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GOODNESS-OF-FIT TESTS VIA CHARACTERIZATIONS

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Abstract: We construct tests of fit using conditions derived from characterizations of continuous distributions via moments of the k-th record values. They are expressed in terms of extremal statistics or in original sample values. The presented tests are universal ones in a sense they can be used practically for all continuous distributions when parameters of distributions are known and for continuous distributions under some regular conditions when parameters are unknown. Moreover, their forms require only elementary evaluations.

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1. Introduction and Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf F and pdf f. For a fixed integer $k \geq 1$ we define a sequence $U_k(1), U_k(2), \ldots$ of k-th upper

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record times of $\{X_n,\, n \geq 1\}$ as follows:

$$\begin{aligned} U_k(1) &= 1, \\ U_k(n+1) &= \min\{j > U_k(n) : X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, \quad n \geq 1. \end{aligned}$$

Write

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1},\ n \geq 1.$$

The sequence $\{Y_n^{(k)},\ n \geq 1\}$ is called the sequence of k-th (upper) record values of the above sequence. For convenience we also take $Y_0^{(k)} = 0$ and note that

$$Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k)$$

(cf. Dziubdziela and Kopociński [5]).

We see that for $k = 1, 2, \ldots$, the sequences $\{Y_n^{(k)},\ n \geq 1\}$ of k–th record values can be obtained from $\{X_n,\ n \geq 1\}$ by inspecting successively the samples X_1, (X_1, X_2), (X_1, X_2, X_3), and so on. For $k = 1$, $Y_1^{(1)} = X_1$, and the following terms are obtained by looking at the maxima of the successive samples; $Y_2^{(1)}$ is the first maximum that exceeds $Y_1^{(1)}$, $Y_3^{(1)}$ is the first maximum that exceeds $Y_2^{(1)}$, and so on. For $k = 2$, $Y_1^{(2)} = \min(X_1, X_2)$, and the following terms are obtained by looking at the next–to–largest values in the successive samples; $Y_2^{(2)}$ is the first next–to–largest value that exceeds $Y_1^{(2)}$, $Y_3^{(2)}$ is the first next–to–largest value that exceeds $Y_2^{(2)}$, and so on. And generally, $Y_1^{(k)} = \min(X_1, \ldots, X_k) = X_{1:k}$, and the following k–th record values are obtained by looking at the k–th largest values in succesive samples, i.e., looking at the order statistics $X_{2:k+1}$ from $(X_1, \ldots, X_{k+1})$, $X_{3:k+2}$ from $(X_1, \ldots, X_{k+2})$, and so on.

The k-th lower record times $L_k(n)$, $n \geq 1$, are defined as

$$\begin{aligned} L_k(1) &= 1, \\ L_k(n+1) &= \min\{j > L_k(n) : X_{k;L_k(n)+k-1} > X_{k:j+k-1}\}, \quad n \geq 1, \end{aligned}$$

and the k-th lower record values as

$$Z_n^{(k)} = X_{k:L_k(n)+k-1}, \qquad n \geq 1.$$

Note that

$$Z_1^{(k)} = X_{k:k} = \max(X_1, \ldots, X_k) \quad \text{(cf. Pawlas and Szynal [17]).}$$

Thus the finite sequence $X_1, \ldots, X_k$ is enough to determine $Y_1^{(k)}$ and $Z_1^{(k)}$, whereas if $n > 1$ then in general an infinite sequence is needed for $Y_n^{(k)}$ and $Z_n^{(k)}$.

The marginal and joint cdf and pdf of the k-th upper and lower record values are given in Dziubdziela and Kopociński [5], Nevzorov [16] and Pawlas and Szynal [17].

We shall use the following recurrence relations for cdf of $Y_n^{(k)}$ and $Z_n^{(k)}$:

$$F_{Y_n^{(k)}}(x) = F_{Y_{n-1}^{(k)}}(x) - \frac{k^{n-1}}{(n-1)!}[1 - F(x)]^k[-\log(1 - F(x))]^{n-1} \tag{1.1}$$

and

$$F_{Z_n^{(k)}}(x) = F_{Z_{n-1}^{(k)}}(x) + \frac{k^{n-1}}{(n-1)!}[F(x)]^k[-\log F(x)]^{n-1} \tag{1.2}$$

for $n \geq 2$ and $k \geq 1$ (cf. Bieniek and Szynal [3], Grudzień and Szynal [6]).

Characterizations of continuous distributions via expected values of two functions of record values were discussed in Grudzień and Szynal [7], Lin [11], Malinowska et al [12]. Tests derived from characterizations in terms of moments of record values were presented in Morris and Szynal [13] and [14]. We used there one equation to construct goodness-of-fit tests. In this paper we derived tests from two equations containing moments of the k-th record values also with non–integer powers.

2. Characterization Conditions via Moments of the k-th Record Values

Let $I(F)$ stand for the minimal interval containing the support of F. Recall that a subdistribution G on $\mathbb{R}$ is nondecreasing and right–continuous function from $\mathbb{R}$ to the interval [0,1].

In what follows we write

$$\begin{aligned} H_t(x) &= (-\log(1 - G(x)))^t, \qquad h_t(x) = (-\log(1 - F(x)))^t, \\ H_t^*(x) &= (-\log G(x))^t, \qquad h_t^*(x) = (-\log F(x))^t, \qquad x \in \mathbb{R},\ t \neq 0. \end{aligned} \tag{2.1}$$

Theorem 1. *Let $\{X_n,\, n \geq 1\}$ be a sequence of i.i.d. random variables with a continuous distribution function F and G subdistribution on $\mathbb{R}$. Futher, let $\boldsymbol{B}$ be a non–singular matrix*

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{2.2}$$

and assume that $n \geq 1$, $k \geq 1$ *and* $s \geq 0$, $s \neq n$, *are given integers and* $r \neq 0$ *a given real number such that* $n + r + 1 > 0$. *Then* $F(x) = G(x)$ *on* $\mathbb{R}$ *iff*

$$\begin{cases} b_{11}k^r EH_r\left(Y_{n+1}^{(k)}\right) + b_{12}k^{r+n-s}EH_{r+n-s}\left(Y_{s+1}^{(k)}\right) \\ \qquad = \left(\frac{1}{n!}b_{11} + \frac{1}{s!}b_{12}\right)\Gamma(n+r+1)\,, \\ b_{21}k^r EH_r\left(Y_{n+1}^{(k)}\right) + b_{22}k^{r+n-s}EH_{r+n-s}\left(Y_{s+1}^{(k)}\right) \\ \qquad = \left(\frac{1}{n!}b_{21} + \frac{1}{s!}b_{22}\right)\Gamma(n+r+1)\,, \end{cases} \tag{2.3}$$

or iff

$$\begin{cases} b_{11}k^r EH_r^*\left(Z_{n+1}^{(k)}\right) + b_{12}k^{r+n-s}EH_{r+n-s}^*\left(Z_{s+1}^{(k)}\right) \\ \qquad = \left(\frac{1}{n!}b_{11} + \frac{1}{s!}b_{12}\right)\Gamma(n+r+1)\,, \\ b_{21}k^r EH_r^*\left(Z_{n+1}^{(k)}\right) + b_{22}k^{r+n-s}EH_{r+n-s}^*\left(Z_{s+1}^{(k)}\right) \\ \qquad = \left(\frac{1}{n!}b_{21} + \frac{1}{s!}b_{22}\right)\Gamma(n+r+1)\,, \end{cases} \tag{2.3*}$$

and also iff

$$(n-m)!k^{2m}EH_{2m}\left(Y_{n-m+1}^{(k)}\right) - 2n!k^m EH_m\left(Y_{n+1}^{(k)}\right) + (n+m)! = 0\,, \tag{2.4}$$

or

$$(n-m)!k^{2m}EH_{2m}^*\left(Z_{n-m+1}^{(k)}\right) - 2n!k^m EH_m^*\left(Z_{n+1}^*\right) + (n+m)! = 0 \tag{2.4*}$$

is satisfied (cf. Malinowska et al [12]).

Theorem 2. *If*

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then $F(x) = G(x)$ *on* $\mathbb{R}$ *iff*

$$\begin{cases} EH_r\left(Y_{n+1}^{(k)}\right) = \dfrac{\Gamma(n+r+1)}{n!k^r}\,, \\ EH_{r+n-s}\left(Y_{s+1}^{(k)}\right) = \dfrac{\Gamma(n+r+1)}{s!k^{r+n-s}}\,, \end{cases} \tag{2.5}$$

or iff

$$\begin{cases} EH_r^*\left(Z_{n+1}^{(k)}\right) = \dfrac{\Gamma(n+r+1)}{n!k^r}\,, \\ EH_{r+n-s}^*\left(Z_{s+1}^{(k)}\right) = \dfrac{\Gamma(n+r+1)}{s!k^{r+n-s}}\,, \end{cases} \tag{2.5*}$$

and also iff (2.4) *or* (2.4*) *is satisfied* (Grudzień and Szynal [7], Lin [11], Malinowska et al [12]).

Letting $n = 1$, $s = 0$ and $m = 1$ we have the following.

Remark 1. $X \sim F$ and F is continuous iff

$$\begin{cases} E\left[h_r\left(Y_2^{(k)}\right)\right] = \dfrac{\Gamma(r+2)}{k^r}, \\ E\left[h_{r+1}\left(Y_1^{(k)}\right)\right] = \dfrac{\Gamma(r+2)}{k^{r+1}}, \end{cases} \tag{2.6}$$

or iff

$$\begin{cases} E\left[h_r^*\left(Z_2^{(k)}\right)\right] = \dfrac{\Gamma(r+2)}{k^r}, \\ E\left[h_{r+1}^*\left(Z_1^{(k)}\right)\right] = \dfrac{\Gamma(r+2)}{k^{r+1}}, \end{cases} \tag{2.6*}$$

and also iff

$$E\left[h_2\left(Y_1^{(k)}\right)\right] - \frac{2}{k}E\left[h_1\left(Y_2^{(k)}\right)\right] + \frac{2}{k^2} = 0, \tag{2.7}$$

or iff

$$E\left[h_2^*\left(Z_1^{(k)}\right)\right] - \frac{2}{k}E\left[h_1^*\left(Z_2^{(k)}\right)\right] + \frac{2}{k^2} = 0. \tag{2.7*}$$

Taking into account that $Y_1^{(k)} = X_{1:k}$ and $Z_1^{(k)} = X_{k:k}$, the formula

$$E(1-F(X))^{\alpha-1}h_{\beta-1}(X) = \frac{\Gamma(\beta)}{\alpha^\beta} \qquad \alpha, \beta > 0,$$

and the relations

$$\begin{aligned} E\left[h_r\left(Y_2^{(k)}\right)\right] &= E\left[h_r\left(Y_1^{(k)}\right)\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r} \\ &= E\left[h_r(X_{1:k})\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r}, \\ E\left[h_r^*\left(Z_2^{(k)}\right)\right] &= E\left[h_r^*\left(Z_1^{(k)}\right)\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r} \\ &= E\left[h_r^*(X_{k:k})\right] - \frac{\Gamma(r+1)}{k^r} + \frac{\Gamma(r+2)}{k^r}, \end{aligned}$$

which follow from (1.1) and (1.2), when $r > -1$, we have from Remark 1.

Remark 2. If $X \sim F$ and F is continuous then

$$\begin{cases} E\left[h_r\left(X_{1:k}\right)\right] = \dfrac{\Gamma(r+1)}{k^r}, \\ E\left[h_{r+1}\left(X_{1:k}\right)\right] = \dfrac{\Gamma(r+2)}{k^{r+1}}, \end{cases} \tag{2.8}$$

$$\begin{cases} E\left[h_r^*\left(X_{k:k}\right)\right] = \dfrac{\Gamma(r+1)}{k^r}\,, \\ E\left[h_{r+1}^*\left(X_{k:k}\right)\right] = \dfrac{\Gamma(r+2)}{k^{r+1}}\,, \end{cases} \tag{2.8*}$$

and also

$$E\left[h_2(X_{1:k})\right] - \frac{2}{k}E\left[h_1(X_{1:k})\right] = 0, \tag{2.9}$$

$$E\left[h_2(X_{k:k})\right] - \frac{2}{k}E\left[h_1(X_{k:k})\right] = 0. \tag{2.9*}$$

For $k = 2$ we have the following remark.

Remark 2′. If $X \sim F$ and F is continuous then

$$\begin{cases} E\left[h_r\left(X_{1:2}\right)\right] = \dfrac{\Gamma(r+1)}{2^r}\,, \\ E\left[h_{r+1}\left(X_{1:2}\right)\right] = \dfrac{\Gamma(r+2)}{2^{r+1}}\,, \end{cases} \tag{2.10}$$

$$\begin{cases} E\left[h_r^*\left(X_{2:2}\right)\right] = \dfrac{\Gamma(r+1)}{2^r}\,, \\ E\left[h_{r+1}^*\left(X_{2:2}\right)\right] = \dfrac{\Gamma(r+2)}{2^{r+1}}\,, \end{cases} \tag{2.10*}$$

and also

$$E\left[h_2(X_{1:2})\right] - E\left[h_1(X_{1:2})\right] = 0, \tag{2.11}$$

$$E\left[h_2^*(X_{2:2})\right] - E\left[h_1^*(X_{2:2})\right] = 0. \tag{2.11*}$$

We say that X has the exponential distribution and write $X \sim \mathrm{Exp}(\alpha)$ or $F \in \mathrm{Exp}(\alpha)$ if its cdf F is given by

$$F(x) = 1 - e^{-\alpha x}, \qquad x > 0;\ \alpha > 0.$$

The pdf is

$$f(x) = \alpha e^{-\alpha x}.$$

Note that

$$X \sim F \Longleftrightarrow -\log(1 - F(X)) = \log\frac{1}{1 - F(X)} \sim \mathrm{Exp}(1).$$

We say that X has the negative exponential distribution if $-X \sim \mathrm{Exp}(\lambda)$ and we write $X \sim \mathrm{N\,Exp}(\lambda)$ or $F \in \mathrm{N\,Exp}(\lambda)$. Then

$$F(x) = e^{\lambda x}, \qquad x < 0;\ \lambda > 0,$$

and

$$f(x) = \lambda e^{\lambda x}.$$

We say that X has the inverse exponential distribution if $\frac{1}{X} \sim \mathrm{Exp}(\theta)$ and we write $X \sim \mathrm{I\,Exp}(\theta)$ or $F \in \mathrm{I\,Exp}(\theta)$. Then

$$F(x) = e^{-\theta/x}, \qquad x > 0;\ \theta > 0,$$

and

$$f(x) = \frac{\theta e^{-\theta/x}}{x^2} \qquad \text{(cf. Klugman et al [9], p. 582).}$$

For the above distributions we have the following remarks.

Remark 3. If $X \sim \mathrm{Exp}(1)$ then

$$\begin{cases} EX_{1:2}^{r} = \dfrac{\Gamma(r+1)}{2^r}, \\ EX_{1:2}^{r+1} = \dfrac{\Gamma(r+2)}{2^{r+1}}. \end{cases} \tag{2.12}$$

Remark 4. If $X \sim \mathrm{N\,Exp}(1)$ then

$$\begin{cases} E\left[-X_{2:2}\right]^{r} = \dfrac{\Gamma(r+1)}{2^r}, \\ E\left[-X_{2:2}\right]^{r+1} = \dfrac{\Gamma(r+2)}{2^{r+1}}. \end{cases} \tag{2.14}$$

Remark 5. If $X \sim \mathrm{I\,Exp}(1)$ then

$$\begin{cases} E[X_{2:2}]^{-r} = \dfrac{\Gamma(r+1)}{2^r}, \\ E[X_{2:2}]^{-(r+1)} = \dfrac{\Gamma(r+2)}{2^{r+2}}. \end{cases}$$

3. Goodness-of-Fit Tests Based on Conditions in Remarks 2–5

3.1. Parameters of F are Specified

Firstly, we construct tests of $H_0 : X \sim F$ based on (2.8)–(2.8*).

We write

$$U_k := X_{1:k} = \min(X_1, \ldots, X_k), \qquad V_k := X_{k:k} = \max(X_1, \ldots, X_k),$$

and we define

$$R_k^{(r)} = k^r[-\log(1-F(U_k))]^r, \qquad {R^*}_k^{(r)} = k^r[-\log F(V_k)]^r.$$

Put

$$Y = R_k^{(r+1)}, \quad Z = R_k^{(r)}, \quad Y^* = R_k^{*(r+1)}, \quad Z^* = R_k^{*(r)}.$$

We use the following statements.

Lemma 1. *Suppose that $X \sim \mathrm{Exp}(1)$. Then if $r > -\frac{1}{2}$ the joint distribution of (Y, Z) is a singular distribution concentrated on the line $C : z = y^{r/(r+1)}$ with df*

$$F(y,z) = P[Y < y] = 1 - \exp\left(-y^{1/(r+1)}\right), \qquad (y,z) \in C.$$

Moreover,

$$\begin{aligned} &EY = ER_k^{(r+1)} = \Gamma(r+2), \qquad EZ = ER_k^{(r)} = \Gamma(r+1),\\ &\operatorname{Var} Y = \operatorname{Var} R_k^{(r+1)} = \Gamma(2r+3) - \Gamma^2(r+2),\\ &\operatorname{Var} Z = \operatorname{Var} R_k^{(r)} = \Gamma(2r+1) - \Gamma^2(r+1),\\ &E(YZ) = \Gamma(2r+2), \quad \operatorname{Cov}(Y,Z) = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2). \end{aligned}$$

Proof. As $X \sim \mathrm{Exp}(1)$ then

$$R_k^{(r)} = k^r X_{1:k}^r \overset{D}{=} W^r \qquad (D - \text{in distribution}),$$

where $W \sim \mathrm{Exp}(1)$.

