International Journal of Pure and Applied Mathematics
Volume 23 No. 2 2005, 151-158

k-ROTUND POINTS AND NESTED SEQUENCE BALLS IN BANACH SPACES

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Abstract: In this paper, the relationships of k-rotund points and k-smooth points and nested sequence balls in Banach spaces are discussed.

AMS Subject Classification: 46B20
Key Words: extreme points, k-extreme points, k-rotund points, k-smooth points, nested sequence of balls

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Let X be a Banach space, $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X, $B_X = \{x \in X : \|x\| \leq 1\}$ be the unit ball of X, C is a convex subset of X, $x \in C$ is said to be an extreme point of convex set C if whenever $x = \lambda y + (1-\lambda)z$ for some λ in (0,1) and $y, z \in C$, then $x = y = z$. As simple examples, every point in the unit sphere of R^2 is an extreme point of B_{R^2}.

In [8], I. Singer introduced the k-strictly convex.

Definition 1. (see [8]) A Banach space X is called k-strictly convex if and only if for any $K+1$ elements $x_0, ..., x_k$ of X the relation

$$\|x_0 + x_1 + ... + x_k\| = \|x_0\| + \|x_1\| + ... + \|x_k\|$$

implies that $x_0, ..., x_k$ are linearly dependent.

Received: June 10, 2005

It is easy to see that if strictly convex space is 1-strictly convex space, and if Banach space X is k-strictly convex, then for any $m \geq k$, X is m-strictly convex.

Theorem 2. (see [8]) *For any Banach space X, then following assertions are equivalent to k-strictly convexity:*

(1) For any k + 1 elements $x_0, x_1, ..., x_k$ of X, $\|x_i\| = 1, \|\sum_{i=0}^{k} x_i\| = \sum_{i=0}^{k} \|x_i\|$ implies that $x_0, ..., x_k$ are linearly dependent.

(2) For any k + 1 elements $x_0, x_1, ..., x_k$ of X, $\|x_i\| = 1, \|\sum_{i=0}^{k} x_i\| < 1$ for all $0 < \alpha_i < 1, \sum_{i=0}^{k} \alpha_i = 1$.

(3) The set $S(X) = \{x : x \in X, \|x\| = 1\}$ contains no convex subsets of dimension $> k - 1$.

(4) For any $x_0 \in X$ and $r > 0$ the set $S(x_0, r) = \{x : x \in X, \|x - x_0\| = r\}$ contains no convex sets of dimension $> k - 1$.

Definition 3. Let k be a positive integer, and dim$X \geq k + 1$. A point $x \in S_X$ is called a k-extreme point of B_X if and only if x cannot be represented as $x = \frac{x_1+x_2+...+x_{k+1}}{k+1}$, where $x_i \in S_X, i = 1, 2, ..., k + 1$, and the set $\{x_1, x_2, ..., x_{k+1}\}$ is linear independent.

It is easy to see that extreme point is 1-extreme point, but the converse is not true. Let $x_0 = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0) \in l_1$, then $\|x_0\| = 1$, set $x^* = (1, 1, 0, ..., 0) \in l_\infty$, then $x^*(x_0) = \|x^*\| = 1$. Assume that $x, y, z \in S_{l_1}$ with $x_0 = \frac{x+y+z}{3}$, then it is easy to see that $x^*(x) = x^*(y) = x^*(z) = 1$, so we have $x^*(x) = x_1 + x_2 = 1$, by $|x_i| \leq 1$ and $\|x\| = 1$, we have $x_1 \geq 0, x_2 \geq 0$ and $x_1 + x_2 = 1$, thus $x_i = 0$ for all $i > 2$. Similarly, we have $y_1 \geq 0$, $y_2 \geq 0$ and $y_1 + y_2 = 1$ and $y_i = 0$ for all $i > 2$, $z_1 \geq 0, z_2 \geq 0$ and $z_1 + z_2 = 1$ and $z_i = 0$ for all $i > 2$. So x, y, z are linear dependent. Hence x_0 is not extreme point of B_{l_1}, and x_0 is a 2-extreme point of B_{l_1}. It is easy to see that for all $k \in N$, set $x_i = \frac{1}{k}$ if $i \leq k$ and $x_i = 0$ if $i > k$, then $x = (x_i) \in B_{l_1}$, x is a k-extreme point of B_{l_1}.

