International Journal of Pure and Applied Mathematics
Volume 23 No. 4 2005, 437-450

THE GROWTH RATE OF SOME DEFICIENCY ZERO KNOT CLASSES

Yuanan Diao[1 §], Claus Ernst[2]

[1]Department of Mathematics and Statistics
University of North Carolina at Charlotte
Charlotte, NC 28223, USA
e-mail: ydiao@uncc.edu

[2]Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101, USA
e-mail: claus.ernst@wku.edu

Abstract: The deficiency of a link K is defined as $d(K) = Cr(K) - 2g(K) - b(K) - \mu(K) + 2$, where $Cr(K)$ is the crossing number of K, $g(K)$ is the genus of K, $b(K)$ is the braid index of K and $\mu(K)$ is the number of components of K. It is known that $d(K) \geq 0$ for any K hence $Cr(K)$ is bounded below by $2g(K) + b(K) + \mu(K) - 2$. It is known that the crossing numbers of deficiency zero links are additive under the connected sum operation. A main result in this paper is a proof that shows the number of deficiency zero links grows exponentially with respect to the crossing numbers of the links. Another part of the paper concerns the problem of whether the braid index and genus of a link can be combined to give an upper bound of the crossing number of the link. We conjecture that for any link K, we have $Cr(K) \leq 2(b(K) + g(K) + \mu(K))$. Using the help of the HOMFLY polynomial, we have verified this inequality for links up to 12 crossings. On the other hand, we also prove that this inequality is indeed true for all two-bridge links. In fact, we prove that if K is a two-bridge link, then $Cr(K) \leq 2g(K) + 2b(K) + \mu(K) - 4$, which is a stronger result than the conjecture.

AMS Subject Classification: 57M25
Key Words: knots, links, crossing numbers, two-bridge knots and links, braid index, genus of a knot

Received: April 28, 2005

[§]Correspondence author

1. Introduction

In this paper, we investigate the relationships among the braid index, the genus and the crossing number of a link. These are well known and thoroughly studied invariants in knot theory and numerous literature exist on these topics.

A well known open question in knot theory is whether the crossing number of a link is additive under the connected sum operation. The answer to this problem is affirmative if we restrict ourselves to the alternating links [4, 6, 9]. On the other hand, it has been shown in [2] that there exist a family of links (called links of deficiency zero) that include both alternating and non-alternating links whose crossing numbers are additive under the connected sum operations within this family. More specifically, a link K is called a link of *deficiency zero* if $d(K) = Cr(K) + 2 - b(K) - 2g(K) - \mu(K) = 0$, where $Cr(K)$ the crossing number of K, $b(K)$ is the braid index of K, $g(K)$ is the genus of K and $\mu(K)$ is the number of components in K. It is known that the torus links and many alternating links are of deficiency zero so there are infinitely many such links. In this paper, we will prove that the number of deficiency zero links actually grows exponentially with respect to the crossing number of the links.

Another part of this paper is devoted to exploring further the relationships among the crossing number, the braid index and the genus of a link. It is well known that $d(K) = Cr(K) + 2 - b(K) - 2g(K) - \mu(K) \geq 0$ for any link K. So in particular we have $Cr(K) \geq 2g(K)$ and $Cr(K) \geq b(K) - 1$. In fact a stronger result due to Ohyama states that $Cr(K) \geq 2(b(K) - 1)$ for any link K [8]. In other word, both the genus and braid index give lower bound of the crossing number. In the second part of this paper, we are interested in the question whether the braid index or the genus may be used to obtain an upper bound of the crossing number. Since there are links whose braid index (or genus) stay constant while the crossing number can grow to infinity, any functions of the braid index or genus of a link alone will not provide an upper bound. Therefore, we will need to look at a combination of the braid index and the genus. The evidence is actually plausible. For all the known knots and links (of a given crossing number), a small genus is usually accompanied by a large braid index and vice versa. We have observed that the inequality $Cr(K) \leq 2(b(K) + g(K) + \mu(K))$ holds for all prime knots and all alternating links up to 12 crossings. For the two-bridge links, we can actually prove that this inequality is generally true.

