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# SOME SUBCATEGORIES OF THE CATEGORY $\mathbf{IRel_R}(H)$

Young Sin Ahn[1], Kul Hur[2 §], Jang Hyun Ryou[3]

[1]Department of Computer Information Science
Dong Kang College
Kwang-Ju, 500-714, KOREA
e-mail: yongsin@dongkang.ac.kr

[2]Division of Mathematics and Informational Statistics
Institute of Basic Natural Science
Wonkwang University
Iksan, Chonbuk, 570-749, KOREA
e-mail: kulhur@wonkwang.ac.kr

[3]Bong Young Girls' Middle School 325-8
Shinjung, 7 Dong, Yangchongu, Seoul, KOREA
e-mail: donggni@hanmail.net

**Abstract:** We introduce the subcategories $\mathbf{IRel_{PR}}(H)$, $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$ of $\mathbf{IRel_R}(H)$ and study their structures in a viewpoint of the topological universe introduce by Nel. In particular, the category $\mathbf{IRel_{PR}}(H)$ (resp. $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$) is a topological universe over **Set**. Moreover, we show that $\mathbf{IRel_E}(H)$ has exponential objects.

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## 1. Introduction

Zadeh [27, 28] introduces the concepts of a fuzzy set and a fuzzy relation as the generalizations of a crisp set and a crisp relation, respectively. In 1983,

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§Correspondence author

Atanassov [1] introduced the notion of an intuitioninstic fuzzy set as the generalization of a fuzzy set and he also investigated many properties of intuitionistic fuzzy sets (cf. [2]). After that time, Çoker [8], Hur and his colleagues [17], and Lee and Lee [24] applied the concept of intuitioninstic fuzzy sets to topology. Also, Hur and his colleagues [16] applied one to topological group. Moreover, Banerjee and Basnet [3], Biswas [5], and Hur and his colleagues [13-15, 18] applied the notion of intuitioninstic fuzzy sets to group theory. Hur [12] investigated categorical structures of $\mathbf{Rel_R}(H)$ consisting of H-fuzzy reflexive relational spaces in the sense of a topological universe, defined by Nel [25]. Also, Hur and his colleagues [19-21] studied categorical structures of the categories $\mathbf{ISet}(H)$, $\mathbf{IRel}(H)$ and $\mathbf{IRel_R}(H)$ in the similar viewpoint.

In this paper, we introduce the subcategories $\mathbf{IRel_{PR}}(H)$, $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$ of $\mathbf{IRel_R}(H)$ and investigate some categorical structures of theses categories in a topological universe viewpoint. Moreover, we show that $\mathbf{IRel_E}(H)$ has exponential objects.

For general background for lattice theory, we refer to [4, 22] and for general categorical background to [10, 11, 23, 25].

## 2. Preliminaries

We introduce some well-known definitions and results which are needed in the later section.

**Definition 1.1.** (see [11]) A category $\mathbf{A}$ is said to be *well-powered* if each $\mathbf{A}$-object has a representative class of subobjects that is a set.

**Dual Notion.** co-(well-powered) (i.e., each object has a representative class of quotient objects which is a set).

**Definition 1.2.** (see [23]) Let $\mathbf{A}$ be a concrete category.

(1) The $\mathbf{A}$-*fibre* of a set $X$ is the class of all $\mathbf{A}$-structures on $X$.

(2) $\mathbf{A}$ is called *properly fibred over* **Set** provided that the following conditions hold:

(i) (Fibre-Smallness) For each set $X$, the $\mathbf{A}$-fibre of $X$ is a set.

(ii) (Terminal Separator Property) For each singleton set $X$, the $\mathbf{A}$-fibre of $X$ has precisely one element.

