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ON THE TRACE OF A PERMUTING TRI-ADDITIVE MAPPING IN LEFT S-UNITAL RINGS

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Abstract: Let R be 2 and 3 torsion-free left s-unital ring. Let $G : R \times R \times R \longrightarrow R$ be a permuting tri-additive mapping and g the trace of G. Let $\alpha : R \longrightarrow R$ be an endomorphism and $\beta : R \longrightarrow R$ an epimorphism. In this paper, we prove the following: a) If g is (α, β)-skew commuting on R, then $G = 0$. b) If g is (β, β)-skew-centralizing on R, then g is (β, β)-commuting on R. c) For $n \geq 2$, let R be also $(n+2)!$-torsion free. If g is n-(α, β)-skew commuting on R, then $G = 0$. d) If g is 2-(α, β)-commuting on R, then $g$is (α, β)-commuting on R.

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1. Preliminaries

Throughout this paper, all rings R will be associative and the center of a ring will be denoted by Z. Let α, β be additive mappings of R into R and $x, y \in R$. As usual, we denote $[x, y] = xy - yx$, $< x, y >= xy + yx$, $[x, y]_{(\alpha,\beta)} = x\alpha(y) -$

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$\beta(y)x$, $< x,y >_{(\alpha,\beta)}= x\alpha(y)+\beta(y)x$. We will use the the following properties in this paper: $[x+y,z]_{(\alpha,\beta)} = [x,z]_{(\alpha,\beta)}+[y,z]_{(\alpha,\beta)}$, $[x,y+z]_{(\alpha,\beta)} = [x,y]_{(\alpha,\beta)}+[x,z]_{(\alpha,\beta)}$, $< x+y,z >_{(\alpha,\beta)}=< x,z >_{(\alpha,\beta)} + < y,z >_{(\alpha,\beta)}$, $< x,y+z >_{(\alpha,\beta)}=< x,y >_{(\alpha,\beta)} + < x,z >_{(\alpha,\beta)}$.

Let f be a mapping from R into R, and S a nonempty subset of R. Then f is called (α,β)-skew-commuting (resp. (α,β)-skew-centralizing) on S if $< f(x),x >_{(\alpha,\beta)} = 0$ (resp. $< f(x),x >_{(\alpha,\beta)}\in Z$) for all $x \in S$. Similarly f is said to be (α,β)-commuting on S if $[f(x),x]_{(\alpha,\beta)} = 0$ for all $x \in S$. If $\alpha=\beta=1$ (the identity map on R), then f is called simply skew-commuting, skew-centralizing and commuting on S, respectively. A mapping $G : R\times R \longrightarrow R$ is said to be symmetric if $G(x,y) = G(y,x)$ for all $x,y \in R$. A mapping $g : R \longrightarrow R$ defined by $g(x) = G(x,x)$ for all $x \in R$, where $G : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of G. The study of such mappings has been investigated by many authors (see, e.g., [1], [2], [7]). In [3], Jung and Chang investigated symmetric biadditive maps and the trace of its with (α,β)-skew-commuting and (α,β)-skew-centralizing maps in rings with left identity.

In [5], Ozturk introduced the notion of a permuting tri-additive mapping and derivations and also proved some results in [5] and [8]. A mapping $G : R\times R\times R \longrightarrow R$ is called tri-additive if $G(x+w,y,z) = G(x,y,z)+G(w,y,z)$, $G(x,y+w,z) = G(x,y,z)+G(x,w,z)$, $G(x,y,z+w) = G(x,y,z)+G(x,y,w)$ holds for all $x,y,z \in R$. A tri-additive mapping $G : R\times R\times R \longrightarrow R$ is called permuting tri-additive if $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x)$ holds for all $x,y,z \in R$. A mapping $g : R \longrightarrow R$ defined by $g(x) = G(x,x,x)$ is called the trace of G where $G : R\times R\times R \longrightarrow R$ is a permuting tri-additive mapping. It is obvious that, if $G : R\times R\times R \longrightarrow R$ is a permuting tri-additive mapping then the trace of G satisfies the relation $g(x+y) = g(x)+g(y)+3G(x,x,y)+3G(x,y,y)$ for all $x,y \in R$. The mapping $g : R \longrightarrow R$ defined by $g(x) = G(x,x,x)$ is an odd function.