The paper is arranged in the following way. In the next section, we will have a brief review of some important results regarding two-bridge links. In Section 3, we will prove some results which relate the various continuous fraction decompositions of a two-bridge link to its genus and crossing number, in particular, we will derive an explicit formula that expresses the deficiency of a two-bridge link as a function of the coefficients obtained from a special continued fraction decomposition using the two-bridge link parameters α, β. We will then use this result in Section 4 to prove that the number of deficiency zero two-bridge links grows exponentially with respect to the crossing number of the two-bridge links. In Section 5, we show that the inequality $Cr(K) \leq 2(b(K)+g(K)+\mu(K))$ holds for all two-bridge links. In fact, we have proven a slightly stronger result, that is, $Cr(K) \leq 2b(K) + 2g(K) + \mu(K) - 4$ for all non-trivial two-bridge links (notice that this stronger inequality cannot be extended to include some other knots, for instance for the trivial knot the right hand side of the inequality becomes negative.) We also discuss how we verified that the inequality $Cr(K) \leq 2(b(K) + g(K) + \mu(K))$ holds for all prime knots and all alternating links of up to 12 crossings.

2. Preliminary Concepts and Lemmas

In this section, we will state some well known results about the crossing number, genus and braid index of two-bridge links. We will also add a new result at the end of the section. Let $K = b(\alpha, \beta)$ be a two-bridge link, where $0 < \beta < \alpha$ and α, β are co-prime. A vector $(a_1, a_2, ..., a_n)$ is called a continued fraction decomposition of $\frac{\beta}{\alpha}$ if there exist integers β_1, β_2, ... , β_{n-2} such that

$$0 < |\beta_{n-2}| < |\beta_{n-3}| < \cdots < |\beta_1| < |\beta|$$

and

$$\begin{aligned}
\alpha &= a_1\beta + \beta_1, \\
\beta &= a_2\beta_1 + \beta_2, \\
\beta_1 &= a_3\beta_2 + \beta_3, \\
&\cdots\cdots \\
\beta_{n-3} &= a_{n-1}\beta_{n-2} + 1, \\
\beta_{n-2} &= a_n \cdot 1.
\end{aligned}$$

For the sake of convenience, we will simply call a continued fraction decomposition of $\frac{\beta}{\alpha}$ a *decomposition* in the rest of this paper. A decomposition $(a_1, a_2, ..., a_n)$ of $\frac{\beta}{\alpha}$ is called a *positive decomposition* if $a_j > 0$ for each j. It is well-known that if $(a_1, a_2, ..., a_n)$ is a positive decomposition of $\frac{\beta}{\alpha}$, then the crossing number of $K = b(\alpha, \beta)$ is $Cr(K) = \sum_{j=1}^{n} a_j$. If the decomposition is of the form

$$(a_1, 2b_2, a_2, 2b_2, ..., a_{n-1}, 2b_{n-1}, a_n),$$

then it is called a *half-even decomposition* of $\frac{\beta}{\alpha}$. Finally, if all entries in the decomposition vector is even, then the decomposition is called an *even decomposition* of $\frac{\beta}{\alpha}$.

Observation About Even and Half-Even Decompositions: 1. It is easy to see that if α and β are both odd, then $\frac{\beta}{\alpha}$ does not admit an even decomposition. On the other hand, if β is even (then α has to be odd since α and β are co-prime), then $\frac{\beta}{\alpha}$ does not admit a half-even decomposition.

2. Each of the positive, half-even and even decompositions (when it exists) is uniquely determined by α and β.

In the following we want to relate the crossing number, the genus and the braid index of 4-plats. It is a well known result that the crossing number of $K = b(\alpha, \beta)$ equals the sum of the entries in the positive decomposition of $\frac{\beta}{\alpha}$. The genus of a 4-plat $K = b(\alpha, \beta)$ can be expressed explicitly using the half-even decomposition of $\frac{\beta}{\alpha}$ provided that β is odd (see Lemma 1). Furthermore, if β is odd, then the braid index of $K = b(\alpha, \beta)$ can also be expressed explicitly using the even decomposition of $\frac{\alpha-\beta}{\alpha}$ (see Lemma 2). The genus and the braid index are invariants of oriented knots and links and they also stay the same when we pass from a knot or a link to its mirror image. Therefore, substituting a knot or a link with its mirror image will not affect our results in this paper. The number of half-even and even decompositions for a given 4-plat will become important to us later, so it is helpful to summarize some basic facts about 4-plats below, see [1] for more details.