(iii) If $\xi$ and $\eta$ are $\mathbf{A}$-structures on a set $X$ such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

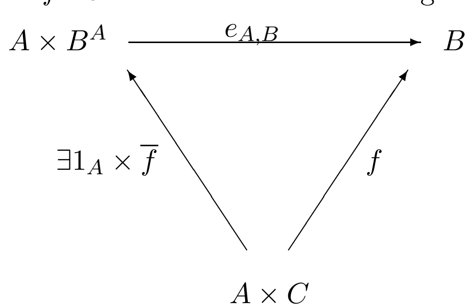
**Result 1.A.** (see [23, Theorem 2.4, 11, Propositions 36.10 and 36.11]) *Let* $\mathbf{A}$ *be a well-powered and co-(well-powered) topological category and let* $\mathbf{B}$ *a subcategory of* $\mathbf{A}$. *Then the following are equivalent:*

*(1)* $\mathbf{B}$ *is epireflective in* $\mathbf{A}$.
*(2)* $\mathbf{B}$ *is closed under the formation of initial monosources.*
*(3)* $\mathbf{B}$ *is closed under the formation of products and pullbacks in* $\mathbf{A}$.

**Result 1.B.** (see [23, Theorem 2.5]) *Let* $\mathbf{A}$ *be a well-powered and co-(well-powered) topological category and let* $\mathbf{B}$ *a subcategory of* $\mathbf{A}$*. Then the following are equivalent:*
*(1)* $\mathbf{B}$ *is bireflective in* $\mathbf{A}$.
*(2)* $\mathbf{B}$ *is closed under the formation of initial sources.*

**Result 1.C.** (see [23, Theorem 2.6]) *If* $\mathbf{A}$ *is a (property fibred, resp.) topological category and* $\mathbf{B}$ *is a bireflective subcategory of* $\mathbf{A}$*, then* $\mathbf{B}$ *is also a (property fibred, resp.) topological category. Moreover, every source in* $\mathbf{B}$ *which is initial in* $\mathbf{A}$ *is initial in* $\mathbf{B}$.

**Definition 1.3.** (see [10]) A category $\mathbf{A}$ is called *Cartesian closed* provided that the following conditions hold:
(1) For any $\mathbf{A}$-objects $A$ and $B$, there exists a product $A \times B$ in $\mathbf{A}$.
(2) Exponential exist in $\mathbf{A}$, i.e., for any $\mathbf{A}$-object $A$, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any $\mathbf{A}$-object $B$, there exists an $\mathbf{A}$-object $B^A$ and a $\mathbf{A}$-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any $\mathbf{A}$-object $C$ and any $\mathbf{A}$-morphism $f : A \times C \to B$, there exists a unique $\mathbf{A}$-morphism $\overline{f} : C \to B^A$ such that the diagram




commutes.

**Definition 1.4.** (see [25]) A category $\mathbf{A}$ is called a *topological universe over* $\mathbf{Set}$ provided that the following conditions hold:
(1) $\mathbf{A}$ is well-structured over $\mathbf{Set}$, i.e., (i) $\mathbf{A}$ is a concrete category; (ii) $\mathbf{A}$ has the fibre-smallness condition; (iii) $\mathbf{A}$ has the terminal separator property.
(2) $\mathbf{A}$ is cotopological over $\mathbf{Set}$.
(3) Final episinks in $\mathbf{A}$ are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any $\mathbf{A}$-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of $f$ and $g_\lambda$ for each $\lambda$, is again a final episink.

Throughout this paper, we use $H$ as a complete Heyting algebra.

**Definition 1.5.** (see [20]) Let $X$ be a set. A pair $R = (\mu_R, \nu_R)$ is called an *intuitionistic H-fuzzy relation* (in shot, *IHFR*) on $X$ if it satisfies the following conditions:

(i) $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ are mappings, where $\mu_R$ and $\nu_R$ denote the *degree of membership* (namely $\mu_R(x, y)$) and the *degree of nonmembership* (namely $\nu_R(x, y)$) of each $(x, y) \in X \times X$ to $R$.

(ii) $\mu_R \leq N(\nu_R)$, i.e., $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$.

In this case, $(X, R)$ or $(X, \mu_R, \nu_R)$ is called an *intuitionistic H-fuzzy relatinal space* (in short, *IHFRS*).