R is called a left s-unital (resp. s-unital) ring if for each $x \in R$ there holds $x \in Rx$ (resp. $x \in Rx \cap xR$). If R is a left s-unital (resp. s-unital) ring then for any finite subset F of R there exists an element e in R such that $ex = x$ (resp. $ex = xe = x$) for all $x \in F$ (see [6], Theorem 1 and [4], Lemma 1). Such an element e will be called a left pseudo-identity (resp. pseudo-identity) of F. Throughout this paper e will be a left pseudo-identity of the set

$$E := \{x, g(x), g(e), \beta(x), G(x,x,e), G(x,e,e)\} \subset R\,,$$

where x is an arbitrary element of R.

In this paper, we investigate permuting tri-additive mapping and the trace of its with (α,β)-skew-commuting and (α,β)-skew-centralizing maps in left s-unital rings.

2. Results

The first result is the following.

Theorem 1. *Let R be 2 and 3-torsion-free left s-unital ring. Let $\alpha : R \longrightarrow R$ be an endomorphism and $\beta : R \longrightarrow R$ an epimorphism. Let $G : R \times R \times R \longrightarrow R$ be a permuting tri-additive mapping and g the trace of G. If g is (α, β)-skew-commuting on R, then $G = 0$.*

Proof. It is given that, for all $x \in R$,

$$< g(x), x >_{(\alpha,\beta)} = g(x)\alpha(x) + \beta(x)g(x) = 0. \qquad (1)$$

$\beta(e)$ is also a left pseudo-identity of R since β is an epimorphism. So from (1), we have

$$< g(e), e >_{(\alpha,\beta)} = g(e)\alpha(e) + g(e) = 0 \qquad (2)$$

and right-multiplying by $\alpha(e)$ gives $g(e)\alpha(e) = 0$ since R is 2-torsion-free. Hence, by (2), we get $g(e) = 0$.

Substituting $x + e$ for x in (1), we obtain, for all $x \in R$,

$$< g(x), e >_{(\alpha,\beta)} +3 < M, x >_{(\alpha,\beta)} +3 < M, e >_{(\alpha,\beta)} +3 < N, x >_{(\alpha,\beta)} \qquad (3)$$

$$+3 < N, e >_{(\alpha,\beta)} = 0\,,$$

where $M = G(x, x, e)$, $N = G(x, e, e)$.

Putting $-x$ instead of x in (3) and comparing (3) with the obtained equation, we have

$$M\alpha(e) + M + N\alpha(e) + N = 0\,, \qquad (4)$$

since g is odd function, R is 2 and 3-torsion-free and $\beta(e)$ is a left pseudo-identity. Right multiplication of (4) by $\alpha(e)$ gives $M\alpha(e) + N\alpha(e) = 0$. From (4), we obtain $M + N = 0$. Hence, we arrive at $g(x + e) = g(x)$ for all $x \in R$. Then, we have

$$0 =< g(x + e), x + e >_{(\alpha,\beta)} =< g(x), e >_{(\alpha,\beta)} = g(x)\alpha(e) + g(x). \qquad (5)$$

Multiplying $\alpha(e)$ from the right, we get $g(x)\alpha(e) = 0$. So from (5), we obtain

$$g(x) = G(x, x, x) = 0 \qquad (6)$$

for all $x \in R$. Then it follows that, for all $x, y \in R$,

$$G(x,x,y) + G(x,y,y) = 0\,, \tag{7}$$

since $G(x+y, x+y, x+y) = 0$, G is permuting tri-additive mapping and R is 3-torsion-free ring. Since $G(x+y+z, x+y+z, x+y+z) = 0$ and R is 2 and 3-torsion free, and using (7), we obtain

$$G(x,y,z) = 0$$

for all $x, y, z \in R$ which gives the conclusion. □

Theorem 2. *Let R be 2 and 3-torsion-free left s-unital ring. Let $\beta : R \longrightarrow R$ be an epimorphism. Let $G : R \times R \times R \longrightarrow R$ be a permuting tri-additive mapping and g the trace of G. If g is (β,β)-skew-centralizing on R, then g is (β,β)-commuting on R.*

Proof. Since g is (β,β)-skew-centralizing on R, we know that

$$< g(x), x >_{(\beta,\beta)} = g(x)\beta(x) + \beta(x)g(x) \in Z \tag{8}$$

for all $x \in R$. Hence

$$g(e)\beta(e) + g(e) \in Z\,, \tag{9}$$

since $\beta(e)$ is a left pseudo-identity. Commuting with $\beta(e)$ gives $g(e) = g(e)\beta(e)$ and we get $2g(e) \in Z$ by (9). Hence $g(e) \in Z$.

