International Journal of Pure and Applied Mathematics
Volume 23 No. 2 2005, 251-253

ON THE MINIMAL FREE RESOLUTION OF GENERAL UNIONS IN P^2 OF A FINITE FIX SUBSET AND MANY GENERAL POINTS

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $A \subset \mathbf{P}^2$ be a finite subset such that no three points of A are collinear. Set $d := \sharp(A)$ and $m_d := 2\lceil d/4 \rceil + 1$. Fix an integer x such that $d + x > m_d(m_d + 1)/2$, i.e. assume that the first integer t such that $d + x \le (t + 2)(t + 1)/2$ satisfies $t \ge m_d$. Let $S \subset \mathbf{P}^2$ be a general subset such that $\sharp(S) = x$. Here we prove that the minimal free resolution of $Z := A \cup S$ is the expected one, i.e. $h^1(\mathbf{P}^2, \mathcal{I}_Z(y)) = 0$ for all $y \ge t$, $h^0(\mathbf{P}^2, \mathcal{I}_Z(x)) = 0$ for all $x < t$ and the homogeneous ideal $\mathbf{I}(Z)$ of Z is minimally generated by $(t + 2)(t + 1)/2 - d - x$ forms of degree t and $\max\{0, 2d + 2x - t^2 - 2t\}$ forms of degree $t + 1$.

AMS Subject Classification: 14N05
Key Words: minimal free resolution, zero-dimensional scheme, postulation, fat point, double point

1. Minimal Free Resolution: Garbage Union General Points

Theorem 1. *Let $A \subset \mathbf{P}^2$ be a finite subset in linearly general position, i.e. assume that no three points of A are collinear. Set $d := \sharp(A)$ and $m_d := 2\lceil d/4 \rceil + 1$. Fix an integer x such that $d + x > m_d(m_d + 1)/2$, i.e. assume*

Received: August 3, 2005

that the first integer t such that $d + x \leq (t+2)(t+1)/2$ satisfies $t \geq m_d$. Let $S \subset \mathbf{P}^2$ be a general subset such that $\sharp(S) = x$. Then the minimal free resolution of $Z := A \cup S$ is the expected one, i.e. $h^1(\mathbf{P}^2, \mathcal{I}_Z(y)) = 0$ for all $y \geq t$, $h^0(\mathbf{P}^2, \mathcal{I}_Z(x)) = 0$ for all $x < t$ and the homogeneous ideal $\mathbf{I}(Z)$ of Z is minimally generated by $(t+2)(t+1)/2 - d - x$ forms of degree t and $\max\{0, 2d + 2x - t^2 - 2t\}$ forms of degree $t+1$.

Let $A \subset \mathbf{P}^2$ be a finite subset in linearly general position. Set $d := \sharp(A)$. It is well-known (Castelnuovo) that $h^1(\mathbf{P}^2, \mathcal{I}_A(y)) = 0$ if $y \geq \lfloor (d+1)/2 \rfloor$ and that this vanishing is sharp if and only if A is contained in a smooth conic. This observation explains the assumption "$t \geq m_d$" made in Theorem 1.

Remark 1. Let $B \subset \mathbf{P}^2$ be a finite subset in linearly general position and such that $\sharp(B) \geq 4$. Fix any 4 points of B, say P_1, P_2, P_3, P_4, and call Γ the pencil of all conics containing $\{P_1, P_2, P_3, P_4\}$. Let D be a general member of Γ. Then D is a smooth conic and $D \cap B = \{P_1, P_2, P_3, P_4\}$.

Remark 2. From the dual of the Euler's sequence we get

$$h^0(\mathbf{P}^2, {\Omega^1}_{\mathbf{P}^2}(t+1)) = t(t+2) \text{ for all } t \geq 2,$$

$$h^0(\mathbf{P}^2, {\Omega^1}_{\mathbf{P}^2}(t+1)) = 0 \text{ for all } t \leq 0,$$

$$h^0(\mathbf{P}^2, {\Omega^1}_{\mathbf{P}^2}(2)) = 3, \quad h^1(\mathbf{P}^2, {\Omega^1}_{\mathbf{P}^2}(t+1)) = 0$$

for all $t \neq -1$ and

$$h^1(\mathbf{P}^2, {\Omega^1}_{\mathbf{P}^2}) = 1.$$

Remark 3. Let $B \subset \mathbf{P}^2$ be a zero-dimensional scheme and t a positive integer. We recall that $h^1(\mathbf{P}^2, \mathcal{I}_B \otimes {\Omega^1}_{\mathbf{P}^2}(t+1)) = h^1(\mathbf{P}^2, \mathcal{I}_B(t)) = 0$ if and only if the minimal free resolution of B is in degree at most t (see e.g. [2]).