The joint df of $Y = R_k^{(r+1)} \overset{D}{=} W^{r+1}$ and $Z = R_k^{(r)} \overset{D}{=} W^r$ is concentrated on the line $C : z = y^{r/(r+1)}$ and has the form

$$\begin{aligned} F(y,z) &= P[Y \le y,\, Z \le z] = P[Y \le y] = P\left[W \le y^{1/(r+1)}\right]\\ &= 1 - \exp\left(-y^{1/(r+1)}\right) \quad \text{for} \quad (y,z) \in C. \end{aligned}$$

Next

$$\begin{aligned} &EY = EW^{r+1} = \Gamma(r+2), \qquad EY^2 = EW^{2(r+1)} = \Gamma(2r+3),\\ &\operatorname{Var} Y = \Gamma(2r+3) - \Gamma^2(r+2),\\ &EZ = \Gamma(r+1), \qquad EZ^2 = \Gamma(2r+1),\\ &\operatorname{Var} Z = \Gamma(2r+1) - \Gamma^2(r+1), \qquad EYZ = \Gamma(2r+2),\\ &\operatorname{Cov}(Y,Z) = \Gamma(2r+2) - \Gamma(r+2)\Gamma(r+1), \end{aligned}$$

which ends the proof of Lemma 1. ∎

Suppose now that we have a sample $X_1, \dots, X_n$ of size $n = kN$. This provides the samples $R_{k1}^{(r)}, \dots, R_{kN}^{(r)}$ and $R^{*(r)}_{k1}, \dots, R^{*(r)}_{kN}$, where

$$R_{kj}^{(r)} = k^r h_r(U_{kj}), \qquad R^{*(r)}_{kj} = k^r h_r^*(V_{kj}), \qquad j = 1, \dots, N,$$

with

$$U_{kj} = \min(X_{k(j-1)+1}, \dots, X_{kj}), \qquad V_{kj} = \max(X_{k(j-1)+1}, \dots, X_{kj}),$$

respectively.

Now we define

$$\boldsymbol{W}_{kj}^{(r)} = \begin{bmatrix} R_{kj}^{(r)} \\ R_{kj}^{(r+1)} \end{bmatrix}, \qquad \boldsymbol{W}^{*(r)}_{kj} = \begin{bmatrix} R^{*(r)}_{kj} \\ R^{*(r+1)}_{kj} \end{bmatrix}, \quad j = 1, \dots, N.$$

We see that

$$\begin{aligned}
\boldsymbol{\mu}_k^{(r)} &:= E\boldsymbol{W}_{k1}^{(r)} = \begin{bmatrix} \Gamma(r+1) \\ \Gamma(r+2) \end{bmatrix} = \Gamma(r+1) \begin{bmatrix} 1 \\ r+1 \end{bmatrix} \\
\Sigma_k^{(r)} &:= \operatorname{Var}\left(\boldsymbol{W}_{k1}^{(r)}\right) \\
&= \begin{bmatrix} \Gamma(2r+1) - \Gamma^2(r+1) & \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \\ \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) & \Gamma(2r+3) - \Gamma^2(r+2) \end{bmatrix}, \\
\boldsymbol{\mu}_k^{*(r)} &= \boldsymbol{\mu}_k^{(r)}, \qquad \Sigma_k^{*(r)} = \Sigma_k^{(r)}.
\end{aligned}$$

We use

$$\begin{aligned}
\Sigma^{*(r)} = \Sigma^{(r)} := \Sigma_k^{(r)} &= \begin{bmatrix} a^{(r)} & b^{(r)} \\ b^{(r)} & c^{(r)} \end{bmatrix} \\
&= \begin{bmatrix} \Gamma(2r+1) - \Gamma^2(r+1) & \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \\ \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) & \Gamma(2r+3) - \Gamma^2(r+2) \end{bmatrix},
\end{aligned}$$

with

$$\begin{aligned}
a^{(r)} &= \Gamma(2r+1) - \Gamma^2(r+1)\,, \\
b^{(r)} &= \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\,, \\
c^{(r)} &= \Gamma(2r+3) - \Gamma^2(r+2)\,.
\end{aligned}$$

Note that

$$\Delta^{(r)} := \det\left(\Sigma^{(r)}\right) = a^{(r)}c^{(r)} - \left(b^{(r)}\right)^2$$

$$= \Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right) = \Delta^{*(r)},$$

and

$$\left(\Sigma^{(r)}\right)^{-1} = \frac{1}{\Delta^{(r)}} \begin{bmatrix} c^{(r)} & -b^{(r)} \\ -b^{(r)} & a^{(r)} \end{bmatrix} = \left(\Sigma^{*(r)}\right)^{-1}.$$

Write

$$\overline{\boldsymbol{W}}^{(r)}_{kN} = \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{W}^{(r)}_{kj}, \qquad \overline{\boldsymbol{W}^*}^{(r)}_{kN} = \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{W}^{*(r)}_{kj}.$$

The CLT says that

$$\sqrt{N}\left(\overline{\boldsymbol{W}}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right) \xrightarrow{D} \boldsymbol{W} \sim N\left(\mathbf{0}, \Sigma^{(r)}\right),$$
$$\sqrt{N}\left(\overline{\boldsymbol{W}^*}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right) \xrightarrow{D} \boldsymbol{W}^* \sim N\left(\mathbf{0}, \Sigma^{(r)}\right),$$

whence

$$T^{(r)}_{kN} = N\left(\overline{\boldsymbol{W}}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right)' \left(\Sigma^{(r)}\right)^{-1} \left(\overline{\boldsymbol{W}}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right) \xrightarrow{D} \chi^2(2), \tag{3.1}$$

and

$$T^{*(r)}_{kN} = N\left(\overline{\boldsymbol{W}^*}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right)' \left(\Sigma^{(r)}\right)^{-1} \left(\overline{\boldsymbol{W}^*}^{(r)}_{kN} - \boldsymbol{\mu}^{(r)}_k\right) \xrightarrow{D} \chi^2(2). \tag{3.2}$$

Provide simple asymptotic tests of H_0. These test–statistics can be written in extended forms as follows

$$T^{(r)}_{kN} = \frac{1}{\Delta^{(r)} N}\Bigg\{ c^{(r)}\left(k^r \sum_{j=1}^{N} h_r(U_{kj}) - N\Gamma(r+1)\right)^2$$
$$- 2b^{(r)}\left(k^r \sum_{j=1}^{N} h_r(U_{kj}) - N\Gamma(r+1)\right)\left(k^{r+1}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - N\Gamma(r+2)\right)$$
$$+ a^{(r)}\left(k^{r+1}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - N\Gamma(r+2)\right)^2\Bigg\},$$

or

$$T^{(r)}_{kN} = \frac{N}{\Delta^{(r)} c^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj})\left(c^{(r)} - b^{(r)} k h_1(U_{kj})\right)\right.$$

$$- \Gamma(r+1)\left(c^{(r)} - (r+1)b^{(r)}\right)\Bigg]^2 + \frac{N}{c^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2)\right]^2, \tag{3.3}$$

or

$$T_{kN}^{(r)} = \frac{N}{\Delta^{(r)}a^{(r)}}\Bigg[\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj})\left(a^{(r)}kh_1(U_{kj}) - b^{(r)}\right)$$

$$- \Gamma(r+1)\left((r+1)a^{(r)} - b^{(r)}\right)\Bigg]^2 + \frac{N}{a^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1)\right]^2, \tag{3.4}$$

and also

$$T_{kN}^{*(r)} = \frac{N}{\Delta^{(r)}c^{(r)}}\Bigg[\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj})\left(c^{(r)} - b^{(r)}kh_1^*(V_{kj})\right)$$

$$- \Gamma(r+1)\left(c^{(r)} - (r+1)b^{(r)}\right)\Bigg]^2 + \frac{N}{c^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N} h_{r+1}^*(V_{kj}) - \Gamma(r+2)\right]^2, \tag{3.5}$$

and

$$T_{kN}^{*(r)} = \frac{N}{\Delta^{(r)}a^{(r)}}\Bigg[\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj})\left(a^{(r)}kh_1^*(V_{kj}) - b^{(r)}\right)$$

$$- \Gamma(r+1)\left((r+1)a^{(r)} - b^{(r)}\right)\Bigg]^2 + \frac{N}{a^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N} h_r^*(V_{kj}) - \Gamma(r+1)\right]^2. \tag{3.6}$$

Replacing $\Delta^{(r)}$, $c^{(r)}$ and $b^{(r)}$ by their values gives

$$T_{kN}^{(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+3) - \Gamma^2(r+2)\right)}$$

$$\cdot \Bigg\{\frac{k^r}{N}\sum_{j=1}^{N} h_r(U_{kj})\left[\Gamma(2r+3) - \Gamma^2(r+2)\right)$$

$$- \left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right) kh_1(U_{kj})\big] - \Gamma(r+2)\Gamma(2r+2)\Bigg\}^2$$

$$+ \frac{N}{\Gamma(2r+3) - \Gamma^2(r+2)} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}(U_{kj}) - \Gamma(r+2) \right]^2 , \quad (3.7)$$

or

$$T_{kN}^{(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+1) - \Gamma^2(r+1)\right)}$$

$$\cdot \Bigg\{ \frac{k^r}{N} \sum_{j=1}^{N} h_r(U_{kj}) \big[\big(\Gamma(2r+1) - \Gamma^2(r+1)\big) kh_1(U_{kj})$$

$$-\Gamma(2r+2) + \Gamma(r+1)\Gamma(r+2)\big] + r\Gamma(r+1)\Gamma(2r+1) \Bigg\}^2$$

$$+ \frac{N}{\Gamma(2r+1) - \Gamma^2(r+1)} \left[\frac{k^r}{N} \sum_{j=1}^{N} h_r(U_{kj}) - \Gamma(r+1) \right]^2 , \quad (3.8)$$

and also

$$T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+3) - \Gamma^2(r+2)\right)}$$

$$\cdot \Bigg\{ \frac{k^r}{N} \sum_{j=1}^{N} h_r^*(V_{kj}) \big[\Gamma(2r+3) - \Gamma^2(r+2)$$

$$- \left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right) kh_1^*(V_{kj})\big] - \Gamma(r+2)\Gamma(2r+2)\Bigg\}^2$$

$$+ \frac{N}{\Gamma(2r+3) - \Gamma^2(r+2)} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} h_{r+1}^*(V_{kj}) - \Gamma(r+2) \right]^2 , \quad (3.9)$$

or

$$T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+1) - \Gamma^2(r+1)\right)}$$

$$\cdot \Bigg\{ \frac{k^r}{N} \sum_{j=1}^{N} h_r^*(V_{kj}) \big[\big(\Gamma(2r+1) - \Gamma^2(r+1)\big) kh_1^*(V_{kj})$$

$$-\Gamma(2r+2) + \Gamma(r+1)\Gamma(r+2)\big] + r\Gamma(r+1)\Gamma(2r+1) \Bigg\}^2$$

$$+\frac{N}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{k^r}{N}\sum_{j=1}^{N}h_r^*(V_{kj})-\Gamma(r+1)\right]^2. \quad (3.10)$$

When $k=1$ the test–statistics are

$$T_{1n}^{(r)}=\frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot\left\{\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left.-\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)h_1(X_j)\right]-\Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+\frac{n}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{1}{n}\sum_{j=1}^{n}h_{r+1}(X_j)-\Gamma(r+2)\right]^2, \quad (3.11)$$

or

$$T_{1n}^{(r)}=\frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot\left\{\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)h_1(X_j)\right.\right.$$
$$\left.\left.-\Gamma(2r+2)+\Gamma(r+1)\Gamma(r+2)\right]+r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+\frac{n}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{1}{n}\sum_{j=1}^{n}h_r(X_j)-\Gamma(r+1)\right]^2, \quad (3.12)$$

and also

$$T_{1n}^{*(r)}=\frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot\left\{\frac{1}{n}\sum_{j=1}^{n}h_r^*(X_j)\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left.-\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)h_1^*(X_j)\right]-\Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+\frac{n}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{1}{n}\sum_{j=1}^{n}h_{r+1}^*(X_j)-\Gamma(r+2)\right]^2, \quad (3.13)$$

or

$$T_{1n}^{*(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot \left\{ \frac{1}{n}\sum_{j=1}^{n} h_r^*(X_j)\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)h_1^*(X_j)\right.\right.$$
$$\left.\left. -\Gamma(2r+2)+\Gamma(r+1)\Gamma(r+2)\right] + r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+ \frac{n}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{1}{n}\sum_{j=1}^{n} h_r^*(X_j) - \Gamma(r+1)\right]^2 . \quad (3.14)$$

Examples. 1. For $H_0 : X \sim \mathrm{Exp}(\alpha)$ the tests in (3.7) and (3.8) are as follows

$$T_{kN}^{(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot \left\{ \frac{k^r\alpha^r}{N}\sum_{j=1}^{N} U_{kj}^r\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left. -\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)k\alpha U_{kj}\right] - \Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+ \frac{N}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{k^{r+1}\alpha^{r+1}}{N}\sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2)\right]^2 ,$$

and

$$T_{kN}^{(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot \left\{ \frac{k^r\alpha^r}{N}\sum_{j=1}^{N} U_{kj}^r\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)kU_{kj}\right.\right.$$
$$\left.\left. -\Gamma(2r+2)+\Gamma(r+1)\Gamma(r+2)\right] + r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+ \frac{N}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{k^r\alpha^r}{N}\sum_{j=1}^{N} U_{kj}^r - \Gamma(r+1)\right]^2 .$$

For $k = 1$

$$T_{1n}^{(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot\left\{\frac{\alpha^r}{n}\sum_{j=1}^{n}X_j\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left.-\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)\alpha X_j\right]-\Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+\frac{n}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{\alpha^{r+1}}{n}\sum_{j=1}^{n}X_j^{r+1}-\Gamma(r+2)\right]^2,$$

and

$$T_{1n}^{(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot\left\{\frac{\alpha^r}{n}\sum_{j=1}^{n}X_j^r\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)\alpha X_j-\Gamma(2r+2)\right.\right.$$
$$\left.\left.+\Gamma(r+1)\Gamma(r+2)\right]+r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+\frac{n}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{\alpha^r}{n}\sum_{j=1}^{n}X_j^r-\Gamma(r+1)\right]^2.$$

2. For $H_0 : X \sim \mathrm{N\,Exp}(\lambda)$ the tests in (3.9) and (3.10) are as follows

$$T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot\left\{\frac{k^r\lambda^r}{N}\sum_{j=1}^{N}(-V_{kj})^r\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left.-\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)k\lambda(-V_{kj})\right]-\Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+\frac{N}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{k^{r+1}\lambda^{r+1}}{N}\sum_{j=1}^{N}(-V_{kj})^{r+1}-\Gamma(r+2)\right]^2,$$

and

$$T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot\left\{\frac{k^r\lambda^r}{N}\sum_{j=1}^{N}(-V_{kj})^r\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)k\lambda(-V_{kj})-\Gamma(2r+2)\right.\right.$$
$$\left.\left.+\Gamma(r+1)\Gamma(r+2)\right]+r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+\frac{N}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{k^r\lambda^r}{N}\sum_{j=1}^{N}(-V_{kj})^r-\Gamma(r+1)\right]^2.$$

For $k=1$

$$T_{1n}^{*(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$
$$\cdot\left\{\frac{\lambda^r}{n}\sum_{j=1}^{n}(-X_j)^r\left[\Gamma(2r+3)-\Gamma^2(r+2)\right.\right.$$
$$\left.\left.-\left(\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right)\lambda X_j\right]-\Gamma(r+2)\Gamma(2r+2)\right\}^2$$
$$+\frac{n}{\Gamma(2r+3)-\Gamma^2(r+2)}\left[\frac{\lambda^{r+1}}{n}\sum_{j=1}^{n}(-X_j)^{r+1}-\Gamma(r+2)\right]^2,$$

and

$$T_{1n}^{*(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)}$$
$$\cdot\left\{\frac{\lambda^r}{n}\sum_{j=1}^{n}(-X_j)^r\left[\left(\Gamma(2r+1)-\Gamma^2(r+1)\right)\lambda(-X_j)\right.\right.$$
$$\left.\left.-\Gamma(2r+2)+\Gamma(r+1)\Gamma(r+2)\right]+r\Gamma(r+1)\Gamma(2r+1)\right\}^2$$
$$+\frac{n}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{\lambda^r}{n}\sum_{j=1}^{n}(-X_j)^r-\Gamma(r+1)\right]^2.$$

3. For $H_0 : X \sim \mathrm{I\,Exp}(\theta)$ the tests in (3.9) and (3.10) are as follows

$$T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2)-\Gamma^2(r+2)\right)\left(\Gamma(2r+3)-\Gamma^2(r+2)\right)}$$

$$
\cdot \left\{ \frac{k^r\theta^r}{N} \sum_{j=1}^{N} \left(\frac{1}{V_{kj}} \right)^r \Big[\Gamma(2r+3) - \Gamma^2(r+2) \right.
$$

$$
\left. - \left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \right) k\theta \frac{1}{V_{kj}} \Big] - \Gamma(r+2)\Gamma(2r+2) \right\}^2
$$

$$
+ \frac{N}{\Gamma(2r+3) - \Gamma^2(r+2)} \left[\frac{k^{r+1}\theta^{r+1}}{N} \sum_{j=1}^{N} \left(\frac{1}{V_{kj}} \right)^{r+1} - \Gamma(r+2) \right]^2,
$$

and

$$
T_{kN}^{*(r)} = \frac{N}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+1) - \Gamma^2(r+1)\right)}
$$

$$
\cdot \left\{ \frac{k^r\theta^r}{N} \sum_{j=1}^{N} \left(\frac{1}{V_{kj}} \right)^r \Big[\left(\Gamma(2r+1) - \Gamma^2(r+1) \right) \frac{k\theta}{V_{kj}} - \Gamma(2r+2) \right.
$$

$$
\left. + \Gamma(r+1)\Gamma(r+2) \Big] + r\Gamma(r+1)\Gamma(2r+1) \right\}^2
$$

$$
+ \frac{N}{\Gamma(2r+1) - \Gamma^2(r+1)} \left[\frac{k^r\theta^r}{N} \sum_{j=1}^{N} \left(\frac{1}{V_{kj}} \right)^r - \Gamma(r+1) \right]^2.
$$

For $k = 1$

$$
T_{1n}^{*(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+3) - \Gamma^2(r+2)\right)}
$$

$$
\cdot \left\{ \frac{\theta^r}{n} \sum_{j=1}^{n} \left(\frac{1}{X_j} \right)^r \Big[\Gamma(2r+3) - \Gamma^2(r+2) \right.
$$

$$
\left. - \left(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \right) \frac{\theta}{X_j} \Big] - \Gamma(r+2)\Gamma(2r+2) \right\}^2
$$

$$
+ \frac{n}{\Gamma(2r+3) - \Gamma^2(r+2)} \left[\frac{\theta^{r+1}}{n} \sum_{j=1}^{n} \left(\frac{1}{X_j} \right)^{r+1} - \Gamma(r+2) \right]^2,
$$

and

$$
T_{1n}^{*(r)} = \frac{n}{\Gamma(2r+1)\left(\Gamma(2r+2) - \Gamma^2(r+2)\right)\left(\Gamma(2r+1) - \Gamma^2(r+1)\right)}
$$

$$
\cdot \left\{ \frac{\theta^r}{n} \sum_{j=1}^{n} \left(\frac{1}{X_j} \right)^r \Big[\left(\Gamma(2r+1) - \Gamma^2(r+1) \right) \frac{\theta}{X_j} - \Gamma(2r+2) \right.
$$

$$+\Gamma(r+1)\Gamma(r+2)] + r\Gamma(r+1)\Gamma(2r+1)\Big\}^2$$

$$+\frac{n}{\Gamma(2r+1)-\Gamma^2(r+1)}\left[\frac{\theta^r}{n}\sum_{j=1}^{n}\left(\frac{1}{X_j}\right)^r - \Gamma(r+1)\right]^2 .$$

3.2. Hypotheses Involving Unknown Parameters

With the usual notation and assumptions F now has the form $F(x;\boldsymbol{\lambda})$ and $\boldsymbol{\lambda}(p\times 1)$ are unknown identifiable parameters with true value $\boldsymbol{\lambda}_0$ in the parameter space Λ, the pdf is denoted by $f(x;\boldsymbol{\lambda})$. In this case we need the following theorem.