Let us replace $x + e$ by e in (8). We get

$$2\beta(x)g(e) + 3\beta(x)M + 3\beta(x)N + g(x) + 3M + 3N + g(x)\beta(e) + 3M\beta(x) \tag{10}$$

$$+3M\beta(e) + 3N\beta(x) + 3N\beta(e) \in Z\,,$$

using (8), (9) and $g(e) \in Z$, where $M = G(x,x,e)$, $N = G(x,e,e)$.

Substituting $-x$ for x in (10) and comparing (10) with the new one, we have

$$\beta(x)N + M + M\beta(e) + N\beta(x) \in Z\,, \tag{11}$$

or

$$2\beta(x)g(e) + 3\beta(x)M + g(x) + 3N + g(x)\beta(e) + 3M\beta(x) + 3N\beta(e) \in Z, \tag{12}$$

since R is 2 and 3 torsion-free ring.

Let us put $x + e$ instead of x in (10). Since $g(e) \in Z$ and $\beta(e)$ is left pseudo-identity, we obtain

$$\beta(x)N + 2\beta(x)g(e) + 3N + M + M\beta(e) + 3N\beta(e) + N\beta(x) \in Z.$$

Using (5), we get

$$2\beta(x)g(e) + 3N + 3N\beta(e) \in Z \tag{13}$$

and commuting with $\beta(e)$, we obtain $N\beta(e) = N$. Writing this in (13), and using 2-torsion free, we have $\beta(x)g(e) + 2N \in Z$. Commuting with $\beta(x)$, using $g(e) \in Z$,we get

$$N = G(x, e, e) \in Z\,, \tag{14}$$

since β is an epimorphism.

Let us commute with $\beta(e)$ the equation (11). We obtain $M\beta(e) = M$ since $N \in Z$. Hence from (11), we have $N\beta(x) + M \in Z$ and commuting again with $\beta(x)$, we obtain

$$M = G(x, x, e) \in Z. \tag{15}$$

In (12), using the equations (14) and (15), we get

$$2\beta(x)g(e) + 6\beta(x)M + 6N + g(x) + g(x)\beta(e) \in Z. \tag{16}$$

Commuting with $\beta(e)$ in (16), we obtain, for all $x \in R$,

$$g(x)\beta(e) = g(x).$$

Using this equality in (16), we have $\beta(x)g(e) + 3\beta(x)M + 3N + g(x) \in Z$. Commuting with $\beta(x)$, it is obtained that

$$g(x)\beta(x) = \beta(x)g(x).$$

Hence g is (β, β)-commuting.

For $n \geq 2$, a mapping $f : R \longrightarrow R$ is called n-(α, β)-skew commuting (resp. n-(α, β)-skew centralizing) on R if $< f(x), x^n >_{(\alpha,\beta)} = 0$ (resp. $< f(x), x^n >_{(\alpha,\beta)} \in Z$) for all $x \in R$. □

Now we extend the results (α, β)-skew-commiting mappings (Theorem 1) to $n-(\alpha, \beta)$-skew-commuting ones.