Knots. If $b(\alpha, \beta)$ is a knot then α is odd and β may be even or odd. Since two-bridge knots are invertible, the orientation of a two-bridge knot does not matter and we can use the classification of un-oriented two-bridge knots. Two 4-plats $b(\alpha, \beta)$ and $b(\alpha', \beta')$ are equivalent if $\alpha = \alpha'$ and $\beta\beta' \equiv 1 \bmod (\alpha)$. The knot $b(\alpha, \alpha - \beta)$ is the mirror image of $b(\alpha, \beta)$. Since we require that $0 < \beta, \beta' < \alpha$, there can be at most one such β' so that $\beta\beta' \equiv 1 \bmod (\alpha)$. Furthermore, if $\beta\beta' \equiv 1 \bmod (\alpha)$, then obviously $(\alpha - \beta)(\alpha - \beta') \equiv 1 \bmod (\alpha)$ as well. So $b(\alpha, \beta) = b(\alpha, \beta')$ implies that $b(\alpha, \alpha - \beta) = b(\alpha, \alpha - \beta')$. Therefore, for a 4-plat knot and its mirror image, there are at most four pairs of numbers to consider: (α, β), (α, β'), $(\alpha, \alpha - \beta)$ and $(\alpha, \alpha - \beta')$.

Notice that exactly two of $\beta, \beta', \alpha - \beta$ and $\alpha - \beta'$ are odd so there are most two even decompositions for a pair consisting of a 4-plat knot and its mirror image (it is possible that the knot is equal to its mirror image in which case

$\beta = \alpha - \beta'$. It is also possible that $\beta = \beta'$. In both cases there are only two pairs of numbers to consider hence there is only one even decomposition).

Links. If $b(\alpha, \beta)$ is a link then α is even and β is odd. For a two-bridge link $b(\alpha, \beta)$ changing the orientation of one of the two components will, in general, result in a different link, which is $b(\alpha, \alpha - \beta)$. The *standard orientation* of a link $b(\alpha, \beta)$ is defined in such a way so that the two middle strings of the 4-plat are parallel (the two middle strings always belong to the two different components). In the following we will assume that all 4-plat links come with the standard orientation. Two oriented 4-plat links $b(\alpha, \beta)$ and $b(\alpha', \beta')$ are equivalent if $\alpha = \alpha'$ and $\beta\beta' \equiv 1 \bmod (2\alpha)$. The knot $b(\alpha, 2\alpha - \beta)$ is the mirror image of $b(\alpha, \beta)$. Similar to the case of a knot, there can be at most one β' satisfying the conditions $0 < \beta' < \alpha$ and $\beta\beta' \equiv 1 \bmod (2\alpha)$. Thus, for a 4-plat link and its mirror image, there are at most eight pairs of numbers to consider: (α, β), (α, β'), $(\alpha, 2\alpha - \beta)$, $(\alpha, 2\alpha - \beta')$, $(\alpha, \alpha - \beta)$, $(\alpha, \alpha - \beta')$, $(\alpha, \alpha + \beta)$ and $(\alpha, \alpha + \beta')$.

If the eight numbers $\beta, \beta', \alpha - \beta, 2\alpha - \beta, 2\alpha - \beta', \alpha + \beta, \alpha + \beta'$ are distinct then there are eight even decompositions belonging to 4 non-equivalent oriented 4-plats. If some of these eight numbers are equal then there will be fewer than 4 non-equivalent oriented 4-plats. In all such cases there are at most two even decompositions for a given oriented 4-plat.

The genus of K can be calculated by the formula given in the following lemma.

Lemma 1. (see [1]) *Let μ be the number of components in a two-bridge knot or link $K = b(\alpha, \beta)$ such that β is odd and $0 < \beta < \alpha$. If*

$$(a_1, 2b_2, a_2, 2b_2, ..., a_{n-1}, 2b_{n-1}, a_n)$$

is the half-even decomposition of $\frac{\beta}{\alpha}$, then the genus $g(K)$ of K is given by

$$g(K) = \frac{1}{2}\big(\sum_{j=1}^{n} |a_j| - \mu\big).$$

Given a vector $v = (b_1, b_2, ..., b_k)$ of integer entries with the property that $b_j \neq 0$, let us define $t(v)$ to be the number of sign changes in the vector v and let $t'(v)$ be the number of sign changes in the vector $(b_1, -b_2, b_3, -b_4, ..., (-1)^{k+1} b_k)$. For instance, if $v = (2, -1, -1, -3, -4, -5, -2)$, then $t(v) = 1$ and $t'(v) = 5$. Using induction, one can easily show that $t(v) + t'(v) = k - 1$ for any vector v of k entries with the given property. We leave the proof of this fact as an exercise to our reader. The braid index of $K = b(\alpha, \beta)$ can be calculated using the formula given in the following lemma.

Lemma 2. (see [7]) *Let $K = b(\alpha, \beta)$ be a two-bridge link such that $0 < \beta < \alpha$ and β is odd. Let $v = (2n_1, 2n_2, ..., 2n_k)$ be the even continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$, then the braid index $b(K)$ of K is given by*

$$b(K) = \sum_{j=1}^{k} |n_j| - k + t'(v) + 2.$$

Notice that $b(K)$ can also be written as

$$b(K) = \sum_{j=1}^{k} |n_j| - t(v) + 1$$

since $t(v) + t'(v) = k - 1$.