Theorem 3. *Let $\widehat{\boldsymbol{T}}_n = \boldsymbol{T}_n(X_1,\ldots,X_n;\widehat{\boldsymbol{\lambda}}_n)$, where $\widehat{\boldsymbol{\lambda}}_n = \widehat{\boldsymbol{\lambda}}_n(X_1,\ldots,X_n)$ is an estimator of a parameter $\boldsymbol{\lambda}$, and moreover let $\boldsymbol{T}_n = \boldsymbol{T}_n(X_1,\ldots,X_n;\boldsymbol{\lambda})$ (here $\boldsymbol{T}_n$, $\boldsymbol{\lambda}$ and $\widehat{\boldsymbol{\lambda}}_n$ may be vectors). Suppose that:*

(i) *For each $\boldsymbol{\lambda}$,*

$$\sqrt{n}\begin{bmatrix}\boldsymbol{T}_n\\ \widehat{\boldsymbol{\lambda}}_n-\boldsymbol{\lambda}\end{bmatrix} \xrightarrow{D} \boldsymbol{T} \sim N(\mathbf{0},\boldsymbol{V}),$$

where

$$\boldsymbol{V} = \begin{bmatrix}\boldsymbol{V}_{11} & \boldsymbol{V}_{12}\\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22}\end{bmatrix}$$

and $\boldsymbol{V}_{22}$ is nonsingular.

(ii) *There is a matrix $\boldsymbol{B}$, possibly depending continuously on $\boldsymbol{\lambda}$, such that*

$$\sqrt{n}\widehat{\boldsymbol{T}}_n = \sqrt{n}\boldsymbol{T}_n + \boldsymbol{B}\sqrt{n}\left(\widehat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}_n\right) + o_p(1).$$

(iii) *$\widehat{\boldsymbol{\lambda}}_n$ is asymptotically efficient. Then*

$$\sqrt{n}\widehat{\boldsymbol{T}}_n \xrightarrow{D} \boldsymbol{T}^* \sim N(\mathbf{0},\boldsymbol{V}_{11}-\boldsymbol{B}\boldsymbol{V}_{22}\boldsymbol{B}')$$

(cf. Pierce [18]).

Note that (ii) is satisfied when $\boldsymbol{T}_n$ is differentiable in $\boldsymbol{\lambda}$, and then

$$\boldsymbol{B} = \lim_{n\to\infty} E\left[\frac{\partial}{\partial\boldsymbol{\lambda}}\boldsymbol{T}_n\right].$$

Next from (**A**) preceding (3.1), when $\boldsymbol{\lambda}=\boldsymbol{\lambda}_0$,

$$D_{kN}^{(r)}(\boldsymbol{\lambda}_0) = \sqrt{N}\left(\overline{\boldsymbol{W}}_{kN}^{(r)}(\boldsymbol{\lambda}_0) - \boldsymbol{\mu}_k^{(r)}\right) \xrightarrow{D} D_k^{(r)} \sim N\left(\mathbf{0},\Sigma^{(r)}\right).$$

When $\boldsymbol{\lambda}$ is replaced by an estimator $\hat{\boldsymbol{\lambda}}_n$ (here we use MLE) for which

$$\sqrt{n}\left(\hat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}_0\right) \xrightarrow{D} N(\mathbf{0}, A(\boldsymbol{\lambda}_0)),$$

where in the regular case $A(\boldsymbol{\lambda}) = \mathcal{I}^{-1}(\boldsymbol{\lambda})$ and $\mathcal{I}$ is the expected information matrix based on a single observation, and in other cases $A(\boldsymbol{\lambda})$ is singular, the quoted theorem gives

$$\Sigma_{k1}^{(r)} = \Sigma^{(r)} - \frac{1}{k}\boldsymbol{B}_k^{(r)} A(\boldsymbol{\lambda}_0)\left(\boldsymbol{B}_k^{(r)}\right)' = \Sigma^{(r)} - \boldsymbol{K}_k^{(r)}\,, \tag{3.15}$$

with

$$\boldsymbol{B}_k^{(r)}(2\times p) = E\left[\left(\frac{\partial \overline{\boldsymbol{W}}_{kN}^{(r)}(\boldsymbol{\lambda})}{\partial \boldsymbol{\lambda}}\right)_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_0}\right],$$

and

$$\boldsymbol{K}_k^{(r)} = \boldsymbol{B}_k^{(r)}\,(k\mathcal{I})^{-1}\left(\boldsymbol{B}_k^{(r)}\right)' \tag{3.16}$$

in the regular case.

Now

$$\begin{aligned} E\left[\frac{\partial \overline{W}_{kN1}^{(r)}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] &= E\left[\frac{\partial}{\partial \lambda_j} k^r h_r(X_{1:k};\boldsymbol{\lambda})\right] \\ &= rk^r E\left[h_{r-1}(X_{1:k};\boldsymbol{\lambda})\frac{1}{1-F(X_{1:k};\boldsymbol{\lambda})}\frac{\partial F(X_{1:k};\boldsymbol{\lambda})}{\partial \lambda_j}\right], \end{aligned}$$

and since $X_{1:k}$ has pdf $k(1-F)^{k-1}f$ then

$$\begin{aligned} E\left[\frac{\partial \overline{W}_{kN1}^{(r)}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] &= rk^{r+1}\int h_{r-1}(x)(1-F(x;\boldsymbol{\lambda}))^{k-2}\frac{\partial F(x;\boldsymbol{\lambda})}{\partial \lambda_j} f(x;\boldsymbol{\lambda})dx \\ &= rk^{r+1}E\left[(1-F(X;\boldsymbol{\lambda}))^{k-2}h_{r-1}(X)\frac{\partial F(X;\boldsymbol{\lambda})}{\partial \lambda_j}\right] = rb_k^{(r)}(\lambda_j)\,, \end{aligned}$$

where we set

$$b_k^{(r)}(\lambda_j) = k^{r+1}E\left[(1-F(X;\boldsymbol{\lambda}))^{k-2}h_{r-1}(X)\frac{\partial F(X;\boldsymbol{\lambda})}{\partial \lambda_j}\right],$$

$j = 1, \ldots, p$.

Similarly,

$$E\left[\frac{\partial \overline{W}_{kN2}^{(r)}(\boldsymbol{\lambda})}{\partial \lambda_j}\right] = (r+1)b_k^{(r+1)}(\lambda_j).$$

Thus

$$\boldsymbol{B}_k^{(r)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{(r)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{(r+1)}\right)' \end{bmatrix}.$$

Then a simple asymptotic test of H_0 is obtained from

$$\hat{T}_{kN}^{(r)} = N\left(\overline{\boldsymbol{W}}_{kN}^{(r)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r)}\right)'\left(\Sigma_{k1}^{(r)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{(r)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r)}\right) \xrightarrow{D} \chi^2(2)\,, \quad (3.17)$$

on H_0, where

$$\overline{\boldsymbol{W}}_{kN}^{(r)}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} R_{kj}^{(r)}(\boldsymbol{\lambda}) \\ R_{kj}^{(r+1)}(\boldsymbol{\lambda}) \end{bmatrix}, \quad R_{kj}^{(r)}(\boldsymbol{\lambda}) = k^r\left[-\log(1 - F(U_{kj};\boldsymbol{\lambda}))\right]^r.$$

The corresponding dual test, obtained by replacing $1 - F(x;\boldsymbol{\lambda})$ by $F(x;\boldsymbol{\lambda})$, and so $1 - F(U_{kj};\boldsymbol{\lambda})$ by $F(V_{kj};\boldsymbol{\lambda})$ uses

$$\hat{T}_{kN}^{*(r)} = N\left(\overline{\boldsymbol{W}}_{kN}^{*(r)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r)}\right)'\left(\Sigma_{k1}^{*(r)}\right)^{-1}\left(\overline{\boldsymbol{W}}_{kN}^{*(r)}(\hat{\boldsymbol{\lambda}}_n) - \boldsymbol{\mu}_k^{(r)}\right) \xrightarrow{D} \chi^2(2) \quad (3.18)$$

on H_0, where

$$\overline{\boldsymbol{W}}_{kN}^{*(r)}(\boldsymbol{\lambda}) = \frac{1}{N}\sum_{j=1}^{N}\begin{bmatrix} (R_{kj}^{*})^{(r)}(\boldsymbol{\lambda}) \\ (R_{kj}^{*})^{(r+1)}(\boldsymbol{\lambda}) \end{bmatrix}, \quad R_{kj}^{*(r)}(\boldsymbol{\lambda}) = k^r\left[-\log(F(V_{kj};\boldsymbol{\lambda}))\right]^r,$$

$$\Sigma_{k1}^{*(r)} = \Sigma^{(r)} - \boldsymbol{K}_k^{*(r)}, \quad (3.19)$$

with

$$\boldsymbol{K}_k^{*(r)} = \frac{1}{k}\boldsymbol{B}_k^{*(r)}(\mathcal{I})^{-1}\left(\boldsymbol{B}_k^{*(r)}\right)' \quad (3.20)$$

and

$$\boldsymbol{B}_k^{*(r)} = \begin{bmatrix} r\left(\boldsymbol{b}_k^{*(r)}\right)' \\ (r+1)\left(\boldsymbol{b}_k^{*(r+1)}\right)' \end{bmatrix},$$

where

$$b_k^{*(r)}(\lambda_j) = -k^{r+1}E\left[F^{k-2}(X;\boldsymbol{\lambda})h_{r-1}^{*}(X;\boldsymbol{\lambda})\frac{\partial F(X;\boldsymbol{\lambda})}{\partial \lambda_j}\right],$$

$j = 1, \ldots, p$.

We write

$$\begin{aligned}
\boldsymbol{K}_k^{(r)} &:= \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)} \end{bmatrix}, \qquad \boldsymbol{K}_k^{*(r)} := \begin{bmatrix} s_k^{*(r)} & t_k^{*(r)} \\ t_k^{*(r)} & u_k^{*(r)} \end{bmatrix}, \\
\triangledown_k^{(r)} &:= \det\left(\boldsymbol{K}_k^{(r)}\right), \qquad \triangledown_k^{*(r)} = \det\left(\boldsymbol{K}_k^{*(r)}\right), \\
\Sigma_{k1}^{(r)} &= \Sigma^{(r)} - \boldsymbol{K}_k^{(r)} = \begin{bmatrix} a_{k1}^{(r)} & b_{k1}^{(r)} \\ b_{k1}^{(r)} & c_{k1}^{(r)} \end{bmatrix} = \begin{bmatrix} a^{(r)} - s_k^{(r)} & b^{(r)} - t_k^{(r)} \\ b^{(r)} - t_k^{(r)} & c^{(r)} - u_k^{(r)} \end{bmatrix}, \\
\Sigma_{k1}^{*(r)} &= \Sigma^{(r)} - \boldsymbol{K}_k^{*(r)} = \begin{bmatrix} a_{k1}^{*(r)} & b_{k1}^{*(r)} \\ b_{k1}^{*(r)} & c_{k1}^{*(r)} \end{bmatrix} = \begin{bmatrix} a^{(r)} - s_k^{*(r)} & b^{(r)} - t_k^{*(r)} \\ b^{(r)} - t_k^{*(r)} & c^{(r)} - u_k^{*(r)} \end{bmatrix}.
\end{aligned}$$

Since in many special cases $\boldsymbol{K}_k^{(r)}$ and $\boldsymbol{K}_k^{*(r)}$ do not depend on $\boldsymbol{\lambda}_0$ and in a consequently $\Sigma_{k1}^{(r)}$ and $\Sigma_{k1}^{*(r)}$ do not depend on $\boldsymbol{\lambda}_0$ we write $a_{k1}^{(r)}$, $b_{k1}^{(r)}$ and $c_{k1}^{(r)}$ instead of $a_{k1}^{(r)}(\boldsymbol{\lambda}_0)$, $b_{k1}^{(r)}(\boldsymbol{\lambda}_0)$, $c_{k1}^{(r)}(\boldsymbol{\lambda}_0)$ and so on.

Note that

$$\begin{aligned}
\Delta_{k1}^{(r)} &= \Delta^{(r)} + \triangledown_k^{(r)} - a^{(r)} u_k^{(r)} - c^{(r)} s_k^{(r)} + 2b^{(r)} t_k^{(r)} \\
&= \Gamma(2r+1)\left[\Gamma(2r+2) - \Gamma^2(r+2)\right] + \triangledown_k^{(r)} \\
&- \left[\Gamma(2r+1) - \Gamma^2(r+1)\right] u_k^{(r)} - \left[\Gamma(2r+3) - \Gamma^2(r+2)\right] s_k^{(r)} \\
&+ 2\left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right] t_k^{(r)}
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\Delta_{k1}^{*(r)} &= \Gamma(2r+1)\left[\Gamma(2r+2) - \Gamma^2(r+2)\right] + \triangledown_k^{*(r)} \\
&- \left[\Gamma(2r+1) - \Gamma^2(r+1)\right] u_k^{*(r)} - \left[\Gamma(2r+3) - \Gamma^2(r+2)\right] s_k^{*(r)} \\
&+ 2\left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right] t_k^{*(r)}.
\end{aligned} \tag{3.22}$$

Using the above notations we get the following tests, which correspond to (3.3)–(3.6)

$$\begin{aligned}
\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} c_{k1}^{(r)}} \Bigg[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) \left(c_{k1}^{(r)} - b_{k1}^{(r)} k \hat{h}_1(U_{kj}) \right) \\
- \Gamma(r+1)\left(c_{k1}^{(r)} - (r+1) b_{k1}^{(r)} \right) \Bigg]^2 \\
+ \frac{N}{c_{k1}^{(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}(U_{kj}) - \Gamma(r+2) \right]^2,
\end{aligned} \tag{3.23}$$

or

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} a_{k1}^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) \left(a_{k1}^{(r)} k \hat{h}_1(U_{kj}) - b_{k1}^{(r)} \right) - \Gamma(r+1) \left((r+1) a_{k1}^{(r)} - b_{k1}^{(r)} \right) \right]^2 + \frac{N}{a_{k1}^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r(U_{kj}) - \Gamma(r+1) \right]^2, \quad (3.24)$$

and also

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} c_{k1}^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) \left(c_{k1}^{*(r)} - b_{k1}^{*(r)} k \hat{h}_1^*(V_{kj}) \right) - \Gamma(r+1) \left(c_{k1}^{*(r)} - (r+1) b_{k1}^{*(r)} \right) \right]^2 + \frac{N}{c_{k1}^{*(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \hat{h}_{r+1}^*(V_{kj}) - \Gamma(r+2) \right]^2, \quad (3.25)$$

or

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} a_{k1}^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) \left(a_{k1}^{*(r)} k \hat{h}_1^*(V_{kj}) - b_{k1}^{*(r)} \right) - \Gamma(r+1) \left((r+1) a_{k1}^{*(r)} - b_{k1}^{*(r)} \right) \right]^2 + \frac{N}{a_{k1}^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \hat{h}_r^*(V_{kj}) - \Gamma(r+1) \right]^2 \quad (3.26)$$

provided $\Delta_{k1}^{(r)} c_{k1}^{(r)} > 0$, $\Delta_{k1}^{(r)} a_{k1}^{(r)} > 0$, $\Delta_{k1}^{*(r)} c_{k1}^{*(r)} > 0$ and $\Delta_{k1}^{*(r)} a_{k1}^{*(r)} > 0$, respectively, where $\hat{h}_r(\cdot) = h_r\left(\cdot, \hat{\boldsymbol{\lambda}}\right)$. For $k = 1$ we have the tests:

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} c_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \hat{h}_r(X_j) \left(c_{11}^{(r)} - b_{11}^{(r)} \hat{h}_1(X_j) \right) \right.$$

$$- \Gamma(r+1)\left(c_{11}^{(r)} - (r+1)b_{11}^{(r)}\right)\Bigg]^2$$

$$+ \frac{n}{c_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}(X_j) - \Gamma(r+2)\right]^2, \quad (3.27)$$

or

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} a_{11}^{(r)}}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j)\left(a_{11}^{(r)}\hat{h}_1(X_j) - b_{11}^{(r)}\right)$$

$$- \Gamma(r+1)\left((r+1)a_{11}^{(r)} - b_{11}^{(r)}\right)\Bigg]^2$$

$$+ \frac{n}{a_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r(X_j) - \Gamma(r+1)\right]^2, \quad (3.28)$$

and also the tests

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} c_{11}^{*(r)}}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r^*(X_j)\left(c_{11}^{*(r)} - b_{11}^{*(r)}\hat{h}_1^*(X_j)\right)$$

$$- \Gamma(r+1)\left(c_{11}^{*(r)} - (r+1)b_{11}^{*(r)}\right)\Bigg]^2$$

$$+ \frac{n}{c_{11}^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_{r+1}^*(X_j) - \Gamma(r+2)\right]^2, \quad (3.29)$$

or

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} a_{11}^{*(r)}}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r^*(X_j)\left(a_{11}^{*(r)}\hat{h}_1^*(X_j) - b_{11}^{*(r)}\right)$$

$$- \Gamma(r+1)\left((r+1)a_{11}^{*(r)} - b_{11}^{*(r)}\right)\Bigg]^2$$

$$+ \frac{n}{a_{11}^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\hat{h}_r^*(X_j) - \Gamma(r+1)\right]^2. \quad (3.30)$$

Note that tests (3.23)-(3.26) and their particular forms (3.27)-(3.30) generate components which can be used as separate test-statistics. We do not discuss those partitions in detail. We discuss only one component for the exponential distribution in Section 5(**i**) to underline the importance of that approach.