Theorem 3. *Let $n \geq 2$. Let R be 2, 3 and $(n+2)!-$torsion free left s-unital ring. Let $\alpha : R \longrightarrow R$ be an endomorphism and $\beta : R \longrightarrow R$ an epimorphism. Let $G : R \times R \times R \longrightarrow R$ be a permuting tri-additive mapping and g the trace of G. If g is $n - (\alpha, \beta)$-skew-commuting on R, then $G = 0$.*

Proof. Since g is $n - (\alpha, \beta)$-skew-commuting on R, we can write

$$< g(x), x^n >_{(\alpha,\beta)} = 0 \tag{17}$$

for all $x \in R$. Using similar processes as in the proof of Theorem 1, we have $g(e) = 0$. Substitution $x + te$ for x in (17) leads to

$$g(x)\alpha(a) + \beta(a)g(x) + 3tM\alpha(a) + 3t\beta(a)M + 3t^2N\alpha(a) + 3t^2\beta(a)N = 0, \quad (18)$$

since $g(x+te) = g(x) + 3tM + 3t^2N$, where $M = G(x, x, e), N = G(x, e, e), a = (x + te)^n$. Using (17) in (18), we obtain a polynomial equation such as

$$P_1(x, e)t + ... + P_{n+1}(x, e)t^{n+1} + P_{n+2}(x, e)t^{n+2} = 0\,,$$

which the coefficients $P_k(x, e)$ for $k = 1, 2, ..., n + 2$ are terms involving x and e.

Replacing t by $1, 2, ..., n + 2$ in turn, and expressing the resulting system of $n + 2$ homogeneous equation with the variables $P_1(x, e), ..., P_{n+2}(x, e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ 2 & 2^2 & ... & 2^{n+2} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ n+2 & (n+2)^2 & ... & (n+2)^{n+2} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n + 2$, and since R is $2, 3$ and $(n + 2)!$ torsion free ring, it follows immediately that for each $k = 1, 2, ..., n + 2$,

$$P_k(x, e) = 0.$$

Hence we have

$$P_{n+2}(x, e) = 3 < N, e^{n+2} >_{(\alpha,\beta)} = 0 =< N, e^{n+2} >_{(\alpha,\beta)}\,. \quad (19)$$

Then, we get

$$N\alpha(e) + N = 0 \quad (20)$$

and multiplying by $\alpha(e)$ from the right, and using 2-torsion free, we have

$$N = G(x, e, e) = 0. \quad (21)$$

Hence, from (18), we obtain

$$g(x)\alpha(a) + \beta(a)g(x) + 3tM\alpha(a) + 3t\beta(a)M = 0. \quad (22)$$

In (22), the coefficient of the term t^{n+1} is

$$P_{n+1}(x,e) = 3 < M, e^{n+1} >= 0.$$

With similar processes as in (19)-(21), we have

$$M = G(x,x,e) = 0.$$

Therefore, from (18), we get

$$P_n(x,e) = g(x)\alpha(e^n) + \beta(e^n)g(x) = g(x)\alpha(e) + g(x) = 0\,, \tag{23}$$

since $N = 0$ and $M = 0$. Multiplying (23) by $\alpha(e)$ with right hand side, we have

$$g(x) = G(x,x,x) = 0.$$

This completes the proof. □

Theorem 4. *Let R be 2 and 3−torsion free left s-unital ring. Let $\alpha : R \longrightarrow R$ be an endomorphism and $\beta : R \longrightarrow R$ an epimorphism. Let $G : R \times R \times R \longrightarrow R$ be a permuting tri-additive mapping and g the trace of G. If g is $2-(\alpha,\beta)$−commuting on R, then g is (α,β)-commuting on R.*

Proof. Let us define a mapping $h : R \longrightarrow R$ by $h(x) = [g(x), x]_{(\alpha,\beta)}$ for all $x \in R$. Note that h is even function. From the hypothesis, we can write

$$< h(x), x >_{(\alpha,\beta)} = [g(x), x^2]_{(\alpha,\beta)} = 0 \tag{24}$$

for all $x \in R$. Since β is an epimorphism, $\beta(e)$ is also a left pseudo-identity. So, we have

$$h(e)\alpha(e) + h(e) = 0 \tag{25}$$

for all $x \in R$. Right multiplying by $\alpha(e)$ gives $h(e)\alpha(e) = 0$ since R is 2-torsion free. Hence, by (25), we get

$$h(e) = [g(e), e]_{(\alpha,\beta)} = 0. \tag{26}$$

Since $g(x+e) = g(x)+g(e)+3M+3N$, where $M = G(x,x,e)$ and $N = G(x,e,e)$, we obtain

$$\begin{aligned} h(x+e) = h(x) + [g(x), e]_{(\alpha,\beta)} + [g(e), x]_{(\alpha,\beta)} + 3[M, x]_{(\alpha,\beta)} \\ + 3[M, e]_{(\alpha,\beta)} + 3[N, x]_{(\alpha,\beta)} + 3[N, e]_{(\alpha,\beta)} \end{aligned} \tag{27}$$