Example. Let $K = b(22, 5)$. The even continued fraction decomposition of $\frac{17}{22}$ is $v = (2, -2, 2, -4, 2)$. We have $t(v) = 4$. Thus $b(K) = 6 - 4 + 1 = 3$. On the other hand, the half-even decomposition of $\frac{5}{22}$ is $(4, 2, 2)$ which is also the positive decomposition of $\frac{5}{22}$, therefore, we have $Cr(K) = 8$ and $g(K) = \frac{1}{2}(6 - 2) = 2$.

It is a well known result that the crossing number of $K = b(\alpha, \beta)$ equals the sum of the entries in the positive decomposition of $\frac{\beta}{\alpha}$. However, this result does not help us in this paper since our gaol is to relate the crossing number of K to its genus and braid index. The following lemma will serve that purpose.

Lemma 3. *Let $K = b(\alpha, \beta)$ be a 2-bridge link such that $0 < \beta < \alpha$ and β is odd. Let $v = (2n_1, 2n_2, ..., 2n_k)$ be the even decomposition of $\frac{\alpha-\beta}{\alpha}$, then the crossing number $Cr(K)$ of K is given by*

$$Cr(K) = 2\sum_{j=1}^{k} |n_j| - k + t'(v) + 1 = 2\sum_{j=1}^{k} |n_j| - t(v).$$

Proof. We will prove the lemma by induction on k. If $k = 1$, then $v = (2n_1)$ is the positive decomposition of $\frac{\alpha-\beta}{\alpha}$. So the crossing number of $b(\alpha, \alpha - \beta)$ is $2n_1 = 2n_1 - 1 + 1$ since $t'(v) = 0$ in this case. Moreover, $\alpha = 2n_1$, $\beta = 2n_1 - 1$ and $b(\alpha, \alpha - \beta) = b(2n_1, 1)$ is equivalent to $b(\alpha, \beta) = b(2n_1, 2n_1 - 1)$. Therefore, we have $Cr(K) = 2n_1$.

Now assume the lemma holds for $k \geq 1$. Let us consider the case of $k+1$, that is, the even decomposition of $\frac{\alpha-\beta}{\alpha}$ is of the form $v = (2n_1, 2n_2, ..., 2n_k, 2n_{k+1})$. Let $v' = (2n_2, ..., 2n_k, 2n_{k+1})$. There are two subcases here.

Subcase 1. $n_2 > 0$. Since $n_1 > 0$, the number of sign changes in $v = (2n_1, 2n_2, ..., 2n_k, 2n_{k+1})$ is the same as that of v'. That is, $t(v) = t(v')$. It follows that $t'(v) = t'(v') + 1$. Let $0 < \beta' < \alpha'$ be such that β' is odd and $v' = (2n_2, ..., 2n_k, 2n_{k+1})$ is the even decomposition of $\frac{\alpha'-\beta'}{\alpha'}$. By the induction assumption, the crossing number of $b(\alpha', \beta')$ (and $b(\alpha', \alpha' - \beta')$) is given by $2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1$. On the other hand, we have (by the definition of continued

$$\frac{\alpha-\beta}{\alpha} = \frac{1}{2n_1 + \frac{\alpha'-\beta'}{\alpha'}} = \frac{\alpha'}{2n_1\alpha' + (\alpha' - \beta')}.$$

It follows that $\alpha = 2n_1\alpha' + (\alpha' - \beta')$, $\alpha - \beta = \alpha'$. Let $(a_1, a_2, ..., a_m)$ be the positive decomposition of $\frac{\alpha'-\beta'}{\alpha'}$, then we have

$$\sum_{j=1}^{m} a_j = 2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1.$$

Furthermore, the positive continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$ is:

$$\begin{aligned} \alpha &= 2n_1\alpha' + (\alpha' - \beta') = 2n_1(\alpha - \beta) + (\alpha' - \beta'), \\ \alpha - \beta &= \alpha' = a_1(\alpha' - \beta') + \cdots . \end{aligned}$$

In other word, the positive decomposition of $\frac{\alpha-\beta}{\alpha}$ is simply $(2n_1, a_1, ..., a_m)$. Therefore, the crossing number of $K = b(\alpha, \alpha - \beta)$ and of $K = b(\alpha, \beta)$ is given by

$$2n_1 + \sum_{j=1}^{m} a_j = 2n_1 + 2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1 = 2\sum_{j=1}^{k+1} |n_j| - (k+1) + t'(v) + 1.$$

This finishes the proof of Subcase 1.