4. Special Cases

4.1. Exponential Distributions: $X \sim \mathrm{Exp}(\alpha)$

Here

$$\begin{aligned} &f(x) = \alpha e^{-\alpha x}, \quad x > 0; \quad \Lambda = \{\alpha : \alpha > 0\} \\ &F(x;\alpha) = 1 - e^{-\alpha x}, \quad h(x) = \alpha x \\ &\frac{\partial F}{\partial \alpha} = x e^{-\alpha x}, \qquad \mathcal{I}^{-1} = \alpha^2 \end{aligned}$$

and

$$\hat{\alpha}_n = 1/\overline{X}_n.$$

Moreover,

$$\begin{aligned} b_k^{(r)}(\alpha) = k^{r+1} E\left[(1 - F(X;\alpha))^{k-2} \log^{r-1}\left(\frac{1}{1 - F(X;\alpha)}\right) \frac{\partial F(X;\alpha)}{\partial \alpha}\right] \\ = \frac{1}{\alpha}\Gamma(r+1), \end{aligned}$$

$$\begin{aligned} \boldsymbol{B}_k^{(r)} = \begin{bmatrix} r\left(b_k^{(r)}(\alpha)\right)' \\ (r+1)\left(b_k^{(r+1)}(\alpha)\right)' \end{bmatrix} = \frac{1}{\alpha}\begin{bmatrix} r\Gamma(r+1) \\ (r+1)\Gamma(r+2) \end{bmatrix} \\ = \frac{\Gamma(r+1)}{\alpha}\begin{bmatrix} r \\ (r+1)^2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{K}_r = \frac{1}{k}\boldsymbol{B}_k^{(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{(r)}\right)' = \frac{1}{k}\frac{\Gamma(r+1)}{\alpha}\begin{bmatrix} r \\ (r+1)^2 \end{bmatrix}\alpha^2\frac{\Gamma(r+1)}{\alpha}\begin{bmatrix} r & (r+1)^2 \end{bmatrix} \\ = \frac{\Gamma^2(r+1)}{k}\begin{bmatrix} r^2 & r(r+1)^2 \\ r(r+1)^2 & (r+1)^4 \end{bmatrix} := \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)} \end{bmatrix}, \end{aligned}$$

and

$$\Sigma_{k1}^{(r)} = \begin{bmatrix} a_{k1}^{(r)} & b_{k1}^{(r)} \\ b_{k1}^{(r)} & c_{k1}^{(r)} \end{bmatrix}$$

with

$$
\begin{aligned}
a_{k1}^{(r)} &= a^{(r)} - s_k^{(r)} = \frac{k\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^2\Gamma^2(r+1)}{k}, \\
b_{k1}^{(r)} &= b^{(r)} - t_k^{(r)} = \frac{k\left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right] - r\Gamma^2(r+2)}{k}, \\
c_{k1}^{(r)} &= c^{(r)} - u_k^{(r)} = \frac{k\left[\Gamma(2r+3) - \Gamma^2(r+2)\right] - (r+1)^2\Gamma^2(r+2)}{k}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\nabla_k^{(r)} &= \det\left(\boldsymbol{K}_r\right) = 0, \\
\Delta_{k1}^{(r)} &= \frac{1}{k}\left\{\left[\Gamma(2r+2) - \Gamma^2(r+2)\right]\left[k\Gamma(2r+1) - \Gamma^2(r+1)\right]\right. \\
&\quad \left. - r^4\Gamma^2(r+1)\Gamma(2r+1)\right\},
\end{aligned}
$$

$$
\Gamma(r+1)\left(c_{k1}^{(r)} - (r+1)b_{k1}^{(r)}\right) = \frac{\Gamma(r+2)}{k}\left[k\Gamma(2r+2) - \Gamma^2(r+2)\right],
$$

$$
\begin{aligned}
&\Gamma(r+1)\left((r+1)a_{k1}^{(r)} - b_{k1}^{(r)}\right) \\
&\qquad = -\frac{r\Gamma(r+1)}{k}\left[k\Gamma(2r+1) - \Gamma(r+1)\Gamma(r+2)\right].
\end{aligned}
$$

Hence for $r = 1$ we have

$$
\Sigma_{k1}^{(1)} = \begin{bmatrix} 1 - \frac{1}{k} & 4\left(1 - \frac{1}{k}\right) \\ 4\left(1 - \frac{1}{k}\right) & 4\left(5 - \frac{4}{k}\right) \end{bmatrix}, \quad \Delta_{k1}^{(1)} = 4\left(1 - \frac{1}{k}\right), \quad \Delta_{11}^{(1)} = 0.
$$

Excluding the case $k = r = 1$, where $\Sigma_{11}^{(1)}$ is singular, (3.23) and (3.24) give

$$
\begin{aligned}
&\hat{T}_{kN}^{(r)} \\
&= \frac{N}{\left[\Gamma(2r+2) - \Gamma^2(r+2)\right]\left[k\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^4\Gamma^2(r+1)\Gamma(2r+1)} \\
&\quad \cdot \frac{1}{k\left[\Gamma(2r+3) - \Gamma^2(r+2)\right] - (r+1)^2\Gamma^2(r+2)} \\
&\quad \cdot \left\{ \left(\frac{k}{\overline{X}_{kN}}\right)^r \frac{1}{N}\sum_{j=1}^{N} U_{kj}^r \left(k\left[\Gamma(2r+3) - \Gamma^2(r+2)\right] - (r+1)^2\Gamma^2(r+2) \right.\right. \\
&\qquad \left.\left. - \left(k\left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right] - r\Gamma^2(r+2)\right)\frac{k}{\overline{X}_{kN}} U_{kj} \right) \right.
\end{aligned}
$$

$$- \Gamma(r+2)\left[k\Gamma(2r+2) - \Gamma^2(r+2)\right] \Bigg\}^2$$

$$+ \frac{kN}{k\left[\Gamma(2r+3) - \Gamma^2(r+2)\right] - (r+1)^2\Gamma^2(r+2)}$$

$$\cdot \left[\left(\frac{k}{\overline{X}_{kN}} \right)^{r+1} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r+1} - \Gamma(r+2) \right]^2$$

and

$$\hat{T}_{kN}^{(r)}$$

$$= \frac{N}{\left[\Gamma(2r+2) - \Gamma^2(r+2)\right]\left[k\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^4\Gamma^2(r+1)\Gamma(2r+1)}$$

$$\cdot \frac{1}{k\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^2\Gamma^2(r+1)}$$

$$\cdot \Bigg\{ \left(\frac{k}{\overline{X}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} \Bigg(\left(k\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^2\Gamma^2(r+1) \right)$$

$$\cdot \frac{k}{\overline{X}_{kN}} U_{kj} - k\left[\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)\right] + r\Gamma^2(r+2) \Bigg)$$

$$+ r\Gamma(r+1)\left[k\Gamma(2r+1) - \Gamma(r+1)\Gamma(r+2)\right] \Bigg\}^2$$

$$+ \frac{kN}{k\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^2\Gamma^2(r+1)}$$

$$\cdot \left[\left(\frac{k}{\overline{X}_{kN}} \right)^{r} \frac{1}{N} \sum_{j=1}^{N} U_{kj}^{r} - \Gamma(r+1) \right]^2,$$

where $\overline{X}_{kN} = \frac{1}{N} \sum_{j=1}^{N} U_{kj}$.

For $k = 1$, $r \neq 1$, we have

$$\hat{T}_{1n}^{(r)}$$

$$= \frac{n}{\left[\Gamma(2r+2) - \Gamma^2(r+2)\right]\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^4\Gamma^2(r+1)\Gamma(2r+1)}$$

$$\cdot \frac{1}{\Gamma(2r+3) - (r^2+2r+2)\Gamma^2(r+1)} \Bigg\{ \left(\frac{1}{\overline{X}_n} \right)^{r} \frac{1}{n} \sum_{j=1}^{n} X_j^{r}$$

$$
\begin{aligned}
&\cdot \Bigg(\Gamma(2r+3) - (r^2+2r+2)\Gamma^2(r+2) \\
&- \big(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) - r\Gamma^2(r+2)\big) \frac{1}{\overline{X}_n} X_j \Bigg) \\
&- \Gamma(r+2)\left[\Gamma(2r+2) - \Gamma^2(r+2)\right] \Bigg\}^2 \\
&+ \frac{n}{\Gamma(2r+3) - (r^2+2r+2)\Gamma^2(r+2)} \\
&\cdot \left[\left(\frac{1}{\overline{X}_n}\right)^{r+1} \frac{1}{n}\sum_{j=1}^{n} X_j^{r+1} - \Gamma(r+2) \right]^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{T}_{1n}^{(r)} \\
&= \frac{n}{[\Gamma(2r+2) - \Gamma^2(r+2)]\,[\Gamma(2r+1) - \Gamma^2(r+1)] - r^4\Gamma^2(r+1)\Gamma(2r+1)} \\
&\cdot \frac{1}{\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)} \Bigg\{ \left(\frac{1}{\overline{X}_n}\right)^{r} \frac{1}{n}\sum_{j=1}^{n} X_j^{r} \\
&\cdot \Bigg(\left[\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)\right] \frac{X_j}{\overline{X}_n} \\
&- [\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)] + r\Gamma^2(r+2) \Bigg) \\
&+ r\Gamma(r+1)\,[\Gamma(2r+1) - \Gamma(r+1)\Gamma(r+2)] \Bigg\}^2 \\
&+ \frac{n}{\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)} \\
&\cdot \left[\left(\frac{1}{\overline{X}_n}\right)^{r} \frac{1}{n}\sum_{j=1}^{n} X_j^{r} - \Gamma(r+1) \right]^2.
\end{aligned}
$$

4.2. Inverse Exponential Distributions: $X \sim \mathrm{I\,Exp}(\theta)$

Here

$$f(x;\theta),\quad x>0,\quad \Lambda=\{\theta:\theta>0\},$$
$$F(x;\theta)=e^{-\theta/x},\qquad h^*(x)=\theta/x,$$
$$\frac{\partial F}{\partial\theta}=-f(x)\frac{x}{\theta},\qquad \mathcal{I}^{-1}=\theta^2$$

and the MLE $\hat{\theta}_n$ is obtained numerically.

Moreover,

$$b_k^{*(r)}(\theta):=-k^{r+1}E\left[F^{k-2}(X;\theta)\log^{r-1}\left(\frac{1}{F(X;\theta)}\right)\frac{\partial F(X;\theta)}{\partial\theta}\right] = \frac{1}{\theta}\Gamma(r+1),$$

and

$$\boldsymbol{B}_k^{*(r)}=\begin{bmatrix} r\left(b_k^{(r)}(\theta)\right)' \\ (r+1)\left(b_k^{(r+1)}(\theta)\right)' \end{bmatrix}=\frac{\Gamma(r+1)}{\theta}\begin{bmatrix} r \\ (r+1)^2\end{bmatrix}.$$

Now, using $\boldsymbol{K}_k^{(r)}$, $\Sigma_{k1}^{(r)}$ and $\Delta_{k1}^{(r)}$ for the exponential distribution, we have

$$\boldsymbol{K}_k^{*(r)}:=\frac{1}{k}\boldsymbol{B}_k^{*(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{*(r)}\right)'=\boldsymbol{K}_k^{(r)} = \frac{1}{k}\begin{bmatrix} r^2\Gamma^2(r+1) & r(r+1)^2\Gamma^2(r+1) \\ r(r+1)^2\Gamma^2(r+1) & (r+1)^4\Gamma^2(r+1)\end{bmatrix}$$

$$\Sigma_{k1}^{*(r)}=\Sigma_{k1}^{(r)},\qquad \Delta_{k1}^{*(r)}=\Delta_{k1}^{r}.$$

As in **1**, the case $k=r=1$ has to be excluded, and we have

$$\hat{T}_{kN}^{*(r)} = \frac{N}{[\Gamma(2r+2)-\Gamma^2(r+2)]\,[k\Gamma(2r+1)-\Gamma^2(r+1)]-r^4\Gamma^2(r+1)\Gamma(2r+1)} \cdot\frac{1}{k\,[\Gamma(2r+3)-\Gamma^2(r+2)]-(r+1)^2\Gamma^2(r+2)} \cdot\left\{\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r\left(k\left[\Gamma(2r+3)-\Gamma^2(r+2)\right]-(r+1)^2\Gamma^2(r+2)\right.\right.$$

$$- \left(k\left[\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right]-r\Gamma^2(r+2)\right)k\frac{\hat{\theta}_n}{V_{kj}}\Bigg)$$

$$-\Gamma(r+2)\left[k\Gamma(2r+2)-\Gamma^2(r+2)\right]\Bigg\}^2$$

$$+\frac{kN}{k\left[\Gamma(2r+3)-\Gamma^2(r+2)\right]-(r+1)^2\Gamma^2(r+2)}$$

$$\cdot\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r+1}-\Gamma(r+2)\right]^2$$

and

$$\hat{T}^{*(r)}_{kN}$$

$$=\frac{N}{\left[\Gamma(2r+2)-\Gamma^2(r+2)\right]\left[k\Gamma(2r+1)-\Gamma^2(r+1)\right]-r^4\Gamma^2(r+1)\Gamma(2r+1)}$$

$$\cdot\frac{1}{k\left[\Gamma(2r+1)-\Gamma^2(r+1)\right]-r^2\Gamma^2(r+1)}$$

$$\cdot\Bigg\{\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r\Bigg(\left(k\left[\Gamma(2r+1)-\Gamma^2(r+1)\right]-r^2\Gamma^2(r+1)\right)k\frac{\hat{\theta}_n}{V_{kj}}$$

$$-k\left[\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right]+r\Gamma^2(r+2)\Bigg)$$

$$+r\Gamma(r+1)\left[k\Gamma(2r+1)-\Gamma(r+1)\Gamma(r+2)\right]\Bigg\}^2$$

$$+\frac{kN}{k\left[\Gamma(2r+1)-\Gamma^2(r+1)\right]-r^2\Gamma^2(r+1)}$$

$$\cdot\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^r-\Gamma(r+1)\right]^2.$$

For $k=1$ we have the tests:

$$\hat{T}^{*(r)}_{1n}$$

$$=\frac{n}{\left[\Gamma(2r+2)-\Gamma^2(r+2)\right]\left[\Gamma(2r+1)-\Gamma^2(r+1)\right]-r^4\Gamma^2(r+1)\Gamma(2r+1)}$$

$$\cdot\frac{1}{\Gamma(2r+3)-(r^2+2r+2)\Gamma^2(r+2)}$$

$$
\cdot \Bigg\{ \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j}\right)^r \Bigg(\Gamma(2r+3) - (r^2+2r+2)\Gamma^2(r+2)
- \Big(\Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) - r\Gamma^2(r+2)\Big) \frac{\hat{\theta}_n}{X_j} \Bigg)
- \Gamma(r+2)\left[\Gamma(2r+2) - \Gamma^2(r+2)\right] \Bigg\}^2
+ \frac{n}{\Gamma(2r+3) - (r^2+2r+2)\Gamma^2(r+2)}
\cdot \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j}\right)^{r+1} - \Gamma(r+2) \right]^2 ,
$$

and

$$
\hat{T}_{1n}^{*(r)}
= \frac{n}{\left[\Gamma(2r+2) - \Gamma^2(r+2)\right]\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - r^4\Gamma^2(r+1)\Gamma(2r+1)}
\cdot \frac{1}{\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)}
\cdot \Bigg\{ \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j}\right)^r \Bigg(\Big(\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1) \Big) \frac{\hat{\theta}_n}{X_j}
- \Gamma(2r+2) + \Gamma(r+1)\Gamma(r+2) + r\Gamma^2(r+2) \Bigg)
+ r\Gamma(r+1)\left[\Gamma(2r+1) - \Gamma(r+1)\Gamma(r+2)\right] \Bigg\}^2
+ \frac{n}{\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)}
\cdot \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j}\right)^{r} - \Gamma(r+1) \right]^2 .
$$

4.3. Weibull Distributions: $X \sim W(\theta, \tau)$

Here

$$f(x) = \frac{\tau}{\theta}\left(\frac{x}{\theta}\right)^{\tau-1} e^{-(x/\theta)^{\tau}}, \quad x > 0;\ \Lambda = \{(\theta,\tau) : \theta > 0,\ \tau > 0\},$$
$$F(x) = 1 - e^{-(x/\theta)^{\tau}}, \qquad h(x) = (x/\theta)^{\tau},$$
$$\frac{\partial F}{\partial \theta} = -\frac{x}{\theta} f(x), \qquad \frac{\partial F}{\partial \tau} = \frac{x}{\tau} f(x) \log\left(\frac{x}{\theta}\right),$$
$$\mathcal{I} = \begin{bmatrix} \frac{\tau^2}{\theta^2} & -\frac{1-\gamma}{\theta} \\ -\frac{1-\gamma}{\theta} & \left(\frac{\pi^2}{6} + (1-\gamma)^2\right)\frac{1}{\tau^2} \end{bmatrix},$$
$$\mathcal{I}^{-1} = \frac{6}{\pi^2}\begin{bmatrix} \left(\frac{\pi^2}{6} + (1-\gamma)^2\right)\left(\frac{\theta}{\tau}\right)^2 & (1-\gamma)\theta \\ (1-\gamma)\theta & \tau^2 \end{bmatrix}$$

and the MLE $\hat{\theta}_n$ and $\hat{\tau}_n$ are obtained numerically.