**Definition 1.6.** (see [20]) Let $(X, R_X)$ and $(Y, R_Y)$ be an IHFRSs. A mapping $f : X \to Y$ is called a *relation preserving mapping* if $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$.

From the above definitions, we can form a concrete category $\mathbf{IRel}(H)$ consisting of all relational spaces and relation preserving mappings between them. Every $\mathbf{IRel}(H)$-mapping will be called an $\mathbf{IRel}(H)$*-mapping*.

**Definition 1.7.** (see [21]) An IHFR $R$ on a set $X$ is said to be *reflexive* if $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$ for each $x \in X$.

The class of all intuitionistic H-fuzzy reflexive relational spaces and $\mathbf{IRel}(H)$-mappings between them forms a subcategory of $\mathbf{IRel}(H)$ and denoted by $\mathbf{IRel_R}(H)$.

## 3. Subcategory of $\mathbf{IRel_R(H)}$

We introduce some subcategories of $\mathbf{IRel_R}(H)$ which are topological universes over **Set**.

**Definition 2.1.** Let $R$ be an IHFR on a set $X$. Then $R$ is said to be:

(1) *symmetric* if $\mu_R(x, y) = \mu_R(y, x)$ and $\nu_R(x, y) = \nu_R(y, x)$ for each $x, y \in X$.

(2) *transitive* if $\mu_{R\circ R} \leq \mu_R$ and $\nu_{R\circ R} \geq \nu_R$, where $\mu_{R\circ R}(x, y) = \bigvee_{z\in X}[\mu_R(x, z) \wedge \mu_R(z, y)]$ and $\nu_{R\circ R}(x, y) = \bigwedge_{z\in X}[\nu_R(x, z) \vee \nu_R(z, y)]$ for any $(x, y) \in X \times X$.

(3) an *intuitionistic H-fuzzy proximity relation* if it is reflexive and symmetric.

(4) an *intuitionistic H-fuzzy preorder relation* if it is reflexive and transitive.

(5) an *intuitionistic H-fuzzy equivalence relation* if it is reflexive, symmetric and transitive.

**Notation 2.1.** (1) $\mathbf{IRel_S}(H)$ denotes the full subcategory of $\mathbf{IRel}(H)$ determined by all intuitionistic H-fuzzy symmetric relational spaces.

(2) $\mathbf{IRel_T}(H)$ denotes the full subcategory of $\mathbf{IRel}(H)$ determined by all intuitionistic H-fuzzy transitive relational spaces.

(3) $\mathbf{IRel_{PR}}(H) = \mathbf{IRel_R}(H) \cap \mathbf{IRel_S}(H)$ denotes the full subcategory of $\mathbf{IRel_R}(H)$ determined by all intuitionistic H-fuzzy proximity relational spaces.

(4) $\mathbf{IRel_P}(H) = \mathbf{IRel_R}(H) \cap \mathbf{IRel_T}(H)$ denotes the full subcategory of $\mathbf{IRel_R}(H)$ determined by all intuitionistic H-fuzzy preorder relational spaces.

(5) $\mathbf{IRel_E}(H) = \mathbf{IRel_R}(H) \cap \mathbf{IRel_S}(H) \cap \mathbf{IRel_T}(H)$ denotes the full subcategory of $\mathbf{IRel_R}(H)$ determined by all intuitionistic H-fuzzy equivalence relational spaces.

It is easy to show that the following result holds.

**Proposition 2.2.** *The category* $\mathbf{IRel_{PR}}(H)$ *(resp.* $\mathbf{IRel_P}(H)$ *and* $\mathbf{IRel_E}(H)$*) is properly fibred over* $\mathbf{Set}$.

**Lemma 2.3.** $\mathbf{IRel_{PR}}(H)$ *(resp.* $\mathbf{IRel_P}(H)$ *and* $\mathbf{IRel_E}(H)$*) is closed under the formation of initial sources in* $\mathbf{IRel_R}(H)$.