Subcase 2. $n_2 < 0$. In this case, we have $t(v) = t(v') + 1$. It follows that $t'(v) = t'(v')$. Let $0 < \beta' < \alpha'$ be such that β' is odd and $-v' = (-2n_2, ..., -2n_k, -2n_{k+1})$ is the even decomposition of $\frac{\alpha'-\beta'}{\alpha'}$. Note that $t'(v') = t'(-v')$. By the induction assumption, the crossing number of $b(\alpha', \beta')$ (and $b(\alpha', \alpha' - \beta')$) is given by $2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1$. On the other hand, we have:

$$\frac{\alpha-\beta}{\alpha} = \frac{1}{2n_1 - \frac{\alpha'-\beta'}{\alpha'}} = \frac{\alpha'}{2n_1\alpha' - \alpha' + \beta'}.$$

It follows that $\alpha = (2n_1-1)\alpha' + \beta'$, $\alpha - \beta = (2n_1-2)\alpha' + \beta'$. Let $(b_1, b_2, ..., b_p)$ be the positive decomposition of $\frac{\beta'}{\alpha'}$, then we have

$$\sum_{j=1}^{p} b_j = 2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1$$

since $\sum_{j=1}^{p} b_j$ is the crossing number of $b(\alpha', \alpha'-\beta')$ and $b(\alpha', \beta')$. Furthermore, the positive continued fraction decomposition of $\frac{\beta}{\alpha}$ in the case of $n_1 > 1$ is:

$$\begin{aligned} \alpha &= 1 \cdot \big((2n_1-2)\alpha' + \beta'\big) + \alpha' = 1 \cdot \beta + \alpha', \\ \beta &= (2n_1-2)\alpha' + \beta', \\ \alpha' &= b_1\beta' + \cdots . \end{aligned}$$

In other word, the positive decomposition of $\frac{\beta}{\alpha}$ is $(1, 2n_1-2, b_1, ..., b_p)$. In the case of $n_1 = 1$ the positive decomposition of $\frac{\beta}{\alpha}$ is $(1, b_1, ..., b_p)$. Therefore, in both cases the crossing number of $K = b(\alpha, \alpha-\beta)$and of $K = b(\alpha, \beta)$ is given by

$$2n_1 - 1 + \sum_{j=1}^{p} b_j = 2n_1 - 1 + 2\sum_{j=2}^{k+1} |n_j| - k + t'(v') + 1 = 2\sum_{j=1}^{k+1} |n_j| - (k+1) + t'(v) + 1.$$

This finishes the proof of Subcase 2 and completes the proof of the lemma. □

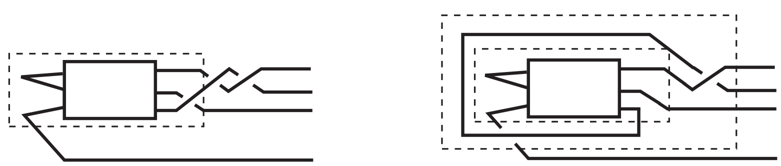

Figure 1: The figure on the left shows the situation before a move is made. The diagram in the dashed box is assumed to be alternating. The figure on the right shows the diagram after the move. The total number of crossings has decreased by one. The smaller dashed box will no be replaced by the bigger one. The diagram in the bigger box is again alternating. Using such moves one can change the diagram of the 4-plat by moving form left to right through the plat. The resulting alternating diagram does not have the structure of a 4-plat any longer

Note that the above Lemma can also be shown by changing the 4-plat diagram based on the even decomposition into an alternating diagram using a move as illustrated in Figure 1 for each change of sign. Each of the $k - t'(v) - 1$ necessary moves decreases the crossing number by one. The advantage of the proof of Lemma 3 is that the technique used in the proof can be used for the proof of the theorems in the next section.

3. Relating the Crossing Number to Genus and Braid Index

In this section, we will state and prove some results concerning the relation among the crossing number, the genus and the braid index of two-bridge links.