Hence

$$\begin{aligned} b_k^{(r)}(\theta) &= k^{r+1} E\left[(1-F(X))^{k-2} \log^{r-1} \frac{1}{1-F(X)} \frac{\partial F(X;\theta,\tau)}{\partial \theta}\right] \\ &= -k^{r+1} E\left[e^{-(k-2)\left(\frac{X}{\theta}\right)^{\tau}} \left(\frac{X}{\theta}\right)^{(r-1)\tau} \frac{X}{\theta} f(X)\right] \\ &= -k^{r+1}\left(\frac{\tau}{\theta}\right)^2 \int_0^{\infty} \left(\frac{x}{\theta}\right)^{(r+1)\tau-1} e^{-k\left(\frac{x}{\theta}\right)^{\tau}} dx \\ &\left(\frac{x}{\theta} = y^{1/\tau}, \qquad dx = \frac{\theta}{\tau} y^{\frac{1}{\tau}-1} dy\right) \\ &= -k^{r+1}\frac{\tau}{\theta}\int_0^{\infty} y^r e^{-ky} dy, = -\frac{\tau}{\theta}\Gamma(r+1) \end{aligned}$$

and

$$\begin{aligned} b_k^{(r)}(\tau) &= k^{r+1} E\left[(1-F(X))^{k-2} \log^{r-1} \frac{1}{1-F(X)} \frac{\partial F(X;\theta,\tau)}{\partial \tau}\right] \\ &= k^{r+1}\frac{\tau}{\theta}\int_0^{\infty} \left(\frac{x}{\theta}\right)^{(r+1)\tau-1} e^{-\left(\frac{x}{\theta}\right)^{\tau}} \log\frac{x}{\theta} dx \\ &\left(\frac{x}{\theta} = y^{1/\tau}, \qquad dx = \frac{\theta}{\tau} y^{\frac{1}{\tau}-1} dy\right) \\ &= k^{r+1}\frac{1}{\tau}\int_0^{\infty} y^r e^{-ky} \log y dy \end{aligned}$$

$$\left(\int_0^\infty x^{p-1}e^{-qx}\ln x dx = \frac{\Gamma(p)}{q^p}\left(\psi(p)-\ln q\right),\right.$$
$$\left.\text{(cf. Ryzyk and Gradsztejn [19], 3.723.1)}\right)$$
$$= \frac{\Gamma(r+1)}{\tau}\left(\psi(r+1)-\log k\right).$$

Hence

$$\boldsymbol{B}_k^{(r)} = \begin{bmatrix} rb_k^{(r)}(\theta) & rb_k^{(r)}(\tau) \\ (r+1)b_k^{(r+1)}(\theta) & (r+1)b_k^{(r+1)}(\tau) \end{bmatrix}$$
$$= \Gamma(r+1)\begin{bmatrix} -r\frac{\tau}{\theta} & \frac{r}{\tau}\left(\psi(r+1)-\log k\right) \\ -(r+1)^2\frac{\tau}{\theta} & \frac{(r+1)^2}{\tau}\left(\psi(r+2)-\log k\right) \end{bmatrix},$$

and

$$\boldsymbol{K}_k^{(r)} = \frac{1}{k}\boldsymbol{B}_k^{(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{(r)}\right)' := \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)} \end{bmatrix},$$

where

$$s_k^{(r)} = \frac{r^2\Gamma^2(r+1)}{k\pi^2}\left[\pi^2 + 6\left(1-\gamma-\psi(r+1)+\log k\right)^2\right],$$
$$t_k^{(r)} = \frac{r\Gamma^2(r+2)}{k\pi^2}\left[\pi^2 + 6(1-\gamma-\psi(r+1)+\log k)\right.$$
$$\left.\cdot\,(1-\gamma-\psi(r+2)+\log k)\right],$$
$$u_k^{(r)} = \frac{(r+1)^2\Gamma^2(r+2)}{k\pi^2}\left[\pi^2 + 6\left(1-\gamma-\psi(r+2)+\log k\right)^2\right].$$

Hence

$$\triangledown_k^{(r)} = \det\left(\boldsymbol{K}_k^{(r)}\right) = \frac{6r^2\Gamma^4(r+2)}{k^2\pi^2}\left(\psi(r+2)-\psi(r+1)\right)^2.$$

Now

$$\Sigma_{k1}^{(r)} = \begin{bmatrix} a_{k1}^{(r)} & b_{k1}^{(r)} \\ b_{k1}^{(r)} & c_{k1}^{(r)} \end{bmatrix},$$

with

$$a_{k1}^{(r)} = a^{(r)} - s_k^{(r)} = \left[\Gamma(2r+1) - \Gamma^2(r+1)\right] - \frac{r^2\Gamma^2(r+1)\left[\pi^2 + 6\left(1-\gamma-\psi(r+1)+\log k\right)^2\right]}{k\pi^2},$$

$$b_{k1}^{(r)} = b^{(r)} - t_k^{(r)} = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) - \frac{r\Gamma^2(r+2)\left[\pi^2 + 6(1-\gamma-\psi(r+1)+\log k)(1-\gamma-\psi(r+2)+\log k)\right]}{k\pi^2},$$

$$c_{k1}^{(r)} = c^{(r)} - u_k^{(r)} = \left[\Gamma(2r+3) - \Gamma^2(r+2)\right] - \frac{(r+1)^2\Gamma^2(r+2)\left[\pi^2 + 6\left(1-\gamma-\psi(r+2)+\log k\right)^2\right]}{k\pi^2}.$$

Hence

$$\begin{aligned}\Delta_{k1}^{(r)} = \Delta^{(r)} + \bigtriangledown_k^{(r)} - a^{(r)}u_k^{(r)} - c^{(r)}s_k^{(r)} + 2b^{(r)}t_k^{(r)} &= \Gamma(2r+1)\left[\Gamma(2r+2) - \Gamma^2(r+2)\right] \\ &+ \frac{6r^2\Gamma^4(r+2)}{k^2\pi^2}\left(\psi(r+2)-\psi(r+1)\right)^2 \\ &- \frac{(r+1)^2\Gamma^2(r+2)}{k\pi^2}\left[\Gamma(2r+1) - \Gamma^2(r+1)\right] \\ &\cdot\left[\pi^2 + 6(1-\gamma-\psi(r+2)+\log k)^2\right] \\ &- \frac{r^2\Gamma^2(r+1)}{k\pi^2}\left[\Gamma(2r+3) - \Gamma^2(r+2)\right] \\ &\cdot\left[\pi^2 + 6(1-\gamma-\psi(r+1)+\log k)^2\right] \\ &+ 2\frac{r\Gamma^2(r+2)}{k\pi^2}\left[\Gamma(3r+2) - \Gamma(r+1)\Gamma(r+2)\right] \\ &\cdot\left[\pi^2 + 6(1-\gamma-\psi(r+1)+\log k)(1-\gamma-\psi(r+2)+\log k)\right].\end{aligned}$$

Using the above notations (3.23) and (3.24) give

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)}c_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}\left(c_{k1}^{(r)} - b_{k1}^{(r)}k\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{\hat{\tau}_n}\right) - \Gamma(r+1)\left(c_{k1}^{(r)} - (r+1)b_{k1}^{(r)}\right)\right]^2$$

$$+\frac{N}{c_{k1}^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{(r+1)\hat{\tau}_n}-\Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{kN}^{(r)}=\frac{N}{\Delta_{k1}^{(r)}a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}\left(a_{k1}^{(r)}k\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{\hat{\tau}_n}-b_{k1}^{(r)}\right)\right.$$
$$\left.-\Gamma(r+1)\left((r+1)a_{k1}^{(r)}-b_{k1}^{(r)}\right)\right]^2$$
$$+\frac{N}{a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{U_{kj}}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}-\Gamma(r+1)\right]^2.$$

For $k=1$ we have

$$\hat{T}_{1n}^{(r)}=\frac{n}{\Delta_{11}^{(r)}c_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_j}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}\left(c_{11}^{(r)}-b_{11}^{(r)}\left(\frac{X_j}{\hat{\theta}_n}\right)^{\hat{\tau}_n}\right)\right.$$
$$\left.-\Gamma(r+1)\left(c_{11}^{(r)}-(r+1)b_{11}^{(r)}\right)\right]^2$$
$$+\frac{n}{c_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_j}{\hat{\theta}_n}\right)^{(r+1)\hat{\tau}_n}-\Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{1n}^{(r)}=\frac{n}{\Delta_{11}^{(r)}a_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_j}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}\left(a_{11}^{(r)}\left(\frac{X_j}{\hat{\theta}_n}\right)^{\hat{\tau}_n}-b_{11}^{(r)}\right)\right.$$
$$\left.-\Gamma(r+1)\left((r+1)a_{11}^{(r)}-b_{11}^{(r)}\right)\right]^2$$
$$+\frac{n}{a_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_j}{\hat{\theta}_n}\right)^{r\hat{\tau}_n}-\Gamma(r+1)\right]^2.$$

4.4. Inverse Weibull Distributions: $X \sim IW(\theta, \tau)$

Here

$$
\begin{aligned}
f(x) &= \frac{\tau}{\theta}\left(\frac{\theta}{x}\right)^{\tau+1} e^{-(\theta/x)^\tau}, \quad x > 0;\ \Lambda = \{(\theta,\tau) : \theta > 0,\ \tau > 0\},\\
F(x) &= e^{-(\theta/x)^\tau}, \qquad h_*(x) = \left(\frac{\theta}{x}\right)^\tau,\\
\frac{\partial F}{\partial \theta} &= -\frac{x}{\theta} f(x), \qquad \frac{\partial F}{\partial \tau} = \frac{x}{\tau} f(x)\log\left(\frac{x}{\theta}\right),\\
\mathcal{I}^{-1} &= \frac{6}{\pi^2}\begin{bmatrix} \left(\frac{\pi^2}{6} + (1-\gamma)^2\right)\left(\frac{\theta}{\tau}\right)^2 & (1-\gamma)\theta \\ (1-\gamma)\theta & \tau^2 \end{bmatrix}.
\end{aligned}
$$

and the MLE $\hat{\theta}_n$ and $\hat{\tau}_n$ are obtained numerically.

Referring to (3.18):

$$
\begin{aligned}
b_k^{*(r)}(\theta) &= -k^{r+1} E\left[F^{k-2}(X)\log^{r-1}\frac{1}{F(X)}\frac{\partial F(X;\theta,\tau)}{\partial \theta}\right]\\
&= -k^{r+1} E\left[e^{-(k-2)(\theta/x)^\tau}\left(\frac{\theta}{x}\right)^{(r-1)\tau}\left(-\frac{x}{\theta}\right)\frac{\tau}{\theta}\left(\frac{\theta}{x}\right)^{\tau+1} e^{-(\theta/x)^\tau}\right]\\
&= k^{r+1}\int_0^\infty e^{-(k-2)(\theta/x)^\tau}\left(\frac{\theta}{x}\right)^{(r-1)\tau}\left(\frac{\tau}{\theta}\right)^2\left(\frac{\theta}{x}\right)^{2\tau+1} e^{-2(\theta/x)^\tau}dx\\
&= k^{r+1}\left(\frac{\tau}{\theta}\right)^2\int_0^\infty e^{-k(\theta/x)^\tau}\left(\frac{\theta}{x}\right)^{(r+1)\tau+1}dx\\
&\quad \left(\frac{\theta}{x} = y^\tau, \qquad dx = -\frac{\theta}{\tau}y^{-\frac{1}{\tau}-1}dy\right)\\
&= k^{r+1}\left(\frac{\tau}{\theta}\right)^2\int_\infty^0 e^{-ky}y^{r+1+\frac{1}{\tau}}\left(-\frac{\theta}{\tau}\right)y^{-\frac{1}{\tau}-1}dy\\
&= \frac{\tau}{\theta}k^{r+1}\int_0^\infty e^{-ky}y^r dy = \frac{\tau}{\theta}\Gamma(r+1)
\end{aligned}
$$

and

$$
\begin{aligned}
b_k^{*(r)}(\tau) &= -k^{r+1} E\left[F^{k-2}(X;\tau)h_{r-1}^*(X;\tau)\frac{\partial F(X;\tau)}{\partial \tau}\right]\\
&= -k^{r+1} E\left[e^{-(k-2)(\theta/x)^\tau}\left(\frac{\theta}{x}\right)^{(r-1)\tau}\left(\frac{x}{\tau}\right)\frac{\tau}{\theta}\left(\frac{\theta}{x}\right)^{\tau+1} e^{-(\theta/x)^\tau}\log\left(\frac{x}{\theta}\right)\right]
\end{aligned}
$$

$$= -k^{r+1} \int_0^\infty e^{-(k-2)(\theta/x)^\tau} \left(\frac{\theta}{x}\right)^{(r-1)\tau} \left(\frac{x}{\theta}\right) \left(\frac{\theta}{x}\right)^{\tau+1} \frac{\tau}{\theta} \left(\frac{\theta}{x}\right)^{\tau+1} \cdot e^{-2(\theta/x)^\tau} \log\left(\frac{x}{\theta}\right) dx$$

$$= -k^{r+1} \frac{\tau}{\theta} \int_0^\infty e^{-k(\theta/x)^\tau} \left(\frac{\theta}{x}\right)^{(r+1)\tau+1} \log\left(\frac{x}{\theta}\right) dx$$

$$\left(\frac{\theta}{x} = y^{\frac{1}{\tau}}, \qquad dx = -\frac{\theta}{\tau} y^{-\frac{1}{\tau}-1} dy\right)$$

$$= -k^{r+1} \frac{1}{\tau} \int_\infty^0 e^{-ky} y^{r+1+\frac{1}{\tau}} y^{-\frac{1}{\tau}-1} dy$$

$$= k^{r+1} \frac{1}{\tau} \int_0^\infty y^r e^{-ky} \log y dy$$

$$\left(\int_0^\infty x^{p-1} e^{-qx} \ln x dx = \frac{\Gamma(p)}{q^p} \left(\psi(p) - \ln q\right)\right)$$

$$= \frac{\Gamma(r+1)}{\tau} \left(\psi(r+1) - \log k\right).$$

Thus

$$b_k^{*(r)}(\theta) = \frac{\tau}{\theta}\Gamma(r+1), \qquad b_k^{*(r)}(\tau) = \frac{\Gamma(r+1)}{\tau}\left(\psi(r+1) - \log k\right),$$

$$\boldsymbol{B}_k^{*(r)} = \begin{bmatrix} rb_k^{*(r)}(\theta) & rb_k^{*(r)}(\tau) \\ (r+1)b_k^{*(r+1)}(\theta) & (r+1)b_k^{*(r+1)}(\theta) \end{bmatrix}$$
$$= \Gamma(r+1) \begin{bmatrix} r\frac{\tau}{\theta} & \frac{r}{\tau}\left(\psi(r+1) - \log k\right) \\ (r+1)^2 \frac{\tau}{\theta} & \frac{(r+1)^2}{\tau}\left(\psi(r+2) - \log k\right) \end{bmatrix}$$

and

$$\begin{aligned}
\boldsymbol{K}_k^{*(r)} &:= \begin{bmatrix} s_k^{*(r)} & t_k^{*(r)} \\ t_k^{*(r)} & u_k^{*(r)} \end{bmatrix} = \frac{1}{k}\boldsymbol{B}_k^{*(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{*(r)}\right)' \\
&= \frac{1}{k}\Gamma(r+1)\begin{bmatrix} r\frac{\tau}{\theta} & \frac{r}{\tau}\left(\psi(r+1)-\log k\right) \\ (r+1)^2\frac{\tau}{\theta} & \frac{(r+1)^2}{\tau}\left(\psi(r+2)-\log k\right) \end{bmatrix} \\
&\cdot \frac{6}{\pi^2}\begin{bmatrix} \left(\frac{\pi^2}{6}+(1-\gamma)^2\right)\left(\frac{\theta}{\tau}\right)^2 & (1-\gamma)\theta \\ (1-\gamma)\theta & \tau^2 \end{bmatrix} \\
&\cdot \Gamma(r+1)\begin{bmatrix} r\frac{\tau}{\theta} & (r+1)^2\frac{\tau}{\theta} \\ \frac{r}{\tau}\left(\psi(r+1)-\log k\right) & \frac{(r+1)^2}{\tau}\left(\psi(r+2)-\log k\right) \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
s_k^{*(r)} &= \frac{r^2\Gamma^2(r+1)}{k\pi^2}\left[\pi^2+6\left(1-\gamma+\psi(r+1)-\log k\right)^2\right] \\
t_k^{*(r)} &= \frac{r\Gamma^2(r+2)}{k\pi^2}\left[\pi^2+6(1-\gamma-\psi(r+1)+\log k)\right. \\
&\cdot \left.(1-\gamma-\psi(r+2)+\log k)\right] \\
u_k^{*(r)} &= \frac{(r+1)^2\Gamma^2(r+2)}{k\pi^2}\left[\pi^2+6(1-\gamma+\psi(r+2)-\log k)^2\right],
\end{aligned}$$

and

$$\begin{aligned}
\triangledown_k^{*(r)} := \det\left(\boldsymbol{K}_k^{*(r)}\right) = s_k^{*(r)}u_k^{*(r)} - \left(t_k^{*(r)}\right)^2 \\
= \frac{6r^2\Gamma^4(r+1)}{k^2\pi^2}\left(\psi(r+2)-\psi(r+1)\right)^2.
\end{aligned}$$

Therefore, referring to (3.22)

$$\begin{aligned}
\Delta_{k1}^{*(r)} = \Delta^{(r)} + \triangledown_k^{*(r)} - a^{(r)}u_k^{*(r)} - c^{(r)}s_k^{*(r)} - 2b_k^{*(r)}t_k^{*(r)} \\
= \Gamma(2r+1)\left[\Gamma(2r+2)-\Gamma^2(r+2)\right] \\
+ \frac{6r^2\Gamma^4(r+2)}{k^2\pi^2}(\psi(r+2)-\psi(r+1))^2 \\
- \frac{(r+1)^2\Gamma^2(r+2)}{k\pi^2}\left[\pi^2+6(1-\gamma+\psi(r+2)-\log k)^2\right] \\
- \frac{r^2\Gamma^2(r+1)}{k\pi^2}\left[\Gamma(2r+3)-\Gamma^2(r+2)\right]
\end{aligned}$$

$$\cdot \left[\pi^2 + 6(2-\gamma+\psi(r+1)-\log k)^2\right]$$
$$- 2\frac{r\Gamma^2(r+2)}{k\pi^2}\left[\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)\right]\left[\pi^2+6(1-\gamma-\log k)^2\right.$$
$$\left.+6(1-\gamma-\log k)(\psi(r+1)-\psi(r+2))+6\psi(r+1)\psi(r+2)\right].$$

Then (3.25) and (3.26) give

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)}c_{k1}^{*(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r\hat{\tau}_n}\left(c_{k1}^{*(r)} - b_{k1}^{*(r)}k\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{\hat{\tau}_n}\right.\right.$$
$$\left.\left. - \Gamma(r+1)\left(c_{k1}^{*(r)}-(r+1)b_{k1}^{*(r)}\right)\right)\right]^2$$
$$+ \frac{N}{c_{k1}^{*(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{(r+1)\hat{\tau}_n} - \Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)}a_{k1}^{*(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r\hat{\tau}_n}\left(a_{k1}^{*(r)}k\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{\hat{\tau}_n} - b_{k1}^{*(r)}\right)\right.$$
$$\left. - \Gamma(r+1)\left((r+1)a_{k1}^{*(r)}-b_{k1}^{*(r)}\right)\right]^2$$
$$+ \frac{N}{a_{k1}^{*(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(\frac{\hat{\theta}_n}{V_{kj}}\right)^{r\hat{\tau}_n} - \Gamma(r+1)\right]^2.$$

For $k=1$ we have

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)}c_{11}^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^{r\hat{\tau}_n}\left(c_{11}^{*(r)} - b_{11}^{*(r)}\left(\frac{\hat{\theta}_n}{X_j}\right)^{\hat{\tau}_n}\right)\right.$$
$$\left. - \Gamma(r+1)\left(c_{11}^{*(r)}-(r+1)b_{11}^{*(r)}\right)\right]^2$$
$$+ \frac{n}{c_{11}^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\hat{\theta}_n}{X_j}\right)^{(r+1)\hat{\tau}_n} - \Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} a_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r\hat{\tau}_n} \left(a_{11}^{*(r)} \left(\frac{\hat{\theta}_n}{X_j} \right)^{\hat{\tau}_n} - b_{11}^{*(r)} \right) \right.$$
$$\left. - \Gamma(r+1) \left((r+1) a_{11}^{*(r)} - b_{11}^{*(r)} \right) \right]^2$$
$$+ \frac{n}{a_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{\hat{\theta}_n}{X_j} \right)^{r\hat{\tau}_n} - \Gamma(r+1) \right]^2 .$$

4.5. Logistic Distributions: $X \sim \mathrm{Log}(\alpha, \beta)$

Here

$$f(x) = \frac{1}{\beta} \left[\exp\left(-\frac{x-\alpha}{\beta} \right) \right] \Big/ \left[\left(1 + \exp\left(-\frac{x-\alpha}{\beta} \right) \right)^2 \right],$$
$$-\infty < x < \infty; \quad \Lambda = \{(\alpha, \beta) : \alpha \in \mathbb{R}, \beta > 0\},$$
$$F(x) = 1 \Big/ \left[1 + \exp\left(-\frac{x-\alpha}{\beta} \right) \right].$$

We use

$$f(x) = \frac{1}{\beta} F(x)(1 - F(x))$$
$$\frac{\partial F}{\partial \alpha} = -f(x), \qquad \frac{\partial F}{\partial \beta} = -\frac{x-\alpha}{\beta} f(x) = -\frac{x-\alpha}{\beta^2} F(x)(1-F(x))$$

$$\mathcal{I}^{-1} = 3\beta^2 \begin{bmatrix} 1 & 0 \\ 0 & 3/(3+\pi^2) \end{bmatrix}.$$

and the MLE $\hat{\alpha}_n$ and $\hat{\beta}_n$ are obtained numerically.