*Proof.* Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be any initial source in $\mathbf{IRel_R}(H)$, for each $(X_\alpha, R_\alpha)$ belongs to $\mathbf{IRel_{PR}}(H)$ (resp. $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$). Then clearly, by the definition of $R$, $R$ is reflexive and symmetric. Thus it is enough to show that $R$ is transitive. By the process of the proof of Lemma 3.3 in [12], since $\mu_{R\circ R} \le \mu_R$, we will show that $\nu_{R\circ R} \ge \nu_R$. Let $x, y \in X$. Then:

$$\begin{aligned}
\nu_{R\circ R}(x,y) &= \bigwedge_{z\in X} [\nu_R(x,z) \vee \nu_R(z,y)] \\
&= \bigwedge_{z\in X} [(\bigvee_{\Gamma} \nu_{R_\alpha} \circ f_\alpha^2(x,z)) \vee (\bigvee_{\Gamma} \nu_{R_\alpha} \circ f_\alpha^2(z,y))] \\
&= \bigvee_{\Gamma}( \bigwedge_{z\in X} [\nu_{R_\alpha}(f_\alpha(x), f_\alpha(z)) \vee \nu_{R_\alpha}(f_\alpha(z), f_\alpha(y))]) = \bigvee_{\Gamma} \nu_{R_\alpha\circ R_\alpha}(f_\alpha(x), f_\alpha(y)) \\
&\ge \bigvee_{\Gamma} \nu_{R_\alpha}(f_\alpha(x), f_\alpha(y)) = \bigvee_{\Gamma} \nu_{R_\alpha} \circ f_\alpha^2(x,y) = \nu_R(x,y).
\end{aligned}$$

Hence $R$ is transitive. This completes the proof. □

From Result 1.B, Result 1.C and Lemma 2.3, we obtain the following result.

**Theorem 2.4.** $\mathbf{IRel_{PR}}(H)$, $\mathbf{IRel_P}(H)$ *and* $\mathbf{IRel_E}(H)$ *are bireflective subcategories of* $\mathbf{IRel_R}(H)$ *and hence topological categories over* $\mathbf{Set}$.

**Theorem 2.5.** $\mathbf{IRel_{PR}}(H)$, $\mathbf{IRel_P}(H)$ *and* $\mathbf{IRel_E}(H)$ *are closed under the formation of final structures in* $\mathbf{IRel_R}(H)$ *and hence all of them are bicoreflective subcategories of* $\mathbf{IRel_R}(H)$.

*Proof.* Let $(f_\alpha : (X_\alpha, R_\alpha) \to (X, R))_\Gamma$ be any final sink in $\mathbf{IRel_R}(H)$ such that each $(X_\alpha, R_\alpha)$ belongs to $\mathbf{IRel_{PR}}(H)$ (resp. $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$). By the definition of $R$, $R$ is reflexive and symmetric. Thus, it is enough to show that $R$ is transitive. By the process of the proof of Theorem 3.5 in [12], $\mu_{R\circ R} \leq \mu_R$. We will show that $\nu_{R\circ R} \geq \nu_R$. Let $x, y \in X$. Then:

$$\nu_{R\circ R}(x,y) = \bigwedge_{z\in X} [\nu_R(x,z) \vee \nu_R(z,y)]$$
$$= \bigwedge_{z\in X} [(\bigwedge_{\Gamma} \bigwedge_{(x_\alpha,z_\alpha)\in f_\alpha^{-1^2}(x,z)} \nu_{R_\alpha}(x_\alpha, z_\alpha)) \vee (\bigwedge_{\Gamma} \bigwedge_{(z_\alpha,y_\alpha)\in f_\alpha^{-1^2}(z,y)} \nu_{R_\alpha}(z_\alpha, y_\alpha))]$$
$$= (\bigwedge_{\Gamma} \bigwedge_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} ( \bigwedge_{z_\alpha\in f_\alpha^{-1}(z)} [\nu_{R_\alpha}(x_\alpha, z_\alpha) \vee \nu_{R_\alpha}(z_\alpha, y_\alpha)])$$
$$= (\bigwedge_{\Gamma} \bigwedge_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} \nu_{R_\alpha\circ R_\alpha}(x_\alpha, y_\alpha) \geq (\bigwedge_{\Gamma} \bigwedge_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} \nu_{R_\alpha}(x_\alpha, y_\alpha)$$
$$= \nu_R(x,y).$$