Theorem 1. *Let $K = b(\alpha, \beta)$ be a two-bridge link with $0 < \beta < \alpha$ and β odd. Let $(2n_1, 2n_2, ..., 2n_k)$ be the even continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$. Let μ be the number of components of K, Then the genus $g(K)$ of K is given by:*

$$g(K) = \frac{k - \mu + 1}{2}.$$

Proof. We use induction on k. Let $v = (2n_1, 2n_2, ..., 2n_k)$ be the even continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$. If $(a_1, 2b_2, a_2, 2b_2, ..., a_{n-1}, 2b_{n-1}, a_n)$ is the half-even continued fraction decomposition of $\frac{\beta}{\alpha}$, then we need to show that

$$2g + \mu = \sum_{j=1}^{n} |a_j| = k + 1 = t(v) + t'(v) + 2.$$

If $k = 1$, we have $\frac{\alpha-\beta}{\alpha} = \frac{1}{2n_1}$ and $t(v) = t'(v) = 0$ ($v = (2n_1)$ has no sign change). If $n_1 = 1$ then $a_1 = 2 = t(v) + t'(v) + 2$. If $n_1 > 1$ then $\frac{\beta}{\alpha} = \frac{2n_1-1}{2n_1}$ has the half-even decomposition $[1, 2n_1 - 2, 1]$. Therefore $\sum_{j=1}^{n} |a_j| = 2 = t(v) + t'(v) + 2$. So now assume that the theorem holds for k and let us consider the case of $k + 1$. Similar to the proof of Lemma 3, we will consider the two subcases of $n_2 > 0$ and $n_2 < 0$. As before we let $v = (2n_1, 2n_2, ..., 2n_{k+1})$ and $v' = (2n_2, 2n_3, ..., 2n_{k+1})$.

Subcase 1. $n_2 > 0$. In this case, $t(v) = t(v')$ and $t'(v) = t'(v') + 1$. Let $0 < \beta' < \alpha'$ be such that β' is odd and $v' = (2n_2, ..., 2n_k, 2n_{k+1})$ is the even decomposition of $\frac{\alpha'-\beta'}{\alpha'}$. By the induction assumption, we have $\sum_{j=1}^{m} |c_j| \geq t(v') + t'(v') + 2$, where $(c_1, 2d_1, c_2, ..., c_m)$ is the half-even decomposition of $\frac{\beta'}{\alpha'}$.

We have
$$\frac{\alpha-\beta}{\alpha}=\frac{1}{2n_1+\frac{\alpha'-\beta'}{\alpha'}}=\frac{\alpha'}{2n_1\alpha'+(\alpha'-\beta')}.$$
It follows that $\alpha = 2n_1\alpha' + (\alpha' - \beta')$ and $\beta = 2n_1\alpha' - \beta'$. The half-even decomposition of $\frac{\beta}{\alpha}$ is therefore of the following form:
$$\begin{aligned}\alpha &= 2n_1\alpha' + (\alpha'-\beta') = 1\cdot(2n_1\alpha'-\beta')+\alpha',\\ \beta &= 2n_1\cdot\alpha'-\beta',\\ \alpha' &= -c_1\cdot(-\beta')+\cdots ....\end{aligned}$$
In other word, the half-even decomposition of $\frac{\beta}{\alpha}$ is
$$v_1=(1,2n_1,-c_1,-2d_1,...,-c_{m-1},-2d_{m-1},-c_m).$$
The claim of the theorem follows trivially since $1+\sum_{j=1}^{m}|c_j| = 1+t(v')+t'(v')+2 = t(v)+t'(v)+2$.

Subcase 2. $n_2 < 0$. Notice that in this case we have $t(v) = t(v') + 1$ and $t'(v) = t'(v')$. Let us first consider the case $n_1 > 1$. Let $0 < \beta' < \alpha'$ be such that β' is odd and $-v' = (-2n_2, ..., -2n_k, -2n_{k+1})$ is the even decomposition of $\frac{\alpha'-\beta'}{\alpha'}$. Notice that $t(v') = t(-v')$ and $t'(v') = t'(-v')$. We have
$$\frac{\alpha-\beta}{\alpha}=\frac{1}{2n_1-\frac{\alpha'-\beta'}{\alpha'}}=\frac{\alpha'}{(2n_1-1)\alpha'+\beta'}.$$
It follows that $\alpha = (2n_1-1)\alpha' + \beta'$ and $\beta = 2(n_1-1)\alpha' + \beta'$.