Moreover, we need

$$b_k^{(r)}(\alpha) = k^{r+1} E \left[(1 - F(X))^{k-2} \log^{r-1} \left(\frac{1}{1 - F(X)} \right) \frac{\partial F(X; \alpha, \beta)}{\partial \alpha} \right]$$
$$= -k^{r+1} \int_{-\infty}^{\infty} (1 - F(x))^{k-2} \log^{r-1} \left(\frac{1}{1 - F(x)} \right) \frac{1}{\beta} F(x)$$
$$\cdot (1 - F(x)) f(x) dx$$

$$= -\frac{k^{r+1}}{\beta}\int_0^1 (1-y)^{k-2}\log^{r-1}\frac{1}{1-y}y(1-y)dy$$
$$= -\frac{k^{r+1}}{\beta}\int_0^1 (1-y)^{k-1}\log^{r-1}\frac{1}{1-y}dy$$
$$+ \frac{k^{r+1}}{\beta}\int_0^1 (1-y)^{k}\log^{r-1}\frac{1}{1-y}dy = -\frac{k}{\beta}\Gamma(r)\left[1-\left(\frac{k}{k+1}\right)^r\right],$$

as

$$\int_0^1 x^{p-1}\log^{q-1}\frac{1}{x}dx = \frac{\Gamma(q)}{p^q}, \qquad p \geq 1, q > 0$$

(cf. Ryzyk and Gradsztejn [19], 3.653.2).

$$b_k^{(r)}(\beta) = k^{r+1}E\left[(1-F(X))^{k-2}\log^{r-1}\left(\frac{1}{1-F(X)}\right)\frac{\partial F(X;\alpha,\beta)}{\partial \beta}\right]$$
$$= -\frac{k^{r+1}}{\beta^2}\int_{-\infty}^{\infty}(1-F(x))^{k-2}\log^{r-1}\frac{1}{1-F(x)}(x-\alpha)F(x) \cdot (1-F(x))f(x)dx$$
$$= \frac{k^{r+1}}{\beta}\int_0^1 (1-y)^{k-2}\log^{r-1}\frac{1}{1-y}\left(\log(1-y)-\ln y\right)y(1-y)dy$$
$$= -\frac{k^{r+1}}{\beta}\int_0^1 (1-y)^{k-2}\log^{r-1}\frac{1}{1-y}\cdot y(1-y)\log\frac{y}{1-y}dy$$
$$= -\frac{k^{r+1}}{\beta}\left\{\int_0^1 (1-y)^{k-1}\log^{r}\frac{1}{1-y}dy - \int_0^1 (1-y)^{k}\log^{r}\frac{1}{1-y}dy - \int_0^1 (1-k)^{k-1}\log^{r-1}\frac{1}{1-y}\log\frac{1}{y}dy + \int_0^1 (1-y)^{k}\log^{r-1}\frac{1}{1-y}\log\frac{1}{y}dy\right\}$$
$$= -\frac{k^{r+1}}{\beta}\left\{\frac{\Gamma(r+1)}{k^{r+1}} - \frac{\Gamma(r+1)}{(k+1)^{r+1}} - \int_0^1 z^{k-1}\log^{r-1}\frac{1}{z}\log\frac{1}{1-z}dz + \int_0^1 z^{k}\log^{r-1}\frac{1}{z}\log\frac{1}{1-z}dz\right\}$$
$$= -\frac{k^{r+1}}{\beta}\left\{\frac{\Gamma(r+1)}{k^{r+1}} - \frac{\Gamma(r+1)}{(k+1)^{r+1}} - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 z^{n+k-1}\log^{r-1}\frac{1}{z}dz + \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 z^{n+k}\log^{r-1}\frac{1}{z}dz\right\}$$

$$= -\frac{k^{r+1}}{\beta}\left\{\frac{\Gamma(r+1)}{k^{r+1}} - \frac{\Gamma(r+1)}{(k+1)^{r+1}} - \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(r)}{(n+k)^r} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(r)}{(n+k+1)^r}\right\}$$

$$= -\frac{k^{r+1}\Gamma(r)}{\beta}\left\{\frac{r}{k^{r+1}} - \frac{r}{(k+1)^{r+1}} - \sum_{n=1}^{\infty}\frac{1}{n(n+k)^r} + \sum_{n=1}^{\infty}\frac{1}{n(n+k+1)^r}\right\}.$$

Thus

$$b_k^{(r)}(\alpha) = -\frac{k\Gamma(r)}{\beta}\left[1 - \left(\frac{k}{k+1}\right)^r\right],$$

$$b_k^{(r)}(\beta) = -\frac{k^{r+1}\Gamma(r)}{\beta}\left\{\frac{r}{k^{r+1}} - \frac{r}{(k+1)^{r+1}} - A_k^{(r)} + A_{k+1}^{(r)}\right\},$$

where

$$A_k^{(r)} = \sum_{n=1}^{\infty}\frac{1}{n(n+k)^r}.$$

Hence

$$\boldsymbol{B}_k^{(r)} = -\frac{k\Gamma(r+1)}{\beta}\cdot \begin{bmatrix} 1-\left(\frac{k}{k+1}\right)^r & k^r\left(\frac{r}{k^{r+1}} - \frac{r}{(k+1)^{r+1}} - A_k^{(r)} + A_{k+1}^{(r)}\right) \\ (r+1)\left(1-\left(\frac{k}{k+1}\right)^{r+1}\right) & (r+1)k^{r+1}\left(\frac{r+1}{k^{r+2}} - \frac{r+1}{(k+1)^{r+2}} - A_k^{(r+1)} + A_{k+1}^{(r+1)}\right)\end{bmatrix}.$$

Now

$$\Sigma_{k1}^{(r)} = \Sigma^{(r)} - \boldsymbol{K}_k^{(r)},$$

with

$$\boldsymbol{K}_k^{(r)} = \frac{1}{k}\boldsymbol{B}_k^{(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{(r)}\right)' = \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)}\end{bmatrix},$$

where

$$s_k^{(r)} = k\Gamma^2(r+1)\left[\left(1-\left(\frac{k}{k+1}\right)^r\right)^2\right.$$

$$+\frac{3}{3+\pi^2}k^{2r}\left(\frac{r}{k^{r+1}}-\frac{r}{(k+1)^{r+1}}-A_k^{(r)}+A_{k+1}^{(r)}\right)^2\Bigg]$$

$$t_k^{(r)}=k\Gamma(r+1)\Gamma(r+2)\left[\left(1-\left(\frac{k}{k+1}\right)^r\right)\left(1-\left(\frac{k}{k+1}\right)^{r+1}\right)\right.$$
$$+\frac{3}{3+\pi^2}k^{2r+1}\left(\frac{r}{k^{r+1}}-\frac{r}{(k+1)^{r+1}}-A_k^{(r)}+A_{k+1}^{(r)}\right)$$
$$\left.\cdot\left(\frac{r+1}{k^{r+2}}-\frac{r+1}{(k+1)^{r+2}}-A_k^{(r+1)}+A_{k+1}^{(r+1)}\right)\right]$$
$$u_k^{(r)}=k\Gamma^2(r+2)\left[\left(1-\left(\frac{k}{k+1}\right)^{r+1}\right)^2\right.$$
$$\left.+\frac{3}{3+\pi^2}k^{2(r+1)}\left(\frac{r+1}{k^{r+2}}-\frac{r+1}{(k+1)^{r+2}}-A_k^{(r+1)}+A_{k+1}^{(r+)}\right)^2\right].$$

Hence

$$\Sigma_{k1}^{(r)}=\begin{bmatrix}a_{k1}^{(r)} & b_{k1}^{(r)}\\ b_{k1}^{(r)} & c_{k1}^{(r)}\end{bmatrix},$$

with

$$a_{k1}^{(r)}=\Gamma(2r+1)-\Gamma^2(r+1)-k\Gamma^2(r+1)\left[\left(1-\left(\frac{k}{k+1}\right)^r\right)^2\right.$$
$$\left.+\frac{3}{3+\pi^2}k^{2r}\left(\frac{r}{k^{r+1}}-\frac{r}{(k+1)^{r+1}}-A_k^{(r)}+A_{k+1}^{(r)}\right)^2\right],$$

$$b_{k1}^{(r)}=\Gamma(2r+2)-\Gamma(r+1)\Gamma(r+2)$$
$$-k\Gamma(r+1)\Gamma(r+2)\left[\left(1-\left(\frac{k}{k+1}\right)^r\right)\left(1-\left(\frac{k}{k+1}\right)^{r+1}\right)\right.$$
$$+\frac{3}{3+\pi^2k^{2r+1}}\left(\frac{r}{k^{r+1}}-\frac{r}{(k+1)^{r+1}}-A_k^{(r)}+A_{k+1}^{(r)}\right)$$
$$\left.\cdot\left(\frac{r+1}{k^{r+2}}-\frac{r+1}{(k+1)^{r+2}}-A_k^{(r+1)}+A_{k+1}^{(r+1)}\right)\right],$$

$$c_{k1}^{(r)} = \Gamma(2r+3) - \Gamma^2(r+2) - k\Gamma^2(r+2)\left[\left(1-\left(\frac{k}{k+1}\right)^{r+1}\right)^2 + \frac{3}{3+\pi^2}k^{2(r+1)}\left(\frac{r+1}{k^{r+2}} - \frac{r+1}{(k+1)^{r+2}} - A_k^{(r+1)} + A_{k+1}^{(r+1)}\right)^2\right].$$

The determinant $\Delta_{k1}^{(r)}$ is given by (3.21) with $s_k^{(r)}$, $t_k^{(r)}$ and $u_k^{(r)}$ given above.

Then (3.23) and (3.24) give

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} c_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\log^r\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \cdot\left(c_{k1}^{(r)} - b_{k1}^{(r)} k\log\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)\right) - \Gamma(r+1)\left(c_{k1}^{(r)} - (r+1)b_{k1}^{(r)}\right)\right]^2 + \frac{N}{c_{k1}^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\log^{r+1}\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+2)\right]^2$$

and

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\log^r\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \cdot\left(a_{k1}^{(r)} k\log\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - b_{k1}^{(r)}\right) - \Gamma(r+1)\left((r+1)a_{k1}^{(r)} - b_{k1}^{(r)}\right)\right]^2 + \frac{N}{a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\log^r\left(1+\exp\left(\frac{U_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+1)\right]^2.$$

For $k=1$

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} c_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\log^r\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \cdot\left(c_{11}^{(r)} - b_{11}^{(r)}\log\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)\right)\right.$$

$$- \Gamma(r+1)\left(c_{11}^{(r)} - (r+1)b_{11}^{(r)}\right)\Bigg]^2$$

$$+ \frac{n}{c_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\log^{r+1}\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)}a_{11}^{(r)}}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\log^{r}\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)$$

$$\cdot\left(a_{11}^{(r)}\log\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - b_{11}^{(r)}\right) - \Gamma(r+1)\left((r+1)a_{11}^{(r)} - b_{11}^{(r)}\right)\Bigg]^2$$

$$+ \frac{n}{a_{11}^{(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\log^{r}\left(1+\exp\left(\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+1)\right]^2.$$

Now consider the dual test in (3.18). Then

$$\begin{aligned}
b_k^{*(r)}(\alpha) &= -k^{r+1}E\left[F^{k-2}(X)\log^{r-1}\frac{1}{F(X)}\frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right] \\
&= -k^{r+1}\int_{-\infty}^{\infty}F^{k-2}(x)\log^{r-1}\frac{1}{F(x)}(-f(x))f(x)dx \\
&= \frac{k^{r+1}}{\beta}\int_{-\infty}^{\infty}F^{k-1}(x)(1-F(x))\log^{r-1}\frac{1}{F(x)}f(x)dx \\
&\qquad \left(F(x)=y, \quad f(x)dx=dy\right) \\
&= \frac{k^{r+1}}{\beta}\int_0^1 y^{k-1}(1-y)\log^{r-1}\frac{1}{y}dy \\
&= \frac{k^{r+1}}{\beta}\left[\int_0^1 y^{k-1}\log^{r-1}\frac{1}{y}dy - \int_0^1 y^{k}\log^{r-1}\frac{1}{y}dy\right] \\
&\qquad \left(\int_0^1 x^{p-1}\log^{q-1}\frac{1}{x}dx = \frac{\Gamma(q)}{p^q}, \quad p\geq 1,\ q>0\right) \\
&= \frac{k^{r+1}}{\beta}\frac{\Gamma(r)}{k^r} - \frac{k^{r+1}}{\beta}\frac{\Gamma(r)}{(k+1)^r} = \frac{k\Gamma(r)}{\beta}\left(1-\left(\frac{k}{k+1}\right)^r\right) = -b_k^{(r)}(\alpha),
\end{aligned}$$

and similarly we show that

$$b_k^{*(r)}(\beta) = -b_k^{(r)}(\beta).$$

Hence

$$\boldsymbol{B}_k^{*(r)} = \begin{bmatrix} r\boldsymbol{b}_k^{*(r)} \\ (r+1)\boldsymbol{b}_k^{*(r+1)} \end{bmatrix} = -B_k^{(r)},$$

and

$$\boldsymbol{K}_k^{*(r)} = \boldsymbol{K}_k^{(r)}.$$

Therefore $s_k^{*(r)} = s_k^{(r)}$, $t_k^{*(r)} = t_k^{(r)}$, $u_k^{*(r)} = u_k^{(r)}$ and $\Delta_{k1}^{*(r)} = \Delta_{k1}^{(r)}$. Thus (3.25) and (3.26) give

$$\begin{aligned}\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} c_{k1}^{*(r)}} &\Bigg[\frac{k^r}{N}\sum_{j=1}^{N}\log^r\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \\ &\cdot\left(c_{k1}^{*(r)} - b_{k1}^{*(r)} k\log\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)\right) \\ &- \Gamma(r+1)\left(c_{k1}^{*(r)} - (r+1)b_{k1}^{*(r)}\right)\Bigg]^2 \\ &+ \frac{N}{c_{k1}^{*(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\log^{r+1}\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+2)\right]^2,\end{aligned}$$

and

$$\begin{aligned}\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} a_{k1}^{*(r)}} &\Bigg[\frac{k^r}{N}\sum_{j=1}^{N}\log^r\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \\ &\cdot\left(a_{k1}^{*(r)} k\log\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - b_{k1}^{*(r)}\right) \\ &- \Gamma(r+1)\left((r+1)a_{k1}^{*(r)} - b_{k1}^{*(r)}\right)\Bigg]^2 \\ &+ \frac{N}{a_{k1}^{*(r)}}\left[\frac{k^{r}}{N}\sum_{j=1}^{N}\log^{r}\left(1+\exp\left(-\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+1)\right]^2.\end{aligned}$$

For $k=1$ we have

$$\begin{aligned}\hat{T}_{1n}^{*(r)} &= \frac{n}{\Delta_{11}^{*(r)} c_{11}^{*(r)}}\Bigg[\frac{1}{n}\sum_{j=1}^{n}\log^r\left(1+\exp\left(-\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right) \\ &\cdot\left(c_{11}^{*(r)} - b_{11}^{*(r)}\log\left(1+\exp\left(-\frac{X_j-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)\right) - \Gamma(r+1)\left(c_{11}^{*(r)} - (r+1)b_{11}^{*(r)}\right)\Bigg]^2\end{aligned}$$

$$+ \frac{n}{c_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \log^{r+1} \left(1 + \exp\left(-\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - \Gamma(r+2) \right]^2,$$

and

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} a_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \log^{r} \left(1 + \exp\left(-\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) \right.$$
$$\cdot \left(a_{11}^{*(r)} \log \left(1 + \exp\left(-\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - b_{11}^{*(r)} \right)$$
$$\left. - \Gamma(r+1) \left((r+1) a_{11}^{*(r)} - b_{11}^{*(r)} \right) \right]^2$$
$$+ \frac{n}{a_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \log^{r} \left(1 + \exp\left(-\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - \Gamma(r+1) \right]^2.$$

4.6. Normal Distributions: $X \sim N(\mu, \sigma^2)$

Here

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty; \quad \Lambda = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 > 0\},$$
$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt,$$
$$\frac{\partial F(x;\mu,\sigma^2)}{\partial \mu} = -f(x), \qquad \frac{\partial F(x;\mu,\sigma^2)}{\partial \sigma^2} = -\frac{x-\mu}{2\sigma^2} f(x),$$
$$\mathcal{I}^{-1} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma^2 \end{bmatrix}$$

and the MLE are $\hat{\mu}_n = \overline{X}_n$ and $\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left(X_i - \overline{X}_n \right)^2$.

