International Journal of Pure and Applied Mathematics
Volume 23 No. 4 2005, 475-477

ON THE ENUMERATION OF HIGHER DIMENSIONAL LATTICE PATHS

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Abstract: In this note we give a decomposition formula for the number of higher dimensional lattice paths with a certain condition.

AMS Subject Classification: 05A15, 05E10
Key Words: lattice path, Young graph

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A *partition* $\lambda = (\lambda_1, \dots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_l > 0$ of integers with $|\lambda| = \sum_{i=1}^{l} \lambda_i = n$, for short we write $\lambda \vdash n$. The *Young diagram* corresponding to λ, is a collection of squares $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, such that $1 \leq j \leq \lambda_i$. We do not distinguish between a partition and its Young diagram. Each $(i, j) \in \lambda$ is called a *node* of λ. A node $(i, \lambda_i + 1)$ is called *addable* (for λ) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$.

The *Young graph* $\mathcal{Y}$ corresponding to λ, is a graph which the vertices are Young diagrams. Two diagrams λ and μ, are connected by an edge going from λ to μ, if $|\mu| = |\lambda| + 1$ and μ can be obtained by adding an addable node to λ.

Received: June 10, 2005

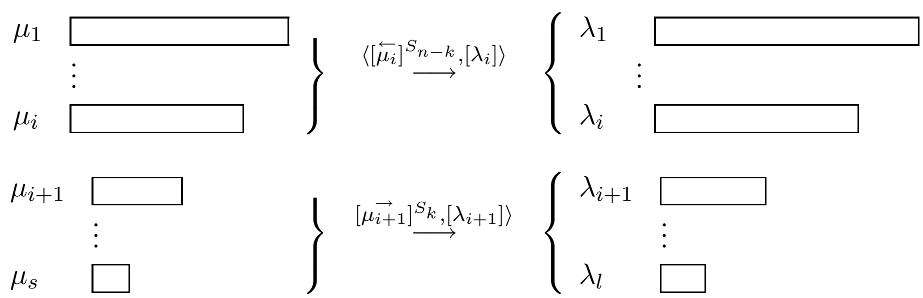
Let $[\lambda]$ denote the irreducible character of the symmetric group S_n labeled by $\lambda \vdash n$. If $\mu \vdash t < n$ then $\langle [\mu]^{S_n}, [\lambda] \rangle$, the multiplicity of $[\lambda]$ in $[\mu]^{S_n}$, equals the number of paths to go from μ to λ in the Young graph of μ. The following theorem gives a decomposition formula for the number of these paths.

Notation. If $\lambda = (\lambda_1, \ldots, \lambda_l)$, we denote $\overrightarrow{\lambda_{i+1}} = (\lambda_{i+1}, \ldots, \lambda_l)$ and $\overleftarrow{\lambda_i} = (\lambda_1, \ldots, \lambda_i)$.

Theorem 1. (Multy Constituent) *Let* $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ *and* $\mu = (\mu_1, \ldots, \mu_s) \vdash n - r$. *Suppose* $\mu_i \geq \lambda_{i+1}$ *for some* i, $\overrightarrow{\lambda_{i+1}} \vdash k$ *and* $\overrightarrow{\mu_{i+1}} \vdash t$. *Then*

$$\langle [\mu]^{S_n}, [\lambda] \rangle = \binom{r}{k-t} \langle [\overleftarrow{\mu_i}]^{S_{n-k}}, [\overleftarrow{\lambda_i}] \rangle \langle [\overrightarrow{\mu_{i+1}}]^{S_k}, [\overrightarrow{\lambda_{i+1}}] \rangle.$$

Proof. The multiplicity of $[\lambda]$ in $[\mu]^{S_n}$ equals to the number of possibilities to fill with addable nodes the rows of the Young diagram of μ to reach to the Young diagram of λ in the corresponding Young graph. Therefore to add nodes to go from μ to λ, we have $\langle [\overrightarrow{\mu_{i+1}}]^{S_k}, [\overrightarrow{\lambda_{i+1}}] \rangle$ possibilities to fill rows $i+1$ to l with $k-t$ nodes (independent of what happens in the first i rows, by assumption that $\mu_i \geq \lambda_{i+1}$). In between these $k-t$ additions in rows $i+1$ to l, we add the $r+t-k$ necessary nodes to the first i rows with $\langle [\overleftarrow{\mu_i}]^{S_{n-k}}, [\overleftarrow{\lambda_i}] \rangle$ different possibilities.



Therefore the number of possibilities to fill μ with addable nodes to reach λ is

$$\binom{(k-t)+(r+t-k)}{r+t-k} \langle [\overleftarrow{\mu_i}]^{S_{n-k}}, [\overleftarrow{\lambda_i}] \rangle \langle [\overrightarrow{\mu_{i+1}}]^{S_k}, [\overrightarrow{\lambda_{i+1}}] \rangle.$$

This proves the theorem. □

Now using the theorem above we give a decomposition formula to the number of l-dimensional lattice paths. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be two points in the l-dimensional plane $\mathbb{Z}^l$ such that $\lambda_1 \geq \cdots \geq \lambda_l \geq 0$,

$\mu_l \geq \cdots \geq \mu_l \geq 0$ and $\lambda_i \geq \mu_i$ for $i = 1, \ldots, l$. A lattice path from λ to μ is a path using steps in the step set $\{e_1, \ldots, e_l\}$, where $e_i \in \mathbb{Z}^l$ and the i-th entry in e_i is 1 and the other entries are zeros. Suppose $\alpha = (\alpha_1, \ldots, \alpha_l)$, where $\alpha_i = \lambda_i - \mu_i$. Then the number of lattice paths from λ to μ which never cross the planes $X_i = X_{i+1}$ equals the number of lattice paths from the origin O in $\mathbb{Z}^l$ to α such that never cross the planes $X_i = X_{i+1} + t_i$ for $t_i = \mu_{i+1} - \mu_i \leq 0$ and $i = 1, \ldots, l-1$. Let $t = (t_1, \ldots, t_{l-1})$. We denote the set of these paths by $L(\alpha : t - \mathbf{1})$, where $t - \mathbf{1} = (t_1 - 1, \ldots, t_{l-1} - 1)$. It is easy to see that there is a one to one corresponding between a path from Ø to the diagram of λ in the corresponding Young graph and a lattice path from O to λ in $\mathbb{Z}^l$ which never cross the planes $X_i = X_{i+1}$ for $i = 1, \ldots, l-1$ (this is a generalization of the case $l = 2$ described in [1] and [2, Chapter 1]). Therefore we can restate the theorem above as follows.

Theorem 2. *Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a point in $\mathbb{Z}^l$ for $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l \geq 0$ and let $t = (t_1, \ldots, t_{l-1})$ such that $t_i \leq 0$. If $\alpha_{i+1} \leq -t_i$ for some i then*

$$|L(\alpha : t - \mathbf{1})| = \binom{|\alpha|}{|\overrightarrow{\alpha_{i+1}}|} |L(\overleftarrow{\alpha_i} : \overleftarrow{t_{i-1}} - \mathbf{1})||L(\overrightarrow{\alpha_{i+1}} : \overrightarrow{t_{i+1}} - \mathbf{1})|.$$

References

[1] D. André, Solution directe du'probléme résolu par M. Bertrand, *C.R. Acad. Sci. Paris*, **105** (1887), 436-437.

[2] S.G. Mohanty, *Lattice Paths Counting and Applications*, Academic Press, New York (1979).