$ and $k \ge 0$ define the integers $a_{n,k}$ and $b_{n,k}$ by the following relations:

$$(n+1)a_{n,k} + b_{n,k} = \binom{n+k}{n}, \quad 0 \le b_{n,k} \le n\,. \qquad (2)$$

Notice that the integers $a_{n,k}$ and $b_{n,k}$ do not depend from the choice of the integers $a_1, \dots, a_n$.

Consider the following statements $A_{n,k}$ and $A'_{n,k}$, $n \ge 1$, $k \ge 1$:

$A_{n,k}$, $n \ge 1$, $k \ge 1$: Let Z be the general union of $a_{n,k}$ general double points and $b_{n,k}$ points. Then $\rho_{n,k,Z}$ is bijective.

$A'_{n,k}$, $n \ge 1$, $k \ge 1$: Let Z be the general union of $a_{n,k}+1$ general double points. Then $\rho_{n,k,Z}$ is injective.

Remark 3. Let $Z, Z', Z'' \subset \mathbf{P}^n$ be zero-dimensional schemes such that $Z' \subseteq Z \subseteq Z$". Notice that the restriction maps $H^0(Z, \mathcal{O}_Z(a_n k)) \to H^0(Z', \mathcal{O}_{Z'}(a_n k))$ are surjective. Hence if $\rho_{n,k,Z}$ is surjective, then $\rho_{n,k,Z'}$ is surjective, while if $\rho_{n,k,Z}$ is injective, then $\rho_{n,k,Z''}$ is injective. Thus to prove Theorem 1 for the integers n, k it is sufficient to prove it when $s = a_{n,k}$ (surjectivity range) and when $s = a_{n,k}+1$ (injective range); the latter case may be dropped if $b_{n,k} = 0$. Hence to prove Theorem 1 for the fixed pair (n,k) it is sufficient to prove the statements $A_{n,k}$ and $A'_{n,k}$.

Remark 4. Let Y be an integral projective variety of dimension at least two, A a zero-dimensional subscheme of Y, D a non-empty integral effective Cartier divisor of Y, $L \in \mathrm{Pic}(Y)$ and $V \subseteq H^0(Y, L)$ a linear subspace. Let $V|D$ denote the image of V by the restriction map $\rho : H^0(Y, L) \to H^0(D, L|D)$. Set $V(-D) := V \cap \mathrm{Ker}(\rho)$ and see it as a linear subspace of $H^0(Y, L(-D)$ dividing by the equation of D. Fix an integer $x > 0$. Let B be a general union of x double points of Y, S the union of x general points of D and E the union of x general double points of D. Since $D \subset Y$, we may see E as a closed subscheme of Y. Let By [1], Lemma 2.3, to check that the restriction map $\rho_{V,A\cup B} : V \to H^0(A \cup B, L|(A \cup B)$ is surjective (resp. injective), it is sufficient to check the following two conditions:

(a) the restriction map $V(-D) \to H^0(\mathrm{Res}_D(A) \cup E, L(-D)|\mathrm{Res}_D(A) \cup E)$ is surjective (resp. injective);

(b) the restriction map $V|D \to H^0((A \cap D) \cup S, L|(A \cap D) \cup S)$ is surjective (resp. injective).

Remark 5. Take $n = 1$. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $p = \mathrm{char}(\mathbb{K}) > 0$ and $a_1 k < p$, Theorem 1 is true in this case (dor any $k \geq 1$) by [5], Theorem 15.

Lemma 2. *Fix integers $a_2 \geq a_1 > 0$ and $k \geq 5$. Then Theorem 1 is true for $n = 2$.*

Proof. We first outline the general inductive step (part (i)), then show the case $k = 5$ is true (part (ii)) and then show why the inductive step works (part (ii)). In parts (i) and (iii) we will only prove $A_{2,k}$, leaving the proof of $A'_{n,k}$ to the interested reader; notice that $A'_{2,k}$ follows from $A_{2,k}$ if $b_{2,k} = 0$, i.e. if $k \equiv 1, 2 \pmod 3$. Recall that H is the line $\{x_2 = 0\}$. Notice that $a_{1,k} = (k+1)/2$ and $b_{1,k} = 0$ if k is odd, while $a_{1,k} = k/2$ and $b_{1,k} = 1$ if k is odd.