Remark 4. Let $D \subset \mathbf{P}^n$ be a rational normal curve. Then ${\Omega^1}_{\mathbf{P}^n}|D$ is isomorphic to the direct sum of n line bundles of degree $-n-1$ ([1], Lemma 1.3). Now assume $n = 2$. Hence D is a smooth conic. For all integers $m \geq 2$ the vector bundle ${\Omega^1}_{\mathbf{P}^2}(m)|D$ is isomorphic to the direct sum of 2 line bundles of degree $2m - 3$. Hence $h^0(D, \mathcal{I}_{B,D} \otimes ({\Omega^1}_{\mathbf{P}^2}(m)|D)) = h^1(D, \mathcal{I}_{B,D} \otimes ({\Omega^1}_{\mathbf{P}^2}(m)|D)) = 0$ for every zero-dimensional scheme $B \subset D$ such that $\text{length}(B) = 2m-2$. Notice that $2m - 2$ is even for every integer m.

Proof of Theorem 1. By semicontinuity it is sufficient to prove the theorem for a specific set $S \subset \mathbf{P}^2$. Every subset B of a set in linearly general position is in linearly general position. If $\sharp(B) \geq 4$ we may apply Remark 1 to it. If $\sharp(B) \leq 3$, then we take any $B' \subset \mathbf{P}^2 \backslash B$ such that $\sharp(B') = 4 - \sharp(B)$ and $B \cup B'$ is in linearly general position and then we may apply Remark 1 to $B \cup B'$.

Take any ordering $P_1, \dots, P_d$ of the points of A. If $d \equiv 1,2,3 \pmod 4$, take general $P_z \in \mathbf{P}^2$, $d < z \le 4\lceil d/4 \rceil$. For each integer y such that $1 \le y \le \lceil d/4 \rceil$ let $D_y \subset \mathbf{P}^2$ be a general conic containing $\{P_{4y-3}, P_{4y-2}, P_{4y-1}, P_{4y}\}$. By Remark 1 we have $D_y \cap A = \{P_{4y-3}, P_{4y-2}, P_{4y-1}, P_{4y}\}$. By assumption we have $t \ge 1+2y$. Set $\epsilon_i := 4$ for $1 \le i\lfloor d/4 \rfloor$, $\epsilon_i := d - 4\lfloor d/4 \rfloor$ if $i = \lceil d/4 \rceil$ and $d \equiv 1,2,3 \pmod 4$, $\epsilon_i := 0$ for all $i >\rceil d/4 \rceil$. For all $y+1 \le z \le \lfloor (t-1)/2 \rfloor$ fix a general conic D_z. For $1 \le i \le \lfloor (t-1)/2$ take any $S_i \subset D_i$ such that $\sharp(S_i) = 2t+2-2i-\epsilon_i$ $S_i \cap D_j = \emptyset$ for all $1 \le j \le i-1$, $D_i \cap \{P_1, \dots, P_{4\lceil d/4 \rceil}\} = \{P_{4y-3}, P_{4y-2}, P_{4y-1}, P_{4y}\}$ if $i \le 4\lceil d/4 \rceil$ and $D_i \cap \{P_1, \dots, P_{4\lceil d/4 \rceil}\} = \emptyset$ if $i > 4\lceil d/4 \rceil$. As in [1], Remark 3.1, or [2], it is sufficient to do the cases $x = \lfloor (t^2+2t)/2 \rfloor$ and $x = \lceil (t^2+2t)/2 \rceil$ (here we use Remark 2)). We will do the first one, the other one being either very similar (case t odd) or exactly the case (case t odd). Notice that $\Omega^1{}_{\mathbf{P}^2}(t+3-2i)|D_i$ is isomorphic to the direct sum of 2 line bundles of degree $2t+1-2i$. If t is odd, then take $S := \bigcup_{i=1}^{(t-1)/2} S_i$; we have $\sharp(S) = x$ by our assumption on x. If t is even, then take as S the union of $\bigcup_{i=1}^{(t-2)/2} S_i$ and another general point of $\mathbf{P}^2$. Then follow steps by steps the proofs in [1] (see in particular [1], Remark 3.1).

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Generators for the homogeneous ideal of s general points of $\mathbb{P}^3$, *J. Algebra*, **106**, No. 1 (1987), 46-52.

[2] E. Ballico, A.V. Geramita, The minimal free resolution of the ideal of s general points in $\mathbb{P}^3$, In: *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry*, CMS Conference Proceedings, Canadian Math. Soc. and Amer. Math. Soc, Providence, RI, Volume **6** (1986), 1-11.