Thus $\nu_{R\circ R} \geq \nu_R$. Hence $R$ is an intuitionistic H-fuzzy transitive relation on $X$. This completes the proof. □

By the similar argument as the process of the proof of Theorem 2.10 in [20], we can easily show the following result.

**Lemma 2.6.** *$\mathbf{IRel_S}(H)$ (resp. $\mathbf{IRel_T}(H)$) is closed under the formation of pullbacks in $\mathbf{IRel}(H)$.*

From Theorem 2.6 in [21], Theorem 2.5 and Lemma 2.6, we obtain the following result.

**Theorem 2.7.** *Final episinks in $\mathbf{IRel_{PR}}(H)$ (resp. $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$) are preserved by pullbacks.*

By Theorem 2.4 and Theorem 2.7, we obtain the following result.

**Theorem 2.8.** *$\mathbf{IRel_{PR}}(H)$ (resp. $\mathbf{IRel_P}(H)$ and $\mathbf{IRel_E}(H)$) is topological universe over $\mathbf{Set}$. Hence each category is a concrete quasitopos in the sense of E.J. Dubuc [5].*

**Lemma 2.9.** *$\mathbf{IRel_{PR}}(H)$ has exponential objects. Hence $\mathbf{IRel_{PR}}(H)$ is Cartesian closed over $\mathbf{Set}$.*

*Proof.* For any $\mathbf{X} = (X, R_X)$, $\mathbf{Y} = (Y, R_Y) \in \mathbf{IRel_{PR}}(H)$, let $R$ be the intuitionistic H-fuzzy reflexive relation on $Y^X = \hom_{\mathbf{IRel_R}(H)}(X, Y)$ defined in the process of the proof of Theorem 2.8 in [21]. Let $f, g \in Y^X$. Since $R_X$ and $R_Y$ are symmetric, $D(f,g) = D(g,f)$ and $E(f,g) = E(g,f)$. Thus, by the definition of $R$, $\mu_R(f,g) = \mu_R(g,f)$ and $\nu_R(f,g) = \nu_R(g,f)$. Hence $R$ is an intuitionistic H-fuzzy proximity relation on $Y^X$. □

**Theorem 2.10.** $\mathbf{IRel_E}(H)$ *is Cartesian closed over* **Set**.

*Proof.* For any $\mathbf{X} = (X, R_X)$, $\mathbf{Y} = (Y, R_Y) \in \mathbf{IRel_E}(H)$, let $R$ be the intuitionistic H-fuzzy reflexive relation on $Y^X = \hom_{\mathbf{IRel_R}(H)}(X,Y)$ defined in the process of the proof of Theorem 2.8 in [21]. Since $\mathbf{IRel_E}(H)$ is a full isomorphism closed subcategory of $\mathbf{IRel_R}(H)$ and $R$ is symmetric by Lemma 2.9, it is sufficient to show that $R$ is transitive, i.e., for any $f, h \in Y^X$

$$\mu_R(f,g) \wedge \mu_R(g,h) \leq \mu_R(f,h) \text{ for each } g \in Y^X, \qquad (*)$$

and

$$\nu_R(f,g) \vee \nu_R(g,h) \geq \nu_R(f,h) \text{ for each } g \in Y^X. \qquad (**)$$

Let $f, g, h \in Y^X$. We consider the four cases:

(i) $D(f,g) = \emptyset, D(g,h) = \emptyset$; $E(f,g) = \emptyset, E(g,h) = \emptyset$.
(ii) $D(f,g) \neq \emptyset, D(g,h) = \emptyset$; $E(f,g) \neq \emptyset, E(g,h) = \emptyset$.
(iii) $D(f,g) = \emptyset, D(g,h) \neq \emptyset$; $E(f,g) = \emptyset, E(g,h) \neq \emptyset$.
(iv) $D(f,g) \neq \emptyset, D(g,h) \neq \emptyset$; $E(f,g) \neq \emptyset, E(g,h) \neq \emptyset$.