(i) Fix the integer $k \geq 2$. It is easy to check that $a_{1,k} + 2b_{1,k} \leq 2a_{1,k-1} + b_{1,k-1}$. First assume $b_{1,k} = 0$, i.e. k odd. Take $(k+1)/2$ general points $P_1, \dots, P_{(k+1)/2} \in H$ and call $Z \subset \mathbf{P}^2$ the union of the double points $2P_i$, $1 \leq i \leq (k+1)/$, of $\mathbf{P}^2$ and a general union W of $a_{2,k} - a_{2,k-1}$ general double points. Set $S := \{P_1, \dots, P_{(k+1)/2}\}$. Notice that $\sharp(Z_{red}) = a_{2,k-1}$ and that $b_{2,k} = b_{2,k-1}$. By Lemma 1 (Horace Lemma) to prove $A_{2,k}$ it is sufficient to prove the surjectivity of the restriction map $\rho_{2,k-1,W\cup S} : V(2; k-1) \to H^0(W \cup S, \mathcal{O}_{W\cup S}(a_2(k-1)))$. By the "inductive assumption" (see part (iii) below) the restriction map $\rho_{2,k-1,W} : V(2, k-1) \to H^0(W, \mathcal{O}_W(a_2(k-1)))$ is surjective. Since S is general in H and $\dim(\mathrm{Ker}(\rho_{2,k-1,W})) \geq \sharp(S)$, to get the surjectivity of $\rho_{2,k-1,W\cup S}$, i.e. to show that S imposes $\sharp(S)$ independent conditions to $\mathrm{Ker}(\rho_{2,k-1,W})$, it is sufficient to show $\dim(\rho_{2,k-2,W}) \leq \mathrm{Ker}(\rho_{2,k-1,W}) - \sharp(S)$. Again, this is true by the "inductive assumption" (see part (iii) below). Now assume $b_{1,k} = 1$, i.e. k even. Take $k/2+1$ general points $P_0, \dots, P_k$ of H. Set $S := \{P_1, \dots, P_{k/2}\}$ and let E the double point of H supported by P_0. By Remark 4 and Remark 5 $A_{2,k}$ is true if $\rho_{2,k-1,W\cup S\cup E}$ is surjective. Again, by the "inductive assumption" (see part (iii) below) we reduce to check $\mathrm{Ker}(\rho_{2,k-1,W})$, it is sufficient to show $\dim(\rho_{2,k-2,W}) \leq \mathrm{Ker}(\rho_{2,k-1,W}) - k/2 - 2$.

(ii) Here we assume $k = 5$. We have $a_{2,5} = 7$, $b_{2,5} = 1$ (and hence $A_{2,5}$ and $A'_{2,5}$ are equivalent), $a_{1,5} = 3$ and $b_{1,5} = 0$. Take 3 general points P_1, P_2, P_3 and make the first reduction step done in part (i). Set $S := \{P_1, P_2, P_3\}$. Let $W \subset \mathbf{P}^2$ be a general union of 4 double points. As in part (i) we get that $A_{2,5}$ is true if the restriction map $\rho_{2,4,W\cup S}$ is surjective. By the second reduction step done in part (i) we see that $\rho_{2,4,W\cup S}$ is surjective if the restriction map $\beta : V(2,3) \to H^0(W, \mathcal{O}_W(3a_2))$ is injective.

(iii) Here we discuss the "inductive assumption". In part (i) of the proof we reduced $A_{2,k}$ to an assertion much weaker than $A_{2,k-1}$ and then to an even weaker form of $A'_{2,k-2}$. Hence by part (ii) it is sufficient to these weak assertions when $k = 6$ and $k = 7$. Take $k = 6$. We have $a_{2,6} = 9$, $b_{2,6} = 1$, $a_{1,6} = 3$, and $b_{1,6} = 1$. Take 4 general points $P_0, \dots, P_3$ of H and set $S := \{P_1, P_2, P_3\}$. Let E be the double points of H such that $E_{red} = \{P\}$. Let W be a general union of 5 double points of $\mathbf{P}^2$. The second construction of part (ii) (case k even) reduce the proof of $A_{2,6}$ to the proof of the surjectivity of the restriction map $\rho_{2,5,W\cup S\cup E} : V(2,5) \to H^0(W \cup E \cup S, \mathcal{O}_{W\cup S\cup E}(5a_2))$. Notice that $\mathrm{length}(H \cap (W \cup E \cup S) = \mathrm{length}(E \cup S) = 5$. Take a specialization of $W \cup E \cup S$ in which we specialize one of the point of W_{red} to a general point $P_4 \in H$. Let $W' \subset W$ be the union of the remaining 4 double points and F is the double point of H such that $F_{red} = \{P_4\}$. By Remark 4 we reduce the proof of the surjectivity of $\rho_{2,5,W\cup S\cup E}$ to the proof of the surjectivity of $\rho_{2,4,W'\cup F}$, which is easy using another induction step. The case $k = 7$ is easier and hence it is omitted. □

Here we collect a few numerical lemmas.