Rotund points were introduced in [6], it is strictly stronger than extreme points. we may generalize the rotund points to k-rotund points.

Definition 4. For $x_0 \in S_X$, if $\|x_0 + x_1 + x_2 + ... + x_k\| = k + 1$ then set $\{x_0, x_1, x_2, ..., x_k\}$ is not linear independent, we say x is a k-rotund point of the unit ball of X.

It is easy to see that if $x \in S_X$ is a k-rotund point of B_X, then x is a k-extreme point of B_X, but the converse is not true. Let $x_0 = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0) \in l_1$, then x_0 is a 2-extreme point of B_{l_1}. It is easy to see that for $x_1 = (0, 0, \frac{1}{3}, \frac{2}{3}, 0, ..., 0), x_2 = (0, 0, 0, 0, \frac{1}{4}, \frac{3}{4}, 0, ..., 0) \in B_{l_1}$, we have $\|x_0 + x_1 + x_2\| = 3$

and x_0, x_1, x_2 are not linear independent. Thus x_0 is a 2-extreme point of B_{l_1} and x_0 is not a 2-rotund point of B_{l_1}.

In [3], a sequence of ball $\{B_n\}$ in Banach space X is called a nested sequence of ball if $B_1 \subseteq B_2 \subseteq ... \subseteq B_n \subseteq ....$

A nested sequence $\{B_n = B(x_n, r_n)\}$ of balls in X is unbounded if $r_n \to \infty$.

Recently, various nested sequence balls in Banach spaces and their relationships with the extremal structures of the unit ball have been studied [2]. We have the following theorm for k-rotund point.

Theorem 5. *Let X be a Banach space. Then the following are equivalent:*

(1) x^* is a k-rotund point of B_{X^*};

(2) for every unbounded nested sequence $\{B_n\}$ of balls such that x^* is bounded on $\bigcup B_n$, set $R = \{y^* \in S_{X^*} : y^*$ is bounded below on $\bigcup B_n\}$, then $dimR \le k+1$;

(3) for every bounded nested sequence $\{B_n\}$ of balls such that x^* is bounded below on $\bigcup B_n$, if for any $\{y_n^*\} \subseteq S_{X^*}$, the sequence $\{\inf y_n^*(B_n)\}$ is bounded below, then the dimension of all w*-limit point of $\{y_n^*\} \le k+1$.

Proof. (1) $\Rightarrow$ (2) For $n \in N$, let $B_n = B(x_n, r_n)$. Since $\inf x^*(B_n) \ge c, n \in N$, we have $x^*(x_n) - r_n \ge c$. Hence

$$x^*(\frac{x_n}{r_n}) - 1 \ge \frac{c}{r_n}, \quad n \in N\,.$$

We may assume that $0 \in B_1$, then $\|\frac{x_n}{r_n}\| \le 1, n \in N$, so we have

$$\lim_{n\to\infty} x^*(\frac{x_n}{r_n}) = 1\,.$$

Similarly, for any $y_i^* \in R = \{y^* \in S_{X^*} : y^*$ is bounded below on $\bigcup B_n\}$, we have

$$\lim_{n\to\infty} y_i^*(\frac{x_n}{r_n}) = 1, \quad i = 1, 2, 3, ...$$

Assume that there exists $y_1^*, y_2^*, ..., y_{k+1}^* \in R$, they are linearly independent, then

$$\lim_{n\to\infty} (y_1^* + y_2^* + ... + y_{k+1}^*)(\frac{x_n}{r_n}) = 1\,.$$

Thus

$$\|y_1^* + y_2^* + ... + y_{k+1}^*\| = k+1$$

but it contradicts X is k-rotund.