It can be easily checked that if $(c_1, 2d_1, c_2, 2d_2, ..., c_{m-1}, 2d_{m-1}, c_m)$ is the half-even decomposition of $\frac{\beta'}{\alpha'}$, then $(1, 2(n_1-1), c_1, 2d_1, ..., c_{m-1}, 2d_{m-1}, c_m)$ is the half-even decomposition of $\frac{\beta}{\alpha}$, in which case we have $1+\sum_{j=1}^{m}|c_j| = 1+t(v')+t'(v')+2 = t(v)+t'(v)+2$. On the other hand, if $n_1 = 1$, then we have
$$\frac{\alpha-\beta}{\alpha}=\frac{1}{2-\frac{\alpha'-\beta'}{\alpha'}}=\frac{\alpha'}{\alpha'+\beta'},$$
that is, $\alpha = \alpha' + \beta'$ and $\beta = \beta'$. In this case, it is also easy to check that the half-even decomposition of $\frac{\beta}{\alpha}$ is $(1+c_1, 2d_1, ..., c_{m-1}, 2d_{m-1}, c_m)$. Since $c_1 > 0$, the result follows trivially. This completes the induction and the proof of the theorem. □

Theorem 2. *Let K be a two-bridge link. Let μ be the number of components of K, $g(K)$ be the genus of K, $Cr(K)$ be the crossing number of K and $b(K)$ be the braid index of K, then we have*
$$Cr(K) = 2g(K) + 2b(K) + \mu - t'(v) - 4,$$

where v is defined as the vector of the even decomposition of $\frac{\alpha-\beta}{\alpha}$ for $0 < \beta < \alpha$ and odd β, and where $b(\alpha, \beta)$ equals K or the mirror image of K.

Proof. By Lemmas 2 and 3 and Theorem 1,

$$\begin{aligned} Cr(K) &= 2\sum_{j=1}^{k} |n_j| - k + t'(v) + 1 \\ &= 2(\sum_{j=1}^{k} |n_j| - k + t'(v) + 2) + (k+1) - t'(v) - 4 \\ &= 2b(K) + 2g(K) + \mu - t'(v) - 4. \end{aligned}$$

In [2] the concept of the deficiency of a link was introduced. The deficiency of a link K is defined as:

$$d(K) = Cr(K) - 2g(K) - b(K) - \mu + 2.$$

It was shown that for any knot or link K $d(K) \geq 0$ and the crossing numbers of knots with deficiency zero are additive under the connected sum operation. So links with deficiency zero form an interesting class of among all links. We have the following corollaries.

Corollary 1. *Let K be a two-bridge link. Then*

$$2g(K) + b(K) + \mu - 2 \leq Cr(K) \leq 2g(K) + 2b(K) + \mu - 4.$$

Corollary 2. *Let $K = b(\alpha, \beta)$ be a two-bridge link such that $0 < \beta < \alpha$ and β is odd. Let $(2n_1, 2n_2, ..., 2n_k)$ be the even continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$, then*

$$d(K) = b(K) - t'(v) - 2 = b(k) + t(v) - k - 1 = \sum_{j=1}^{k} |n_j| - k.$$

In particular, K is of deficiency zero if and only $|n_j| = 1$ for each j, i.e., the absolute value of each entry in the even continued fraction decomposition of $\frac{\alpha-\beta}{\alpha}$ is 2.

The last statement of the above corollary gives a complete characterization of deficiency zero two-bridge links.

4. The Growth Rate of the Number of Deficiency Zero Links

Let us first prove a relatively easy result.

Theorem 3. *There are infinitely many two-bridge links with any given deficiency d.*

Proof. Let $d \geq 0$ be any given integer. Let $K_j = b(\alpha_j, \beta_j)$ be a two-bridge link such that $0 < \beta < \alpha$, β is odd and such that the even decomposition of $\frac{\alpha-\beta}{\alpha}$ is given by $v_j = (2(d+1), 2, ..., 2)$ (with j entries). Since $t(v_j) = 0$, $Cr(K_j) = 2j+2d$. By Corollary 2, $d(K_j) = d$. Since $Cr(K_j) = 2j+2d \neq 2i+2d = Cr(K_i)$ for any $i \neq j$, the result of the theorem follows. □

In the above proof, we showed that there are infinitely many links with different crossing numbers sharing the same deficiency. However, what happens if we restrict us to links with the same crossing number? How many such links may share the same deficiency? The following theorem answers part of this question.

Theorem 4. *For any given integer $d \geq 0$, the number of links with deficiency d and crossing number c is at least $e^{\theta(c-2d)}$, where $\theta > 0$ is a constant that is independent of d and c.*

Proof. It suffices to show that the theorem holds for two-bridge links. We will count the number of even decompositions which then will be divided by two to obtain a lower bound on the number of non equivalent 4-plat links (see the observations about links before Lemma 1). We will prove the case for $d = 0$ first.