If we replace x by $x+e$ in (24) and using (24), (26) and permutting tri-additivity of G, we have, for all $x \in R$,

$$\begin{aligned} h(x)\alpha(e) &+ [g(x),e]_{(\alpha,\beta)}\alpha(x) + [g(x),e]_{(\alpha,\beta)}\alpha(e) + [g(e),x]_{(\alpha,\beta)}\alpha(x) \\ &+ [g(e),x]_{(\alpha,\beta)}\alpha(e) + 3[M,x]_{(\alpha,\beta)}\alpha(x) + 3[M,x]_{(\alpha,\beta)}\alpha(e) \\ &+ 3[M,e]_{(\alpha,\beta)}\alpha(x) + 3[M,e]_{(\alpha,\beta)}\alpha(e) + 3[N,x]_{(\alpha,\beta)}\alpha(x) + 3[N,x]_{(\alpha,\beta)}\alpha(e) \\ &+ 3[N,e]_{(\alpha,\beta)}\alpha(x) + 3[N,e]_{(\alpha,\beta)}\alpha(e) \\ &+ h(x) + \beta(x)[g(x),e]_{(\alpha,\beta)} + [g(x),e]_{(\alpha,\beta)} + \beta(x)[g(e),x]_{(\alpha,\beta)} \\ &+ [g(e),x]_{(\alpha,\beta)} + 3\beta(x)[M,x]_{(\alpha,\beta)} + 3[M,x]_{(\alpha,\beta)} + 3\beta(x)[M,e]_{(\alpha,\beta)} \\ &+ 3[M,e]_{(\alpha,\beta)} + 3\beta(x)[N,x]_{(\alpha,\beta)} + 3[N,x]_{(\alpha,\beta)} + 3\beta(x)[N,e]_{(\alpha,\beta)} \\ &+ 3[N,e]_{(\alpha,\beta)} = 0\,. \qquad (28) \end{aligned}$$

Substituting $-x$ for x in (28) and comparing (28) with the obtained result, we get, for all $x \in R$,

$$\begin{aligned} [g(x),e]_{(\alpha,\beta)}\alpha(e) &+ [g(e),x]_{(\alpha,\beta)}\alpha(e) + 3[M,x]_{(\alpha,\beta)}\alpha(e) + 3[M,e]_{(\alpha,\beta)}\alpha(x) \\ &+ [N,x]_{(\alpha,\beta)}\alpha(x) + 3[N,e]_{(\alpha,\beta)}\alpha(e) + [g(x),e]_{(\alpha,\beta)} + [g(e),x]_{(\alpha,\beta)} \\ &+ 3[M,x]_{(\alpha,\beta)} + 3\beta(x)[M,e]_{(\alpha,\beta)} + 3\beta(x)[N,x]_{(\alpha,\beta)} + 3[N,e]_{(\alpha,\beta)} = 0\,, \qquad (29) \end{aligned}$$

since h and M are even, g and N are odd, R is 2-torsion free ring.

Right multiplication of (29) by $\alpha(e)$ gives

$$\begin{aligned} 2[g(x),e]_{(\alpha,\beta)}\alpha(e) &+ 2[g(e),x]_{(\alpha,\beta)}\alpha(e) + 6[M,x]_{(\alpha,\beta)}\alpha(e) \\ &+ 6[N,e]_{(\alpha,\beta)}\alpha(e) + 3[M,e]_{(\alpha,\beta)}\alpha(x)\alpha(e) + 3[N,x]_{(\alpha,\beta)}\alpha(x)\alpha(e) \\ &+ 3\beta(x)[M,e]_{(\alpha,\beta)}\alpha(e) + 3\beta(x)[N,x]_{(\alpha,\beta)}\alpha(e) = 0\,. \qquad (30) \end{aligned}$$