Next we use

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

Moreover, we need

$$b_k^{(r)}(\mu) = k^{r+1} E \left[(1 - F(X))^{k-2} \log^{r-1} \frac{1}{1 - F(X)} \frac{\partial F(X;\mu,\sigma^2)}{\partial \mu} \right]$$

$$
\begin{aligned}
&= -k^{r+1} E\left[(1-F(X))^{k-2} \log^{r-1} \frac{1}{1-F(X)} f(X)\right] \\
&= -k^{r+1} \int_{-\infty}^{\infty} \left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{k-2} \log^{r-1} \frac{1}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)} \\
&\qquad \cdot \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)^2 dx \\
&\qquad \left(\frac{x-\mu}{\sigma} = z, \qquad dx = \sigma dz\right) \\
&= -k^{r+1} \frac{1}{\sigma} \int_{-\infty}^{\infty} (1-\Phi(z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(z)} \phi^2(z) dz \\
&= -k^{r+1} \frac{1}{\sigma} \int_{-\infty}^{\infty} \phi(z)\,(1-\Phi(z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(z)} d\Phi(z) \\
&= -k^{r+1} \frac{1}{\sigma} E\left[\phi(Z)\,(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)}\right] = -k^{r+1} \frac{1}{\sigma} E_1(k,r)\,,
\end{aligned}
$$

where

$$
E_1(k,r) = E\left[\phi(Z)\,(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)}\right], \quad Z \sim N(0,1).
$$

Next

$$
\begin{aligned}
b_k^{(r)}(\sigma^2) &= k^{r+1} E\left[(1-F(X))^{k-2} \log^{r-1} \frac{1}{1-F(X)} \frac{\partial F(X;\mu,\sigma^2)}{\partial \sigma^2}\right] \\
&- k^{r+1} \int_{-\infty}^{\infty} \left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{k-2} \log^{r-1} \frac{1}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)} \\
&\qquad \cdot \left(\frac{x-\mu}{2\sigma^2}\right) \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)^2 dx \\
&\qquad \left(\frac{x-\mu}{\sigma} = z, \qquad dx = \sigma dz\right) \\
&= -k^{r+1} \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} z\phi(z)(1-\Phi(z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(z)} d\Phi(z) \\
&= -k^{r+1} \frac{1}{2\sigma^2} E\left[Z\phi(Z)(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)}\right] \\
&\qquad = -k^{r+1} \frac{1}{2\sigma^2} E_2(k,r)\,,
\end{aligned}
$$

where

$$
E_2(k,r) = E\left[Z\phi(Z)\,(1-\Phi(Z))^{k-2} \log^{r-1} \frac{1}{1-\Phi(Z)}\right],
$$

$Z \sim N(0,1)$.

Hence

$$\boldsymbol{B}_k^{(r)} = \begin{bmatrix} rb_k^{(r)}(\mu) & rb_k^{(r)}(\sigma^2) \\ (r+1)b_k^{(r+1)}(\mu) & (r+1)b_k^{(r+1)}(\sigma^2) \end{bmatrix}$$
$$= -k^{r+1} \begin{bmatrix} \frac{r}{\sigma}E_1(k,r) & \frac{r}{2\sigma^2}E_2(k,r) \\ k\frac{r+1}{\sigma}E_1(k,r+1) & k\frac{r+1}{2\sigma^2}E_2(k,r+1) \end{bmatrix}$$

and

$$\boldsymbol{K}_k^{(r)} = \frac{1}{k}\boldsymbol{B}_k^{(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{(r)}\right)'$$
$$= \frac{k^{2(r+1)}}{k} \begin{bmatrix} \frac{r}{\sigma}E_1(k,r) & \frac{r}{2\sigma^2}E_2(k,r) \\ k\frac{r+1}{\sigma}E_1(k,r+1) & k\frac{r+1}{2\sigma^2}E_2(k,r+1) \end{bmatrix} \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$
$$\cdot \begin{bmatrix} \frac{r}{\sigma}E_1(k,r) & k\frac{r+1}{\sigma}E_1(k,r+1) \\ \frac{r}{2\sigma^2}E_2(k,r) & k\frac{r+1}{2\sigma^2}E_2(k,r+1) \end{bmatrix} = \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)} \end{bmatrix},$$

with

$$s_k^{(r)} = r^2k^{2r+1}\left[E_1^2(k,r) + \frac{1}{2}E_2^2(k,r)\right],$$
$$t_k^{(r)} = r(r+1)k^{2r+2}\left[E_1(k,r)E_1(k,r+1) + \frac{1}{2}E_2(k,r)E_2(k,r+1)\right],$$
$$u_k^{(r)} = (r+1)^2k^{2r+3}\left[E_1^2(k,r+1) + \frac{1}{2}E_2^2(k,r+1)\right].$$

Therefore

$$\bigtriangledown_k^{(r)} = \frac{r^2(r+1)^2k^{2(2r+2)}}{2}\left[E_1(k,r)E_2(k,r+1) - E_1(k,r+1)E_2(k,r)\right]^2.$$

Now

$$
\begin{aligned}
a_{k1}^{(r)} &= a^{(r)} - s_k^{(r)} = \Gamma(2r+1) - \Gamma^2(r+1) \\
&\quad - r^2 k^{2r+1} \left[E_1^2(k,r) + \frac{1}{2} E_2^2(k,r) \right] , \\
b_{k1}^{(r)} &= b^{(r)} - t_k^{(r)} = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2) \\
&\quad - r(r+1) k^{2r+2} \left[E_1(k,r)E_1(k,r+1) + \frac{1}{2} E_2(k,r)E_2(k,r+1) \right] , \\
c_{k1}^{(r)} &= c^{(r)} - u_k^{(r)} = \Gamma(2r+3) - \Gamma^2(r+2) \\
&\quad - (r+1)^2 k^{2r+3} \left[E_1^2(k,r+1) + \frac{1}{2} E_2^2(k,r+1) \right] , \\
\Delta_{k1}^{(r)} &= \Delta^{(r)} + \bigtriangledown_k^{(r)} - a^{(r)} u_k^{(r)} - c^{(r)} s_k^{(r)} + 2 b^{(r)} t_k^{(r)} .
\end{aligned}
$$

Using the above quantities (3.23) and (3.24) give

$$
\begin{aligned}
\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} c_{k1}^{(r)}} \Bigg[& \frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^r \\
& \cdot \left(c_{k1}^{(r)} + b_{k1}^{(r)} k \log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right) \right) \\
& - \Gamma(r+1) \left(c_{k1}^{(r)} - (r+1) b_{k1}^{(r)} \right) \Bigg]^2 \\
& + \frac{N}{c_{k1}^{(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2) \right]^2 ,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} a_{k1}^{(r)}} \Bigg[& \frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^r \\
& \cdot \left(a_{k1}^{(r)} k \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right) - b_{k1}^{(r)} \right) \\
& - \Gamma(r+1) \left((r+1) a_{k1}^{(r)} - b_{k1}^{(r)} \right) \Bigg]^2 \\
& + \frac{N}{a_{k1}^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(1 - F(U_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^r - \Gamma(r+1) \right]^2 .
\end{aligned}
$$

For $k=1$

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} c_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^r \cdot \left(c_{11}^{(r)} + b_{11}^{(r)} \log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right) - \Gamma(r+1)\left(c_{11}^{(r)} - (r+1) b_{11}^{(r)} \right) \right]^2 + \frac{n}{c_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r+1} - \Gamma(r+2) \right]^2 ,$$

or

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} a_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^r \cdot \left(a_{11}^{(r)} \left(-\log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right) - b_{11}^{(r)} \right) - \Gamma(r+1)\left((r+1) a_{11}^{(r)} - b_{11}^{(r)} \right) \right]^2 + \frac{n}{a_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(1 - F(X_j; \hat{\mu}_n, \hat{\sigma}_n^2)\right)\right)^{r} - \Gamma(r+1) \right]^2 .$$

Referring now to the dual test, we see similarly that

$$b_k^{*(r)}(\mu) = -k^{r+1} \frac{1}{\sigma} E\left[\phi(Z) \Phi^{k-2}(Z) \log^{r-1} \frac{1}{\Phi(Z)} \right] = -b_k^{(r)}(\mu) ,$$

and

$$b_k^{*(r)}(\sigma^2) = -k^{r+1} \frac{1}{2\sigma^2} E\left[Z\phi(Z) \Phi^{k-2}(Z) \log^{r-1} \frac{1}{\Phi(Z)} \right] = -b_k^{(r)}\left(\sigma^2\right) .$$

Hence we conclude that $\boldsymbol{K}_k^{*(r)} = \boldsymbol{K}_k^{(r)}$ and then (3.25) and (3.26) give

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} c_{k1}^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^r \right.$$

$$\cdot \left(c_{k1}^{*(r)} + b_{k1}^{*(r)} k \log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right) - \Gamma(r+1)\left(c_{k1}^{*(r)} - (r+1) b_{k1}^{*(r)}\right)\Bigg]^2$$

$$+ \frac{N}{c_{k1}^{*(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^{r+1} - \Gamma(r+2)\right]^2,$$

or

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{*(r)} a_{k1}^{*(r)}} \Bigg[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^r$$

$$\cdot \left(a_{k1}^{*(r)} k \left(-\log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right) - b_{k1}^{*(r)}\right)$$

$$- \Gamma(r+1)\left((r+1) a_{k1}^{*(r)} - b_{k1}^{*(r)}\right)\Bigg]^2$$

$$+ \frac{N}{a_{k1}^{*(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log F\left(V_{kj}; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^r - \Gamma(r+1)\right]^2.$$

For $k = 1$

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} c_{11}^{*(r)}} \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log F\left(X_j; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^r$$

$$\cdot \left(c_{11}^{*(r)} + b_{11}^{*(r)} \log F\left(X_j; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right) - \Gamma(r+1)\left(c_{11}^{*(r)} - (r+1) b_{11}^{*(r)}\right)\Bigg]^2$$

$$+ \frac{n}{c_{11}^{*(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log F\left(X_j; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^{r+1} - \Gamma(r+2)\right]^2,$$

and

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{*(r)} a_{11}^{*(r)}} \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log F\left(X_j; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right)^r$$

$$\cdot \left(a_{11}^{*(r)} \left(-\log F\left(X_j; \hat{\mu}_n, \hat{\sigma}_n^2\right)\right) - b_{11}^{*(r)}\right)$$

$$- \Gamma(r+1)\left((r+1) a_{11}^{*(r)} - b_{11}^{*(r)}\right)\Bigg]^2$$

$$+\frac{n}{a_{11}^{*(r)}}\left[\frac{1}{n}\sum_{j=1}^{n}\left(-\log F\left(X_j;\hat{\mu}_n,\hat{\sigma}_n^2\right)\right)^r-\Gamma(r+1)\right]^2.$$

4.7. Cauchy Distributions: $X \sim C(\alpha,\beta)$

Here

$$f(x;\alpha,\beta)=\frac{1}{\pi\beta}\frac{1}{1+((x-\alpha)/\beta)^2},\quad -\infty<x<\infty,$$
$$\boldsymbol{\lambda}=\{(\alpha,\beta):\ \alpha\in\mathbb{R},\beta>0\},$$
$$F(x;\alpha,\beta)=\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta},$$
$$\frac{\partial F}{\partial\alpha}=-f(x),\qquad \frac{\partial F}{\partial\beta}=-\frac{x-\alpha}{\beta}f(x),$$
$$\mathcal{I}^{-1}=2\beta^2\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

and the MLE $\hat{\alpha}_n$ and $\hat{\beta}_n$ are obtained numerically.

Here

$$b_k^{(r)}(\alpha)=k^{r+1}E\left[(1-F(X))^{k-2}\log^{r-1}\frac{1}{1-F(X)}\frac{\partial F(X;\alpha,\beta)}{\partial\alpha}\right]$$
$$-k^{r+1}E\left[\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}\right)^{k-2}\log^{r-1}\frac{1}{\frac{1}{2}-\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}}f(X)\right]$$
$$=-k^{r+1}\frac{1}{\pi^2\beta^2}\int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)^{k-2}$$
$$\cdot\log^{r-1}\frac{1}{\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}}\frac{1}{\left(1+\left(\frac{x-\alpha}{\beta}\right)^2\right)^2}dx$$
$$\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}=y,\quad dy=-\frac{1}{\pi\beta}\frac{1}{1+\left(\frac{x-\alpha}{\beta}\right)^2}dx\right.$$
$$\left.\frac{x-\alpha}{\beta}=\tan\left(\pi\left(\frac{1}{2}-y\right)\right)\right)$$
$$=k^{r+1}\frac{1}{\pi\beta}\int_0^1 y^{k-2}\left(\log^{r-1}\frac{1}{y}\right)\cos^2\pi\left(\frac{1}{2}-y\right)dy$$

$$= -k^{r+1}\frac{1}{\pi\beta}\int_0^1 y^{k-2}\left(\log^{r-1}\frac{1}{y}\right)\sin^2 \pi y dy$$

$$\left(\sin^2 x = \sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j-1}x^{2j}}{(2j)!}\right.$$

$$\left.\text{(cf. Ryzyk and Grandsztejn [19], 1.412.1)}\right)$$

$$= -\frac{1}{\pi\beta}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j-1}\pi^{2j}}{(2j)!}\int_0^1 y^{2j+k-2}\left(\log\frac{1}{y}\right)^{r-1}dy$$

$$\left(\int_0^1 x^{p-1}\log^{q-1}\frac{1}{x}dx = \frac{\Gamma(q)}{p^q},\ p \geq 1, q > 0\right)$$

$$= -k^{r+1}\frac{\Gamma(r)}{\pi\beta}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{2^{2j-1}\pi^{2j}}{(2j+k-1)^r(2j)!}$$

$$= k^{r+1}\frac{\Gamma(r)}{2\pi\beta}\sum_{j=1}^{\infty}(-1)^{j}\frac{(2\pi)^{2j}}{(2j+k-1)^r(2j)!}$$

$$= k^{r+1}\frac{\Gamma(r)}{\beta}\sum_{j=1}^{\infty}(-1)^{j}\frac{(2\pi)^{2j-1}}{(2j+k-1)^r(2j)!}$$

$$= k^{r+1}\frac{\Gamma(r)}{\beta}S_k^{(r)}(2\pi)\,,$$

where

$$S_k^{(r)}(z) = \sum_{j=1}^{\infty}(-1)^{j}\frac{z^{2j-1}}{(2j+k-1)^r(2j)!}.$$

Next

$$b_k^{(r)}(\beta) = k^{r+1}E\left[(1-F(X))^{k-2}\log^{r-1}\frac{1}{1-F(X)}\frac{\partial F(X;\alpha\beta)}{\partial\beta}\right]$$

$$= -k^{r+1}E\left[\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}\right)^{k-2}\right.$$

$$\left.\cdot\left(\log^{r-1}\frac{1}{\frac{1}{2}-\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}}\right)\frac{X-\alpha}{\beta}f(X)\right]$$

$$= -k^{r+1}\frac{1}{\pi^2\beta^2}\int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)^{k-2}$$

$$\cdot \left(\log^{r-1} \frac{1}{\frac{1}{2} - \frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}}\right) \frac{x-\alpha}{\beta} \frac{1}{\left(1+\left(\frac{x-\alpha}{\beta}\right)^2\right)^2} dx$$

$$= k^{r+1}\frac{1}{\pi\beta}\int_0^1 y^{k-2}\left(\log\frac{1}{y}\right)^{r-1} \tan\left(\pi\left(\frac{1}{2}-y\right)\right)\cos^2\pi\left(\frac{1}{2}-y\right)dy$$

$$= -k^{r+1}\frac{1}{\pi\beta}\int_0^1 y^{k-2}\left(\log\frac{1}{y}\right)^{r-1} \sin\left(\pi\left(\frac{1}{2}-y\right)\right)$$

$$\cdot \cos\left(\pi\left(\frac{1}{2}-y\right)\right)dy$$

$$= -k^{r+1}\frac{1}{2\pi\beta}\int_0^1 y^{k-2}\left(\log\frac{1}{y}\right)^{r-1}\sin 2\pi y dy$$

$$\left(\sin x = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x^{2j-1}}{(2j-1)!}\right)$$

$$= -k^{r+1}\frac{1}{2\pi\beta}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2\pi)^{2j-1}}{(2j-1)!}\int_0^1 y^{k+2j-3}\log^{r-1}\frac{1}{y}dy$$

$$= -k^{r+1}\frac{\Gamma(r)}{2\pi\beta}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2\pi)^{2j-1}}{(2j+k-2)^r(2j-1)!}$$

$$= k^{r+1}\frac{\Gamma(r)}{\beta}\sum_{j=1}^{\infty}(-1)^{j}\frac{(2\pi)^{2j-2}}{(2j+k-2)^r(2j-1)!} = k^{r+1}\frac{\Gamma(r)}{\beta}C_k^{(r)}(2\pi)\,,$$

where

$$C_k^{(r)}(z) = \sum_{j=1}^{\infty}(-1)^j\frac{z^{2j-2}}{(2j+k-2)^r(2j-1)!}.$$

Thus

$$b_k^{(r)}(\alpha) = k^{r+1}\frac{\Gamma(r)}{\beta}S_k^{(r)}(2\pi)\,,$$

$$b_k^{(r)}(\beta) = k^{r+1}\frac{\Gamma(r)}{\beta}C_k^{(r)}(2\pi).$$

Hence

$$\boldsymbol{B}_k^{(r)} = \begin{bmatrix} rb_k^{(r)}(\alpha) & rb_k^{(r)}(\beta) \\ (r+1)b_k^{(r+1)}(\alpha) & (r+1)b_k^{(r+1)}(\beta)\end{bmatrix}$$

$$= k^{r+1}\frac{\Gamma(r+1)}{\beta}\begin{bmatrix} S_k^{(r)}(2\pi) & C_k^{(r)}(2\pi) \\ (r+1)kS_k^{(r+1)}(2\pi) & (r+1)kC_k^{(r+1)}(2\pi)\end{bmatrix}\,,$$

$$\boldsymbol{K}_k^{(r)} = \frac{1}{k}\boldsymbol{B}_k^{(r)}\mathcal{I}^{-1}\left(\boldsymbol{B}_k^{(r)}\right)'$$

$$= k^{2(r+1)}\frac{\Gamma^2(r+1)}{\beta^2}\begin{bmatrix} S_k^{(r)}(2\pi) & C_k^{(r)}(2\pi) \\ (r+1)kS_k^{(r+1)}(2\pi) & (r+1)kC_k^{(r+1)}(2\pi)\end{bmatrix}$$

$$\cdot 2\beta^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix} S_k^{(r)}(2\pi) & (r+1)kS_k^{(r+1)}(2\pi) \\ C_k^{(r)}(2\pi) & (r+1)kC_k^{(r+1)}(2\pi)\end{bmatrix} = \begin{bmatrix} s_k^{(r)} & t_k^{(r)} \\ t_k^{(r)} & u_k^{(r)}\end{bmatrix},$$

with

$$s_k^{(r)} = 2k^{2(r+1)}\Gamma^2(r+1)\left[\left(S_k^{(r)}(2\pi)\right)^2 + \left(C_k^{(r)}(2\pi)\right)^2\right],$$

$$t_k^{(r)} = 2k^{2r+3}\Gamma(r+1)\Gamma(r+2)$$
$$\cdot\left[S_k^{(r)}(2\pi)S_k^{(r+1)}(2\pi) + C_k^{(r)}(2\pi)C_k^{(r+1)}(2\pi)\right],$$

$$u_k^{(r)} = 2k^{2(r+2)}\Gamma^2(r+2)\left[\left(S_k^{(r+1)}(2\pi)\right)^2 + \left(C_k^{(r+1)}(2\pi)\right)^2\right].$$