*Case (i).* Suppose $D(f,g) = \emptyset, D(g,h) = \emptyset$; $E(f,g) = \emptyset$ and $E(g,h) = \emptyset$. Let $(x,y) \in X \times X$. Since $R_X$ is symmetric and $g \in Y^X$,

$$\mu_{R_X}(x,y) = \mu_{R_X}(y,x) \leq \mu_{R_Y}(g(y), g(x))$$

and

$$\nu_{R_X}(x,y) = \nu_{R_X}(y,x) \geq \nu_{R_Y}(g(y), g(x)).$$

Since $D(f,g) = \emptyset$ and $R_Y$ is transitive,

$$\mu_{R_X}(x,y) \leq \mu_{R_Y}(f(x), g(y)) \wedge \mu_{R_Y}(g(y), g(x)) \leq \mu_{R_Y}(f(x), g(x)).$$

Since $D(g,h) = \emptyset$ and $R_Y$ is transitive,

$$\mu_{R_X}(x,y) \leq \mu_{R_Y}(f(x), g(x)) \wedge \mu_{R_Y}(g(x), h(y)) \leq \mu_{R_Y}(f(x), h(y)).$$

Thus $D(f,h) = \emptyset$. So, by the definition of $R$, $\mu_R(f,g) \wedge \mu_R(g,h) = 1 = \mu_R(f,h)$.

On the other hand, since $E(f,g) = \emptyset$ and $R_Y$ is transitive,

$$\nu_{R_X}(x,y) \geq \nu_{R_Y}(f(x), g(y)) \vee \nu_{R_Y}(g(y), g(x)) \geq \mu_{R_Y}(f(x), g(x)).$$

Since $E(g,h) = \emptyset$ and $R_Y$ is transitive,

$$\nu_{R_X}(x,y) \geq \mu_{R_Y}(f(x), g(x)) \vee \nu_{R_Y}(g(x), h(y)) \geq \nu_{R_Y}(f(x), h(y)).$$

Thus $E(f,h) = \emptyset$. So, by the definition of $R$, $\nu_R(f,g) \vee \nu_R(g,h) = 0 = \nu_R(f,h)$.

*Case (ii).* Suppose $D(f,g) \neq \emptyset, D(g,h) = \emptyset$; $E(f,g) \neq \emptyset$ and $E(g,h) = \emptyset$. Then $D(f,h) \subset D(f,g)$ and $E(f,h) \subset E(f,g)$ : Let $(a,b) \notin D(f,g)$. Then

$\mu_{R_X}(a,b) \le \mu_{R_Y}(f(a),g(b))$.
Since $g \in Y^X$ and $R_Y$ is symmetric,
$\mu_{R_X}(a,b) \le \mu_{R_Y}(g(a),g(b)) = \mu_{R_Y}(g(b),g(a))$.
Since $R_Y$ is transitive,
$\mu_{R_X}(a,b) \le \mu_{R_Y}(f(a),g(b)) \wedge \mu_{R_Y}(g(b),g(a)) \le \mu_{R_Y}(f(a),g(a))$.
Since $D(g,h) = \emptyset$,
$\mu_{R_X}(a,b) \le \mu_{R_Y}(g(a),h(b))$.
Since $R_Y$ is transitive,
$\mu_{R_X}(a,b) \le \mu_{R_Y}(f(a),g(b)) \wedge \mu_{R_Y}(g(a),h(b)) \le \mu_{R_Y}(f(a),h(b))$.
Thus $(a,b) \notin D(f,h)$. So $D(f,h) \subset D(f,g)$. By the similar argument as the above proof, we can see that $E(f,h) \subset E(f,g)$.