By Corollary 2, any two-bridge link $K = b(\alpha, \beta)$ (with $0 < \beta < \alpha$) with the even decomposition $(2n_1, 2n_2, ..., 2n_k)$ for $\frac{\alpha-\beta}{\alpha}$ is of deficiency *zero* if $|n_j| = 1$ for each j. Thus by Lemma 3, the crossing number $c = 2\sum_{j=1}^{k} |n_j| - t(v) = 2k - t(v)$ or $2k = c + t$. Let $t(v) = t = \lfloor \frac{k}{4} \rfloor - 1$. Then $k \geq 4t + 4$. Furthermore, t is of the order of $\frac{c}{7}$ and k is of the order of $\frac{4c}{7}$. The question is, how many different vectors of the form $(2n_1, 2n_2, ..., 2n_k)$ we can produce under the constrain that $|n_j| = 1$ and that there are $t(v)$ sign changes? Since $n_1 = 1$ and the $t(v) = t$ sign changes can occur at any other $k-1$ entries, we have ${}_tC_{k-1}$ such vectors. Furthermore, we have the following estimate:

$$
\begin{aligned}
{}_tC_{k-1} = \frac{k!}{t!(k-t-1)!} &= \frac{(k-1)(k-2)\cdots(k-t)}{t(t-1)\cdots 1} \\
&\geq \frac{(4t+3)(4t+2)\cdots(3t+4)}{t(t-1)\cdots 1} > 3^t \approx 3^{c/7}.
\end{aligned}
$$

The result now follows.

Finally, for the general case of $d > 0$. Applying the above argument to the number of even decompositions $(2n_1, 2n_2, ..., 2n_k)$ such that $n_1 = 1$ and $|n_j| = 1$, but with c replaced by $c - 2d$. The argument above shows that there are at least $3^{(c-2d)/7}$ such decompositions. Now we can simply replace $n_1 = 1$ by $n_1 = d + 1$ in each of these decompositions. We leave it to our reader to verify that these decompositions do produce the two-bridge links with the desired properties. □

5. The Upper Bounds of the Crossing Numbers by Combined Braid Index and Genus

We had conjectured that some combination of the genus and braid index (together with the number of components) of a link K may bound above the crossing number of K in the introduction section. We note the inequality $Cr(K) \leq 2g(K) + 2b(K) + \mu - 4$ proved in Section 3 for two-bridge links imply that $Cr(K) \leq 2(g(K) + b(K) + \mu)$ for all two-bridge links, which we state as the following conjecture for the general case.

Conjecture. *For any link K, we have $Cr(K) \leq 2(g(K) + b(K) + \mu)$.*

A weaker form of the conjecture would be that $Cr(K) \leq M(g(K)+b(K)+\mu)$ for some constant $M > 0$ for any link K.

To verify this conjecture for the prime knots and links that we could access at this point, we proceeded as follows.

First, we generate all prime knots and alternating links up to 12 crossings. Then we calculate the HOMFLY and Alexander polynomials for each of these knots and links. Using the facts that the braid index of K is bounded below by half the ℓ-span of the HOMFLY polynomial of K plus one ([3, 5]) and the genus of K is bounded below by half the span of the Alexander polynomial of K, we are able to calculate a lower bound of $2(g(K) + b(K) + \mu)$ (with the help of Morwen Thistlethwaite). This lower bound turns out to be greater or equal to the corresponding crossing number for all the knots and alternating links we have checked. It remains an interesting question whether this approach will meet a counterexample and if so, what kind of counterexamples are there and whether such counterexamples would actually produce real counterexamples to our conjecture.

References

[1] G. Burde, H. Zieschang, *Knots*, De Gruyter (1985).

[2] Y. Diao, The additivity of crossing numbers, *J. Knot Theory Ramifications*, **13**, No. 7 (2004), 857-866.

[3] J. Franks, R.F. Williams, Braids and the Jones polynomial, *Trans. Amer. Math. Soc.*, **303**, No. 1 (1987), 97-108.

[4] L. Kauffman, State models and the Jones polynomial, *Topology*, **26**, No. 3 (1987), 395-407.

[5] H. Morton, Seifert circles and knot polynomials, *Proc. Cambridge Phil. Soc.*, **99** (1986), 107-109.

[6] K. Murasugi, Jones polynomials and classical conjectures in knot theory, *Topology*, **26**, No. 2 (1987), 187-194.

[7] K. Murasugi, On the braid index of alternating links, *Trans. Am. Math. Soc.*, **326** (1991), 237-260.

[8] Y. Ohyama, On the minimal crossing number and the braid index of links, *Canad. J. Math.*, **45**, No. 1 (1993), 117-131.

[9] M. Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology*, **26**, No. 3 (1987), 297-309.