(2) $\Rightarrow$ (3) Since B_{X^*} is weak*-compact, $\{y_n^*\}$ have some w^*-cluster points in X^*. Suppose that $\{y_n^*\}$ have w^*-cluster points $e_1^*, e_2^*, ..., e_{k+1}^*$ are linearly independent, then it is easy to see

$$e_i^*(x_n) - r_n \geq c \text{ for all } n \in N\,.$$

Thus for each i

$$\inf\ e_i^*(B_n) \geq c \text{ for all } n \in N\,,$$

a contradiction.

(3) $\Rightarrow$ (2) Easy.

(2) $\Rightarrow$ (1) Suppose there exists $y_i^* \in S_{X^*}$ with $\|x^*+y_1^*+y_2^*+...+y_k^*\| = k+1$. Let $\{x_n\} \in S_X$ such that $(x^*+y_1^*+y_2^*+...+y_k^*)(x_n) \to k+1$. Then $y_i^*(x_n) \to 1$ for $i = 1, 2, ..., k$ and $x^*(x_n) \to 1$. Choose a sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all n and $\sum_{n=1}^{\infty} \delta_n < 1$. We may assume that $y_i^*(x_n) > 1-\delta_n$ for $i = 1, 2, ..., k+1$ and $x^*(x_n) > 1 - \delta_n$.

Let $B_n = B(\sum_{i=1}^{n} x_i, n+\sum_{i=1}^{n} \delta_i)$. Obviously, $\{B_n\}$ is an unbounded nested sequence of balls. For any $n \in N$, we have

$$\begin{aligned}\inf y_i^*(B_n) = y_i^*(\sum_{j=1}^{n} x_j) - n - \sum_{j=1}^{n} \delta_j &= \sum_{j=1}^{n}[y_i^*(x_j) - 1 - \delta_j] \\ &= -\sum_{j=1}^{*}[1 - y_i^*(x_j) + \delta_j] > -\sum_{j=1}^{n} 2\delta_i > -2\,.\end{aligned}$$

Thus we have $\inf y_i^*(B_n) > -2, i = 1, 2, ..., k$. Similarly, $\inf x^*(B_n) > -2$, so $\{x^*, y_1^*, y_2^*, ..., y_k^*\}$ are linear dependant, hence Banach space x^* is a k-rotund point of B_{X^*}. □

A classical result of Taylor [9] and Foguel [5] showed that X^* is strictly convex if and only if every subspace Y is a U-subspace of X. We recall that a subspace Y of a Banach space X is said to be a U-subspace of X if each $y^* \in Y^*$ has a unique Hahn-Banach (i.e., norm preserving) extension in X^*. We may generalizes the theorem about the uniqueness of Hahn-Banach extension.

Theorem 6. *Let X be a Banach space. Then the following are equivalent:*

(1) x^* is a k-rotund point of B_{X^*};

(2) for any subspace $Y \subseteq X$ such that $\|x^*|_Y\| = 1, x^*|_Y$ have at most k linearly independent norm-preserving linear extensions to X.

Proof. (1) $\Rightarrow$ (2) Suppose x^* is a k-rotund point of B_{X^*}, Y is a subspace of X such that $\|x^*|_Y\| = 1$, and $x^*|_Y$ have $x^*, x_1^*, x_2^*, ..., x_k^*$ as linearly independent norm-preserving linear extensions to X, thus $\|x^* + x_1^* + ... + x_k^*\| = k+1$, it contradicts x^* is a k-rotund point of B_{X^*}.