Let $(x,z) \in D(f,g)$. Then $\mu_{R_X}(x,y) > \mu_{R_Y}(f(x),g(y))$.
Since $D(g,h) = \emptyset$, $\mu_{R_X}(x,y) \le \mu_{R_Y}(g(x),h(y))$.
Since $g \in Y^X$ and $R_Y$ is symmetric and transitive,

$$\begin{aligned}\mu_{R_X}(x,y) &\le \mu_{R_Y}(g(x),g(y)) \wedge \mu_{R_Y}(g(x),h(y)) \\ &= \mu_{R_Y}(g(y),g(x)) \wedge \mu_{R_Y}(g(x),h(y)) \le \mu_{R_Y}(g(y),h(y)).\end{aligned}$$

Thus $\mu_{R_Y}(f(x),g(y)) < \mu_{R_X}(x,y) \le \mu_{R_Y}(g(y),h(y))$. So

$$\begin{aligned}\mu_{R_Y}(f(x),g(y)) &= \mu_{R_Y}(f(x),g(y)) \wedge \mu_{R_Y}(g(y),h(y)) \\ &\le \mu_{R_Y}(f(x),h(y)) \text{ for each } (x,y) \in D(f,g).\end{aligned}$$

Since $D(f,h) \subset D(f,g)$,

$$\begin{aligned}&\mu_R(f,g) \wedge \mu_R(g,h) \\ &\quad = [\bigwedge_{(x,y)\in D(f,g)} \mu_{R_Y}(f(x),g(y))] \wedge 1 = \bigwedge_{(x,y)\in D(f,g)} \mu_{R_Y}(f(x),g(y)) \\ &\quad \le \bigwedge_{(x,y)\in D(f,g)} \mu_{R_Y}(f(x),h(y)) \le \bigwedge_{(x,y)\in D(f,h)} \mu_{R_Y}(f(x),h(y)) = \mu_R(f,h).\end{aligned}$$

Now let $(x,y) \in E(f,g)$. Then $\nu_R(x,y) < \nu_{R_Y}(f(x),g(y))$. Since $E(g,h) = \emptyset$, $\nu_{R_X}(x,y) \ge \nu_{R_Y}(g(x),h(y))$. Since $g \in Y^X$ and $R_Y$ is symmetric and transitive,

$$\begin{aligned}\nu_{R_X}(x,y) &\ge \mu_{R_Y}(g(x),g(y)) \vee \nu_{R_Y}(g(x),h(y)) \\ &= \nu_{R_Y}(g(y),g(x)) \vee \nu_{R_Y}(g(x),h(y)) \ge \nu_{R_Y}(g(y),h(y)).\end{aligned}$$

Thus $\nu_{R_Y}(f(x),g(y)) > \nu_R(x,y) \ge \nu_{R_Y}(g(y),h(y))$. So

$$\nu_{R_Y}(f(x),g(y)) = \nu_{R_Y}(f(x),g(y)) \vee \nu_{R_Y}(g(y),h(y)) \geq \nu_{R_Y}(f(x),h(y)) \text{ for each } (x,y) \in E(f,g).$$

Since $E(f,h) \subset E(f,g)$,

$$\begin{aligned}\nu_R(f,g) \vee \nu_R(g,h) &= [\bigvee_{(x,y)\in E(f,g)} \nu_{R_Y}(f(x),g(y))] \vee 0 \\ &= \bigvee_{(x,y)\in E(f,g)} \nu_{R_Y}(f(x),g(y)) \\ &\geq \bigvee_{(x,y)\in E(f,g)} \nu_{R_Y}(f(x),h(y)) \geq \bigvee_{(x,y)\in E(f,h)} \nu_{R_Y}(f(x),h(y)) = \nu_R(f,h).\end{aligned}$$

*Case (iii).* Suppose $D(f,g) = \emptyset, D(g,h) \neq \emptyset;\ E(f,g) = \emptyset$ and $E(g,h) \neq \emptyset$. Then it is dual of Case (ii).