Lemma 3. *We have $a_{n,k} \geq a_{n-1,k} + b_{n-1,k}$ for all $n \geq 3$ and $k \geq 4$.*

Proof. Subtracting the equation in (2) for the pair $(n-1,k)$ from the same equation for the pair (n,k) we obtain:

$$n(a_{n,k} - b_{n-1,k}) + a_{n,k} + b_{n,k} - b_{n-1,k} = \binom{n+k-1}{n}, \tag{3}$$

and hence it is sufficient to check the inequality

$$(n-1)b_{n-1,k} + a_{n,k} + b_{n,k} \leq \binom{n+k-1}{n}. \tag{4}$$

Since $a_{n,k} \leq \binom{n+k}{n}/(n+1)$ (use (2)), $b_{n-1,k} \leq n-1$ and $b_{n,k} \leq n$, we easily conclude, at least if $n \geq 4$ and $k \geq 5$. For low n and k compute explicitly the values of the integers $a_{n,k}$, $a_{n-1,k}$ and $b_{n-1,k}$. □

Lemma 4. *We have*

$$a_{n-1,k} + n + nb_{n-1,k} \leq \binom{n+k-2}{n-1} \tag{5}$$

for all $n \geq 3$ and $k \geq 4$

Proof. By the equation in (2) for the pair $(n-1,k)$ we obtain that it is sufficient to use the trivial inequality $a_{n-1,k} \geq n-1$. □

Proof of Theorem 1. By Remark 3 it is sufficient to prove $A_{n,k}$ and $A'_{n,k}$ for all (n,k) such that $n \geq 2$ and $k \geq 5$. As in Lemma 2 both proofs are very similar, the proof of $A'_{n,k}$ being usually easier. We will often use induction on n; when we talk about $V(n-1;k)$, $A_{n-1,k}$ and $A'_{n-1,k}$ we will means these assertions with respect to the integers $1 \leq a_1 < \cdots < a_n - 1$. By Lemma 2, Remark 5 and Remark we may assume $n \geq 3$. We only give an outline of proof, the detail being left to the reader (they are very similar, but numerically easier) to the ones in the proof of [2], Theorem 1). Assume $n \geq 3$, $k \geq 5$. First, (as in part (i) of the proof of Lemma 2) we will write the inductive step of the proof $A_{n,k}$, i.e. the proof of the bijectivity of the restriction map $\rho_{n-1,k,A\cup B}$ for a general union $A \cup B \subset H$ with A union of $a_{n-1,k}$ double points of H and B union of $b_{n-1,k}$ points. Let A' be the union of the double points of $\mathbf{P}^n$ with A_{red} its support and A'' the union of $a_{n,k} - a_{n-1,k} - b_{n-1,k}$ general double points of $\mathbf{P}^n$; here we use that $a_{n,k} \geq a_{n-1,k} + b_{n-1,k}$ (Lemma 3). By semicontinuity to prove $A_{n,k}$ it is sufficient to prove $\dim(h^0(\mathbf{P}^n, \mathcal{I}_{A'\cup A''}(a_n k))) \cap V_{n,k}) = b_{n,k}$. Now we made on H the construction given in part (i) of the proof of Lemma 2. The construction may be done (i.e. the prescribed number of certain connected components of certain zero-dimensional schemes are non-empty), by Lemma 3 and Lemma 4. As in parts (ii) and (iii) of the proof of Lemma 2 the initial cases $k = 5, 6, 7$ may be done in the same way. we just remark that $b_{m,5} = 0$ if and only if $(m+5)(m+4)(m+3)(m+2) \equiv 0 \pmod{120}$ and this is true if $m = 6, 7, 8$. □

3. Birkhoff Weighted Interpolation on $\mathbb{A}^1_{\mathbb{K}}$

Fix a prime p. For any non-negative integers a, b, let $a = \sum_{i\geq 0} a_i p^i$, $0 \leq a_i \leq p-1$, and $b = \sum_{i\geq 0} b_i p^i$, $0 \leq b_i \leq p-1$, be their p-adic expansions. The integer a is said to be p-adically bigger than or equal to b if $a_i \geq b_i$ for all i ([4], p. 9). Notice that $\geq_p$ is only a partial ordering of $\mathbb{N}$. We have $\binom{a}{b} \equiv 0 \pmod p$ if and only if a is not p-adically bigger than or equal ([4], 3.6).