(2) $\Rightarrow$ (1) Assume that there exist $x^*, x_1^*, x_2^*, ..., x_k^* \in S_{X^*}$ are linearly independent with $\|x^* + x_1^* + ... + x_k^*\| = k+1$, then for the subspace $Y \subseteq X$ such that $\|\frac{x^*+x_1^*+...+x_k^*}{k+1}|_Y\| = 1$, we have $\|x^*|_Y\| = 1$, and x has $k+1$ linearly independent norm-preserving linear extensions to X, a contradiction. □

Corollary 7. (see [7]) *Let $k \in N$, and* $\dim X \geq k+1$. *Then all bounded linear functionals defined on subspaces of X have at most k linearly independent norm-preserving linear extensions to X if and only if X^* is k-rotund.*

Corollary 8. (see [9], [5]) *Let X be a Banach space with* dim ≥ 2. *Then all bounded linear functionals define on a subspaces of X have unique norm-preserving linear extensions to X if and only if X^* is rotund.*

It is rather surprising that the unit ball B_X of Banach space X has extreme points if Banach space X has some k-extreme points of its unit ball.

Theorem 9. *Let X be a Banach space, B_X has a k-rotund point for some k if and only if B_X has an extreme point.*

Proof. Let x_0 be a k-rotund point of B_X, then $\|x_0 + x_1 + x_2 + ... + x_k\| = k+1$ implies the set $\{x_0, x_1, x_2, ..., x_k\}$ is not linear independent, thus for any $x^* \in S_{X^*}$, $\dim S(x^*, 0) = \dim\{x \in S_X : x^*(x) = 1\} \leq k$, so $S(x^*, 0)$ is a bounded convex subset of k-dimensional subspace. So we have $e_0 \in \text{ext} S(x^*, 0)$, since $S(x^*, 0)$ is a fact of B_X, hence e_0 is an extreme point of B_X. □

Corollary 10. *Let X be a Banach space, if for any equivalent norm $|||.|||$, $B_{(X,|||.|||)}$ has some k-rotund points for some k, then do not contains an isomorphic copy of c_0.*

A convex subset $C \subseteq X$ is a cone with vertex 0 if it is closed under multiplication by positive scalars. In [2], it is proved that if X^* is k-rotund, then the unit of any unbounded nested sequence of ball is a cone. It is easy to see for any finite dimension Banach space X ($\dim X \leq k$), X must be k-rotund for any equivalent norm. For the converse, we have the following theorem.

Theorem 11. *Let X be a Banach space, if for any equivalent norm $|||.|||$ on X, $(X^*, |||.|||)$ is k-rotund, then Banach space X must be finite dimensional.*

Proof. Since for any equivalent norm $|||.|||$ on X, $(X^*, |||.|||)$ is k-rotund, by above theorem, we know that the unit of any unbounded nested sequence of ball is a cone, so by [2], Banach space X is finite dimensional if and only if for any equivalent norm, the unit of each unbounded nested sequence of balls is a cone, thus Banach space X is finite dimensional. □

Theorem 12. *Banach space X is finite dimensional if and only if for any subspace M of X, for any equivalent norm $|||.|||$ on M, $(M, |||.|||)$ is k-rotund for some $k \in N$.*

Proof. It is easy to see if Banach space X is finite dimensional, then for any subspace M of X, for any equivalent norm $|||.|||$ on M, $(M, |||.|||)$ is k-rotund for some $k \in N$.

Suppose that X is infinite dimensional, then there exists $\|e_i\| = 1$ for all $i \in N$, and $\{e_i\}$ are linear independent. Let $M = \text{span}\{e_i\}$, then M is a subspace of X, set $|||x||| = \sum_{k=1}^{m} |x_{i_k}|$ whenever $x = \sum_{k=1}^{m} x_{i_k} e_{i_k}$, it is easy to see $(M, |||.|||)$ is not k-rotund for any $k \in N$. □

Theorem 13. *For arbitrary Banach space X (dim ≥ 2), there exist an equivalent norm $|||.|||$, such that the unit sphere $S_{(X,|||.|||)}$ has a point x_0 is not extreme point of the unit ball $B_{(X,|||.|||)}$.*