Substituting again $x+e$ instead of x in (30) and using (30), we obtain

$$\begin{aligned} 4[g(e),x]_{(\alpha,\beta)}\alpha(e) &+ 12[N,e]_{(\alpha,\beta)}\alpha(e) + 6[M,e]_{(\alpha,\beta)}\alpha(e) + 6[N,x]_{(\alpha,\beta)}\alpha(e) \\ &+ 3[N,e]_{(\alpha,\beta)}\alpha(x)\alpha(e) + [g(e),x]_{(\alpha,\beta)}\alpha(x)\alpha(e) + 3\beta(x)[N,e]_{(\alpha,\beta)}\alpha(e) \\ &+ \beta(x)[g(e),x]_{(\alpha,\beta)}\alpha(e) = 0\,, \qquad (31) \end{aligned}$$

since R is 2-torsion free ring.

Putting $-x$ for x and comparing (31), we get

$$[g(e),x]_{(\alpha,\beta)}\alpha(e) + 3[N,e]_{(\alpha,\beta)}\alpha(e) = 0\,. \qquad (32)$$

Furthermore we get

$$[g(e),x]_{(\alpha,\beta)}\alpha(x) + 3[N,e]_{(\alpha,\beta)}\alpha(x) = [g(e),x]_{(\alpha,\beta)}\alpha(ex) + 3[N,e]_{(\alpha,\beta)}\alpha(ex)$$

$$= \left([g(e), x]_{(\alpha,\beta)}\alpha(e) + 3[N, e]_{(\alpha,\beta)}\alpha(e)\right) \alpha(x) = 0\,. \quad (33)$$

According to (32) and (33), the relation (31) becomes

$$[M, e]_{(\alpha,\beta)}\alpha(e) + [N, x]_{(\alpha,\beta)}\alpha(e) = 0. \quad (34)$$

With similar process as obtaining of (33), we have

$$[M, e]_{(\alpha,\beta)}\alpha(x) + [N, x]_{(\alpha,\beta)}\alpha(x) = 0. \quad (35)$$

Using the obtained equations (32), (34) and (35) in (30), we get

$$[g(x), e]_{(\alpha,\beta)}\alpha(e) + 3[M, x]_{(\alpha,\beta)}\alpha(e) = 0.$$

Therefore (29) becomes

$$[g(x), e]_{(\alpha,\beta)} + [g(e), x]_{(\alpha,\beta)} + 3[M, x]_{(\alpha,\beta)} + \beta(x)[M, e]_{(\alpha,\beta)} + 3\beta(x)[N, x]_{(\alpha,\beta)} + 3[N, e]_{(\alpha,\beta)} = 0. \quad (36)$$

If we put $x + e$ instead of x in (36), and compare with (36), we get

$$2[g(e), x]_{(\alpha,\beta)} + 3[M, e]_{(\alpha,\beta)} + 6[N, e]_{(\alpha,\beta)} + 3[N, x]_{(\alpha,\beta)} + 3\beta(x)[N, e]_{(\alpha,\beta)} + \beta(x)[g(e), x]_{(\alpha,\beta)} = 0. \quad (37)$$

Substituting $-x$ for x and comparing (36) we write

$$[g(e), x]_{(\alpha,\beta)} + 3[N, e]_{(\alpha,\beta)} = 0. \quad (38)$$

So, the equation (37) becomes

$$[M, e]_{(\alpha,\beta)} + [N, x]_{(\alpha,\beta)} = 0. \quad (39)$$

Hence from (36), we have

$$[g(x), e]_{(\alpha,\beta)} + 3[M, x]_{(\alpha,\beta)} = 0. \quad (40)$$

Using (38), (39), (40) in (27), we obtain $h(x + e) = h(x)$. Since

$$< h(x), x >_{(\alpha,\beta)} = 0 \ \text{ for all } \ x \in R,$$

the relation $h(x + e)\alpha(x + e) + \beta(x + e)h(x + e) = 0$ becomes

$$h(x)\alpha(e) + h(x) = 0 \quad (41)$$

for all $x \in R$. Right multiplying (41) by $\alpha(e)$ we have $h(x)\alpha(e) = 0$ since R is 2-torsion free. Hence from (41), we obtain $h(x) = 0$ for all $x \in R$ which gives the conclusion. □

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