Therefore

$$\bigtriangledown_k^{(r)} = 4k^{4(r+1)}\Gamma^2(r+1)\Gamma^2(r+2)$$
$$\cdot\left[S_k^{(r)}(2\pi)C_k^{(r+1)}(2\pi) - S_k^{(r+1)}(2\pi)C_k^{(r)}(2\pi)\right]^2.$$

Now

$$a_{k1}^{(r)} = a^{(r)} - s_k^{(r)} = \Gamma(2r+1) - \Gamma^2(r+1)$$
$$- 2k^{2(r+1)}\Gamma^2(r+1)\left[\left(S_k^{(r)}(2\pi)\right)^2 + \left(C_k^{(r)}(2\pi)\right)^2\right],$$

$$b_{k1}^{(r)} = b^{(r)} - t_k^{(r)} = \Gamma(2r+2) - \Gamma(r+1)\Gamma(r+2)$$
$$- 2k^{2r+3}\Gamma(r+1)\Gamma(r+2)\left[S_k^{(r+1)}(2\pi)S_k^{(r)}(2\pi) + C_k^{(r)}C_k^{(r+1)}(2\pi)\right],$$

$$c_{k1}^{(r)} = c^{(r)} - u_k^{(r)} = \Gamma(2r+3) - \Gamma^2(r+2)$$
$$- 2k^{2(r+2)}\Gamma^2(r+2)\left[\left(S_k^{(r+1)}(2\pi)\right)^2 + \left(C_k^{(r+1)}(2\pi)\right)^2\right],$$

$$\Delta_{k1}^{(r)} = \Delta^{(r)} + \bigtriangledown_k^{(r)} - a^{(r)}u_k^{(r)} - c^{(r)}s_k^{(r)} + 2b^{(r)}t_k^{(r)}.$$

Using the above quantities (3.23) and (3.24) give

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} c_{k1}^{(r)}} \Bigg[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^r \cdot \left(c_{k1}^{(r)} + b_{k1}^{(r)} k \log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - \Gamma(r+1) \left(c_{k1}^{(r)} - (r+1) b_{k1}^{(r)} \right) \Bigg]^2 + \frac{N}{c_{k1}^{(r)}} \left[\frac{k^{r+1}}{N} \sum_{j=1}^{N} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_N}{\hat{\beta}_n} \right) \right)^{r+1} - \Gamma(r+2) \right]^2,$$

or

$$\hat{T}_{kN}^{(r)} = \frac{N}{\Delta_{k1}^{(r)} a_{k1}^{(r)}} \Bigg[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^r \cdot \left(a_{k1}^{(r)} k \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - b_{k1}^{(r)} \right) - \Gamma(r+1) \left((r+1) a_{k1}^{(r)} - b_{k1}^{(r)} \right) \Bigg]^2 + \frac{N}{a_{k1}^{(r)}} \left[\frac{k^r}{N} \sum_{j=1}^{N} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{U_{kj} - \hat{\alpha}_N}{\hat{\beta}_n} \right) \right)^r - \Gamma(r+1) \right]^2.$$

For $k = 1$

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} c_{11}^{(r)}} \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^r \cdot \left(c_{11}^{(r)} + b_{11}^{(r)} \log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - \Gamma(r+1) \left(c_{11}^{(r)} - (r+1) b_{11}^{(r)} \right) \Bigg]^2 + \frac{n}{c_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^{r+1} - \Gamma(r+2) \right]^2,$$

or

$$\hat{T}_{1n}^{(r)} = \frac{n}{\Delta_{11}^{(r)} a_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^r \right.$$
$$\cdot \left(a_{11}^{(r)} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) - b_{11}^{(r)} \right)$$
$$\left. - \Gamma(r+1) \left((r+1) a_{11}^{(r)} - b_{11}^{(r)} \right) \right]^2$$
$$+ \frac{n}{a_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right)^r - \Gamma(r+1) \right]^2 .$$

Referring to the dual test

$$b_k^{*(r)}(\alpha) = -k^{r+1} E\left[(F(X))^{k-2} \log^{r-1} \frac{1}{F(X)} \frac{\partial F(X;\alpha,\beta)}{\partial \alpha} \right]$$
$$= k^{r+1} E\left[\left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{X-\alpha}{\beta} \right)^{k-2} \log^{r-1} \frac{1}{\frac{1}{2} + \frac{1}{\pi} \arctan \frac{X-\alpha}{\beta}} f(X) \right]$$
$$= k^{r+1} \frac{1}{\pi^2 \beta^2} \int_{-\infty}^{\infty} \left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{x-\alpha}{\beta} \right)^{k-2}$$
$$\cdot \left(-\log\left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{x-\alpha}{\beta} \right) \right)^{r-1} \frac{1}{\left(1 + \left(\frac{x-\alpha}{\beta} \right)^2 \right)^2} dx$$
$$\left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{x-\alpha}{\beta} = y, \quad dy = \frac{1}{\pi\beta} \frac{1}{1 + ((x-\alpha)/\beta)^2} dx \right.$$
$$\left. \frac{x-\alpha}{\beta} = -\tan \pi \left(\frac{1}{2} - y \right) \right)$$
$$= k^{r+1} \frac{1}{\pi\beta} \int_0^1 y^{k-2} \left(\log^{r-1} \frac{1}{y} \right) \cos^2 \pi \left(\frac{1}{2} - y \right) dy$$
$$= k^{r+1} \frac{1}{\pi\beta} \int_0^1 y^{k-2} \left(\log^{r-1} \frac{1}{y} \right) \sin^2 \pi y dy = -k^{r+1} \frac{\Gamma(r)}{\beta} S_k^{(r)}(2\pi) = -b_k^{(r)}(\alpha).$$

Next

$$b_k^{*(r)}(\beta) = -k^{r+1} E\left[(F(X))^{k-2} \log^{r-1} \frac{1}{F(X)} \frac{\partial F(X;\alpha,\beta)}{\partial \beta} \right]$$

$$= k^{r+1} E\left[\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}\right)^{k-2}\log^{r-1}\frac{1}{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{X-\alpha}{\beta}} f(X)\right]$$

$$= k^{r+1}\frac{1}{\pi^2\beta^2}\int_{-\infty}^{\infty}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)^{k-2}$$

$$\cdot\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x-\alpha}{\beta}\right)\right)^{r-1}\frac{x-\alpha}{\beta}\frac{1}{\left(1+\left(\frac{x-\alpha}{\beta}\right)^2\right)^2}dx$$

$$= -k^{r+1}\frac{\Gamma(r)}{\beta}C_k^{(r)}(2\pi).$$

Hence

$$\boldsymbol{K}_k^{*(r)} = \boldsymbol{K}_k^{(r)},$$

and (3.25) and (3.26) give

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{(r)}c_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^r\right.$$

$$\cdot\left(c_{k1}^{(r)}+b_{k1}^{(r)}k\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)$$

$$\left.-\Gamma(r+1)\left(c_{k1}^{(r)}-(r+1)b_{k1}^{(r)}\right)\right]^2$$

$$+\frac{N}{c_{k1}^{(r)}}\left[\frac{k^{r+1}}{N}\sum_{j=1}^{N}\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_N}{\hat{\beta}_n}\right)\right)^{r+1}-\Gamma(r+2)\right]^2,$$

or

$$\hat{T}_{kN}^{*(r)} = \frac{N}{\Delta_{k1}^{(r)}a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^r\right.$$

$$\cdot\left(a_{k1}^{(r)}k\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_n}{\hat{\beta}_n}\right)\right)-b_{k1}^{(r)}\right)$$

$$\left.-\Gamma(r+1)\left((r+1)a_{k1}^{(r)}-b_{k1}^{(r)}\right)\right]^2$$

$$+\frac{N}{a_{k1}^{(r)}}\left[\frac{k^r}{N}\sum_{j=1}^{N}\left(-\log\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{V_{kj}-\hat{\alpha}_N}{\hat{\beta}_n}\right)\right)^r-\Gamma(r+1)\right]^2.$$

For $k = 1$

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{(r)} c_{11}^{(r)}} \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^r \cdot \left(c_{11}^{(r)} + b_{11}^{(r)} \log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - \Gamma(r+1)\left(c_{11}^{(r)} - (r+1) b_{11}^{(r)}\right) \Bigg]^2 + \frac{n}{c_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^{r+1} - \Gamma(r+2) \right]^2,$$

and

$$\hat{T}_{1n}^{*(r)} = \frac{n}{\Delta_{11}^{(r)} a_{11}^{(r)}} \Bigg[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^r \cdot \left(a_{11}^{(r)} \left(-\log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right) - b_{11}^{(r)} \right) - \Gamma(r+1)\left((r+1) a_{11}^{(r)} - b_{11}^{(r)}\right) \Bigg]^2 + \frac{n}{a_{11}^{(r)}} \left[\frac{1}{n} \sum_{j=1}^{n} \left(-\log\left(\frac{1}{2} + \frac{1}{\pi}\arctan\frac{X_j - \hat{\alpha}_n}{\hat{\beta}_n}\right)\right)^{r} - \Gamma(r+1) \right]^2.$$

5. Comparisons with Other Tests

5.1. Tests of Exponentiality

When $n = 20$ the statistics $\hat{T}_{kN}^{(r)}$ were investigated for $k = 1, 2, 4, 5$ and $r = -0.1, 0.5, 1, 1.5, 3, 5$ (with the exception of $k = 1$ and $r = 1$ as noted in Subsection 4.1). The critical values for 5% tests were simulated using 100,000 samples. Moreover, we consider the component

$$\hat{T}_{1n;c}^{(r)} := \frac{n}{\Gamma(2r+1) - (r^2+1)\Gamma^2(r+1)} \left[\left(\frac{1}{\overline{X}_n}\right)^r \frac{1}{n} \sum_{j=1}^{n} X_j^r - \Gamma(r+1) \right]^2$$

	k	r	$W(\theta,\tau):\tau=$			$G(\lambda,\beta):\beta=$			$Ln(\nu,\sigma):\sigma=$			$Par(\theta,\alpha):\alpha=$		
			2	1.5	0.5	2	1.5	0.5	0.775	1	1.2	1	2	4
$\hat{T}_{kN}^{(r)}$	1	-0.1	67	14	96	14	3	77	11	13	31	82	46	21
		0.5	86	32	96	35	11	73	35	20	33	84	49	23
		1.5	93	44	92	42	15	58	29	17	35	**85**	50	23
		3	68	16	84	13	4	45	10	19	**41**	84	**53**	**27**
		5	0	0	74	0	1	35	5	20	38	80	49	25
	2	-0.1	47	12	94	12	4	74	8	4	11	70	30	12
		0.5	78	29	90	33	12	59	29	6	13	72	32	13
		1	74	27	76	27	10	39	19	7	16	68	30	12
		1.5	49	16	57	14	7	26	9	7	16	60	26	11
		3	0	1	9	1	3	10	2	4	6	6	6	6
		5	0	1	11	1	2	11	2	5	6	7	8	6
	4	-0.1	59	20	88	20	8	65	17	2	1	46	13	6
		0.5	71	28	83	31	13	52	32	5	3	50	14	6
		1	78	35	64	39	16	31	39	7	3	43	10	4
		1.5	78	36	33	38	17	11	33	6	2	22	3	3
		3	37	16	1	14	9	2	8	3	2	1	2	3
		5	14	9	1	7	6	3	4	3	2	1	2	3
	5	-0.1	61	21	86	23	9	62	21	3	1	39	11	6
		0.5	71	29	81	32	13	50	33	5	2	44	11	5
		1	79	37	60	43	19	29	**45**	9	2	34	7	4
		1.5	80	39	0	43	20	1	41	8	2	0	1	2
		3	52	22	0	21	11	2	14	4	2	0	1	3
		5	31	15	1	13	8	2	7	3	2	0	2	3
WE			94	48	78	43	17	35	25	17	33	79	44	20
G			95	51	91	48	19	55	31	12	30	84	47	20
$L(0.5)$			92	48	93	47	19	61	33	7	17	79	37	14
P			93	48	90	45	18	52	28	12	29	82	44	18
CO			**96**	**53**	96	**56**	**22**	73	42	11	22	82	44	18
$\hat{T}_{1n;c}^{(r)}$		-0.1	87	37	**97**	41	13	**79**	35	5	11	79	37	15
		0.5	**96**	52	95	54	21	67	39	12	26	84	46	20
		1.5	92	40	87	36	12	48	22	18	37	84	50	24
		3.0	34	4	80	3	1	40	6	**21**	40	82	51	**27**
		5.0	0	0	73	0	1	34	5	**21**	38	80	48	25

Table 1: Powers of 5% tests

of $\hat{T}_{1n}^{(r)}$ from Section 4 ($\mathbf{1}^0$). The alternative distributions are:

Weibull: $X \sim W(\theta, \tau)$

$$f(x) = \frac{\tau}{\theta}\left(\frac{x}{\theta}\right)^{\tau-1} e^{-(x/\theta)^{\tau}}, \qquad \theta,\ \tau > 0\,.$$

Gamma: $X \sim G(\lambda, \beta)$

$$f(x) = \frac{\lambda^{\beta}}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x}, \qquad \lambda,\ \beta > 0\,.$$

	k	r	1	7	14	32	37	38	44	50	56
$\hat{T}^{(r)}_{kN}$	1	-0.1	1	1	11	77	4	11	10	42	98
		0.5	1	1	11	76	3	13	17	49	98
		1.0	1	1	10	71	1	9	**19**	49	96
		1.5	0	0	10	62	0	0	18	45	94
		3.0	1	1	10	72	0	4	**19**	**50**	97
		5.0	0	0	10	46	0	0	17	43	91
	2	-0.1	8	5	8	67	6	13	10	35	97
		0.5	14	7	5	60	6	6	6	7	55
		1.0	11	7	5	32	5	6	7	9	5
		1.5	10	7	4	21	5	7	8	10	9
		3.0	10	7	4	1	5	4	6	7	5
		5.0	7	6	4	2	4	2	7	8	6
$\hat{T}^{*(r)}_{kN}$	1	-0.1	1	1	11	77	2	10	17	47	96
		0.5	1	1	11	76	4	13	13	47	99
		1.0	1	1	10	71	4	12	4	32	99
		1.5	0	1	10	62	4	10	0	0	31
		3.0	1	1	10	72	5	13	3	25	97
		5.0	0	0	10	46	4	9	0	0	0
	2	-0.1	8	5	8	67	6	15	13	40	96
		0.5	14	8	5	60	7	9	5	9	57
		1.0	11	7	5	33	**8**	12	5	11	71
		1.5	10	7	4	21	**8**	12	5	10	64
		3.0	9	7	4	1	7	9	4	2	0
		5.0	7	6	4	2	7	10	3	1	0
K^2			38	**14**	**13**	79	4	12	14	40	96
R			40	4	10	80	5	11	14	39	96
W			**48**	2	**13**	91	**8**	**29**	**19**	46	**100**
Y			8	7	10	**92**	6	6	7	24	96

Table 2: Powers of 5% tests

Log–normal: $X \sim \mathrm{Ln}(\nu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}\exp\left(-\frac{1}{2\sigma^2}(\log x - \nu)^2\right), \qquad \nu \in \mathbb{R},\ \sigma > 0\,.$$

Pareto: $X \sim Par(\theta, \alpha)$

$$f(x) = \alpha\theta^{\alpha}\frac{1}{(x+\theta)^{\alpha+1}}, \qquad \theta,\ \alpha > 0.$$

The powers are shown in the Table 1, together with the following tests, taken from Ascher [2].

Hahn and Spurrier's test

$$WE = (n-1)S_n^2/n^2\overline{X}_n^2, \qquad \text{where } S_n^2 \text{ is the sample variance}.$$

Gini's statistic

$$G = \sum_{i=1}^{n-1} iD_{i+1}/n(n-1)\overline{X}_n, \qquad D_{i+1} = (n-i)\left(X_{i+1:n} - X_{i:n}\right).$$

Lorenz statistic. For given $p \in (0,1)$

$$L(p) = \sum_{i=1}^{[np]} X_{i:n}/n\overline{X}_n .$$

Pietra's test.

$$P = \sum_{i=1}^{n} |X_i - \overline{X}_n|/2n\overline{X}_n .$$

Cox and Oakes' test.

$$CO = n + \sum_{i=1}^{n} \log X_i - \left(\sum_{i=1}^{n} X_i \log X_i\right)/\overline{X}_n.$$

We note that when $k = 1$ and for some values of r our tests compare favorably with known recommended tests.

5.2. Tests of Normality

Here the statistics $\hat{T}_{kN}^{(r)}$ and $\hat{T}_{kN}^{*(r)}$ were investigated for a sample of size 20, when $k = 1, 2$ and $r = -0.1, 0.5, 1, 1.5, 4, 5$. The critical values for 5% tests were simulated using 100,000 samples. The alternatives used were chosen from paper Bowman et al [4] with the numbering used there, they are:

The powers against these alternatives were simulated using 25,000 samples. The Table 2 also shows the powers of the following tests taken from Bowman et al [4], where only 200 samples were used for simulation.

K^2: The Bowman & Shenton test,

R: The rectangle test,

W: The Shapiro–Wilk test,

Y: The D'Agostino tests.

Here the Shapiro-Wilk test is usually the best. But our tests perform also quite well against some alternatives.

	Symmetric		Skew
1	X, where $Y = \frac{1}{2}\ln\left(\frac{X}{1-X}\right)$ and $Y \sim N(0,1)$	37	$X \sim$Beta$(3,2)$
7	$X \sim$Beta$(2,2)$	38	$X \sim$Beta$(2,1)$
14	$X \sim$t(10)	44	$X \sim$Weibull$(1,2)$
32	$X \sim$t(1)	50	$X \sim \chi^2(4)$
		56	$X \sim$Weibull$\left(1,\frac{1}{2}\right)$

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