Proof. Since $\dim X \geq 2$, we have $x_0 \in S_X$, and $f_0 \in A_{x_0}$, such that $f_0(x_0) = 1$. Set $Y = \{x \in X : f_0(x) = 0\}$, then X is isomorphic to $Y \bigoplus R$, so we definite a norm on $Y \bigoplus R$ as $\|(y, r)\| = \max\{\|y\|, |r|\}$, and it is easy to see that (0,1) is not an extreme point of $B_{(Y \bigoplus R, \|.\|)}$. Hence the unit ball of Banach space X has an extreme point in the equivalent norm $|||x||| = \|(y, r)\| = \max\{\|y\|, |r|\}$. □

So, it is easy to see the following corollary hold.

Corollary 14. *Let X be a Banach space, if every equivalent norm on X is rotund, then X is 1-dimensional Banach space.*

Another way to study the properties of k-rotund points and k-extreme points is via their dual notions.

Definition 15. $x \in S_X$ is called a k-smooth point of Banach space X, if $\dim\{x^* \in S_{X^*} : x^*(x) = 1\} \leq k + 1$.

Let $x_0 = (1, 1, 1, 0, ..., 0) \in c_0$, then $\|x_0\| = 1$ and $\dim\{x^* \in S_{X^*} : x^*(x) = 1\} \leq 3$, so x_0 is a 2-smooth point of c_0, and x_0 is not a smooth point of c_0.

Theorem 16. *For $x \in S_X$, if any $x^* \in D(x)$ is a k-rotund point of B_{X^*}, x is a k-smooth point, but the converse is not true.*

Proof. If $y_1^*, y_2^*, ..., y_{k+1}^* \in \{x^* \in S_{X^*} : x^*(x) = 1\}$, then $\|y_1^* + y_2^* + ... + y_{k+1}^*\| = k + 1$, since any $x^* \in D(x)$ is a k-rotund point of B_{X^*}, we have $y_1^*, y_2^*, ..., y_{k+1}^*$ are linearly linearly independent. Thus x is a k-smooth point.

Let $x_0 = (1, 1, 1, 0, ..., 0) \in c_0$, then $\|x_0\| = 1$ and x_0 is a 2-smooth point of c_0, but $x^* = (1, 0, ..., 0) \in D(x_0) \subseteq l_1$, x^* is not a 2-rotund point of B_{l_1}. In fact, x^* is not a k-rotund point of B_{l_1} for any $k \in N$. □

Followings directly from the definitions, we have the following result.

Theorem 17. *For $x \in S_X$, if any $x^* \in D(x)$ is a k-smooth point of B_{X^*}, x is a k-rotund point.*

k-smooth points can be characterized in terms of perfect nested sequence of balls similar to smooth point.

Theorem 18. *Let $x \in S_X$ in Banach space X. The following are equivalent:*

(1) x is a k-smooth point;

(2) for every perfect nested sequence of balls $\{B_n\}$ with unbounded radii in the direction of x, set $R = \{y^* \in S_{X^*} : y^*$ is bounded below on $\bigcup B_n\}$, then $\dim R \leq k+1$*;*

(3) for every perfect nested sequence of balls $\{B_n\}$ with unbounded radii in the direction of x and for any x, for any $\{y_n^*\} \subseteq S_{X^*}$, the sequence $\{\inf\ y_n^*(B_n)\}$ is bounded below, then the dimension of all w^*-limit point of $\{y_n^*\} \leq k+1$.

Acknowledgements

It is a pleasure to thank Professor Lin Bor-Luh for informative discussions about his comprehensive work. This research was done when the author was visiting the University of Iowa, the author is grateful to the University of Iowa for support and hospitality.

Supported by the Foundation of Sun Yat-sen University Advanced Research Centre.

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