Proof of Proposition 1. For all integers $a \geq b \geq 0$ we have $D^b(t^a) = \binom{a}{b} x^{a-b}$ ([4], Example 3.1). Hence the matrix of the Hasse derivatives has as its determinant a polynomial in t whose leading term is βt^α, where $\alpha := \sum_{i=1}^n (a_i - b_i)$ and $\beta := \prod_{i=1}^n \binom{a_i}{b_i}$ (seen as an element of $\mathbb{K}$, not as an integer). If $\beta \neq 0$, then the associated matrix has non-zero determinant and hence invertible determinant over the function field $\mathbb{K}(t)$. Notice that $\beta \neq 0$ and hence none of the integers $\binom{a_i}{b_i}$ is divisible by p ([4], 3.6). □

When $b_i = a_i$ for all i, then the matrix used in the proof of Proposition 1 is in a triangular form and hence as a corollary of the proof just given we get the following result.

Proposition 2. *Fix integers $n > 0$ and $0 < a_1 < \cdots < a_n$. Let $E := (E_{1,k})_{1\leq k\leq a_n}$ be the $1 \times (a_n + 1)$ Birkhoff interpolation matrix with one row and $E_{1,k} = 1$ if and only if $k \in \{1, a_1, \ldots, a_n\}$. Let V the linear subspace of $\mathbb{K}[t]$ spanned by $1, t^{a_1}, \ldots, t^{a_n}$. Then the Birkhoff interpolation problem for the matrix E with respect to the Hasse derivatives and the linear subspace V is almost-regular over $\mathbb{K}$ if and only if either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $p := \mathrm{char}(\mathbb{K}) > 0$ *and none of the integers $\binom{a_i}{b_i}$ is divisible by p.*

Remark 6. Assume $p := \mathrm{char}(\mathbb{K}) > 0$ and take $n = 1$. We get that the first order Birkhoff interpolation problem for the evaluation and the first derivative with respect to the linear subspace $1, t^a$ is not almost regular if and only if $a \equiv 0 \pmod p$.

Proposition 3. *Fix integers $n > 0$, $0 < a_1 < \cdots < a_n$ and $0 < b_1 < \cdots < b_n \leq a_n$. Let $E := (E_{1,k})_{1\leq k\leq a_n}$ be the $1\times(a_n+1)$ Birkhoff interpolation matrix with one row and $E_{1,k} = 1$ if and only if $k \in \{0, b_1, \ldots, b_n\}$. Let V the linear subspace of $\mathbb{K}[t]$ spanned by $1, t^{a_1}, \ldots, t^{a_n}$. Then the Birkhoff interpolation problem for the matrix E with respect to the ordinary derivatives and the linear subspace V is almost-regular over $\mathbb{K}$ if and only if either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $p := \mathrm{char}(\mathbb{K}) > 0$*, $b_i \leq a_i$ for all i and $a_n < p$. If $b_i \neq a_i$ for at least one index i and either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $p := \mathrm{char}(\mathbb{K}) > 0$*, $b_i \leq a_i$ for all i and $a_n < p$, then this interpolation problem is not always solvable on $\mathbb{A}^1_{\mathbb{K}}$ and the set B of all $P \in \mathbb{A}^1_{\mathbb{K}}$ is not uniquely solbable satisfies $\sharp(S) \leq \sum_{i=1}^n (a_i - b_i)$. If $b_i \neq a_i$ for* *all* $1 \leq i \leq n$ *and either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $p := \mathrm{char}(\mathbb{K}) > 0$*,* $b_i \leq a_i$ *for all* i *and* $a_n < p$*, then this interpolation problem is uniquely solvable at each point* $\mathbb{A}^1_{\mathbb{K}}$

Proof. It is sufficient to use that $\partial^b(t^a) = 0$ if and only if either $b > a$ or $b > 0$ and $a \geq p$. For the last two assertion notice that the associated determinant is a polynomial in t with exact degree $\sum_{i=1}^n (a_i - b_i)$. $\square$

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