*Case (iv).* Suppose $D(f,g) \neq \emptyset, D(g,h) \neq \emptyset;\ E(f,g) \neq \emptyset$ and $E(g,h) \neq \emptyset$. Then we can show that $D(f,h) \subset D(f,g) \cup D(g,h)$ and $E(f,h) \subset E(f,g) \cup E(g,h)$ by the similar argument as in the proof in Case (ii).

Suppose $(a,b) \in D(f,g) \cap D(g,h)$. Then, by the definition of $R$,

$$\mu_R(f,g) \wedge \mu_R(g,h) \leq \mu_{R_Y}(f(a),g(b))) \wedge \mu_{R_Y}(g(a),h(b)).$$

Suppose $(a,b) \in D(f,g) - D(g,h)$. Then,

$$\mu_{R_Y}(f(a),g(b)) < \mu_{R_X}(a,b) \leq \mu_{R_Y}(g(a),h(b)).$$

Thus, by the definition of $R$,

$$\begin{aligned}\mu_R(f,g) \wedge \mu_R(g,h) &= [\bigwedge_{(x,y)\in D(f,g)} \mu_{R_Y}(f(x),g(y))] \leq \mu_{R_Y}(f(a),g(b)) \\ &= \mu_{R_Y}(f(a),g(b)) \wedge \mu_{R_Y}(g(a),h(b)).\end{aligned}$$

Suppose $(a,b) \in D(g,h) - D(f,g)$. Then, by the similar way,

$$\mu_R(f,g) \wedge \mu_R(g,h) \leq \mu_{R_Y}(f(a),g(b)) \wedge \mu_{R_Y}(g(a),h(b))$$

In all, for any $(a,b) \in D(f,g) \cup D(g,h)$.

$$\mu_R(f,g) \wedge \mu_R(g,h) \leq \mu_{R_Y}(f(x),g(y)) \wedge \mu_{R_Y}(g(x),h(y)).$$

Now let $(a,b) \in D(f,g) \cup D(g,h)$. Then:

$$\mu_{R_Y}(f(x),g(y)) < \mu_{R_X}(x,y) \text{ or } \mu_{R_Y}(g(x),h(y)) < \mu_{R_X}(x,y).$$

Since $g \in Y^X$ and $R_Y$ is symmetric and transitive,

$$\begin{aligned}&\mu_{R_Y}(f(x),g(x)) \wedge \mu_{R_Y}(g(x),h(y)) \\ &\qquad = \mu_{R_Y}(f(x),g(y)) \wedge \mu_{R_Y}(g(x),h(y)) \wedge \mu_{R_X}(x,y) \\ &\qquad \leq \mu_{R_Y}(f(x),g(x)) \wedge \mu_{R_Y}(g(x),h(y)) \wedge \mu_{R_Y}(g(x),g(y))\end{aligned}$$

$$= \mu_{R_Y}(f(x), g(x)) \wedge \mu_{R_Y}(g(y), g(x)) \wedge \mu_{R_Y}(g(x), h(y))$$
$$\leq \mu_{R_Y}(f(x), g(y)) \wedge \mu_{R_Y}(g(y), h(y)) \leq \mu_{R_Y}(f(x), h(y)).$$

Thus

$$\mu_R(f,g) \wedge \mu_R(g,h) \leq \bigwedge_{(x,y)\in D(f,g)\cup D(g,h)} \mu_{R_Y}(f(x), h(y))$$
$$\leq \bigwedge_{(x,y)\in D(f,h)} \mu_{R_Y}(f(x), h(y)) = \mu_{R_Y}(f,h).$$

By the similar argument, we can see that:

$$\nu_R(f,g) \vee \nu_R(g,h) \geq \nu_R(f,h).$$

In either case, (*) and (**) hold. Hence $R$ is transitive. This completes the proof. □

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