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A FUNCTIONAL EQUATION FOR SQUARING A SUM OF FOUR NUMBERS

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Abstract: We introduce a new functional equation which is deduced from an expansion formula for the square of a sum of four real numbers, $f(x+y, z+w) = f(x, y) + f(y, z) + f(z, w) + f(w, x) - f(x, -z) - f(y, -w)$. We will solve this equation and investigate its Hyers-Ulam stability.

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1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [18]). Among those was the question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

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If the answer is affirmative, the functional equation for homomorphisms is said to be stable in the sense of Hyers and Ulam because the first result concerning the stability of functional equations was presented by D. H. Hyers. Indeed, he has answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces (see [9].)

We may find a number of papers concerning the stability results of various functional equations (see [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16] and the references cited therein).

It is easy to check that the following equality

$$(x+y+z+w)^2 = (x+y)^2 + (y+z)^2 + (z+w)^2 + (w+x)^2 - (x-z)^2 - (y-w)^2$$

holds true for all $x, y, z, w \in \mathbb{R}$. From this equality we can deduce a functional equation

$$f(x+y, z+w) = f(x,y) + f(y,z) + f(z,w) + f(w,x) - f(x,-z) - f(y,-w), \quad (1)$$

where $(x+y)^2$ is the model for $f(x,y)$.

In this paper, we will prove that a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution of the equation (1) if and only if there exists a quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = q(x+y)$ for all $x, y \in \mathbb{R}$. Furthermore, we will prove the Hyers-Ulam stability of the functional equation (1).

2. Solutions of (1)

We will first solve the functional equation (1) in the following theorem. Indeed, the functional equation (1) is 'equivalent' to the quadratic functional equation, $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for the class of real functions. Every solution of the quadratic functional equation is called a quadratic function.

Theorem 1. *A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (1) *for all $x, y, z, w \in \mathbb{R}$ if and only if there exists a quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = q(x+y)$ for all $x, y \in \mathbb{R}$.*

Proof. If we put $x = y = z = w = 0$ in (1), then we immediately obtain

$$f(0,0) = 0. \quad (2)$$

If we set $y = z = w = 0$ in (1), then we have

$$f(x, 0) = f(0, x) \tag{3}$$

for each $x \in \mathbb{R}$. Put $z = w = 0$ in (1) and make use of (2) and (3) to get

$$f(x + y, 0) = f(x, y) \tag{4}$$

for all $x, y \in \mathbb{R}$. If we put $y = w = 0$ in (1) and apply (2), (3) and (4) to this case, then we obtain

$$f(x + z, 0) + f(x - z, 0) = 2f(x, 0) + 2f(z, 0) \tag{5}$$

for any $x, z \in \mathbb{R}$.

If we define $q(x) = f(x, 0)$ for every $x \in \mathbb{R}$ and if we change the notation of z by y, then the last equality yields

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)$$

for all $x, y \in \mathbb{R}$, i.e., q is a quadratic function.

It follows from (3) and (5) that

$$f(0, x + z) + f(0, x - z) = 2f(0, x) + 2f(0, z)$$

for all $x, z \in \mathbb{R}$. If we define $q_1(x) = f(0, x)$ for $x \in \mathbb{R}$, then q_1 is also a quadratic function. In view of (3), we get $q(x) = q_1(x)$ for each $x \in \mathbb{R}$.

Consequently, the equality (4) implies that $f(x, y) = q(x+y)$ holds true for all $x, y \in \mathbb{R}$.

Now, let $q : \mathbb{R} \to \mathbb{R}$ be an arbitrary quadratic function. Then, q is an even function and it satisfies

$$\begin{aligned}
&q(x + y + z + w) + q(x + y - z - w) = 2q(x + y) + 2q(z + w),\\
&q(y + z + w + x) + q(y + z - w - x) = 2q(y + z) + 2q(w + x),\\
&-q(x - z + y - w) - q(x - z - y + w) = -2q(x - z) - 2q(y - w)
\end{aligned}$$

for all $x, y, z, w \in \mathbb{R}$. Summing up the above three equalities with consideration of the evenness of q, we have

$$\begin{aligned}
q(x + y + z + w) \quad = \quad & q(x + y) + q(y + z) + q(z + w) + q(w + x)\\
& - q(x - z) - q(y - w)
\end{aligned}$$

for any $x, y, z, w \in \mathbb{R}$.

If we define $f(x, y) := q(x+y)$ for all $x, y \in \mathbb{R}$, then the last equality implies that f satisfies the functional equation (1) for all $x, y, z, w \in \mathbb{R}$. □

3. Hyers-Ulam Stability of (1)

We will now investigate a stability problem of the functional equation (1). As a matter of convenience, we will set

$$\begin{aligned} Df(x,y,z,w) &:= f(x+y,z+w) \\ &\quad - f(x,y) - f(y,z) - f(z,w) - f(w,x) + f(x,-z) + f(y,-w) \end{aligned}$$

for all $x, y, z, w \in \mathbb{R}$.

In view of Theorem 1, we can guess that the stability of (1) is strongly connected with quadratic functions.

Theorem 2. *If a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality*

$$|Df(x,y,z,w)| \leq \varepsilon \tag{6}$$

for all $x, y, z, w \in \mathbb{R}$ and for some $\varepsilon \geq 0$, then there exists a quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x,y) - q(x+y)| \leq 7\varepsilon$$

for all $x, y \in \mathbb{R}$.

Proof. Put $x = y = z = w = 0$ in (6) to get

$$|f(0,0)| \leq \varepsilon. \tag{7}$$

Now, we set $y = z = w = 0$ in (6) and we obtain

$$|f(x,0) - f(0,x) - f(0,0)| \leq \varepsilon \tag{8}$$

for any $x \in \mathbb{R}$. If we put $z = w = 0$ in (6), then we have

$$|f(x+y,0) - f(x,y) - f(0,0) - f(0,x) + f(x,0)| \leq \varepsilon \tag{9}$$

for all $x, y \in \mathbb{R}$. Let $y = w = 0$ in (6). Then, we get

$$\begin{aligned} &|f(x,z) - f(x,0) - f(0,z) - f(z,0) \\ &\quad - f(0,x) + f(x,-z) + f(0,0)| \leq \varepsilon \end{aligned} \tag{10}$$

for all $x, z \in \mathbb{R}$.

By (7), (8), (9) and (10), we obtain

$$\begin{aligned} &|f(x+z,0) + f(x-z,0) - 2f(x,0) - 2f(z,0)| \\ &\quad \leq |f(x+z,0) - f(x,z) - f(0,0) - f(0,x) + f(x,0)| \end{aligned}$$

$$
\begin{aligned}
&+ |f(x-z,0) - f(x,-z) - f(0,0) - f(0,x) + f(x,0)| \\
&+ |-3f(x,0) + 3f(0,x) + 3f(0,0)| + |-f(z,0) + f(0,z) + f(0,0)| \\
&+ |f(x,z) - f(x,0) - f(0,z) - f(z,0) - f(0,x) \\
&+ f(x,-z) + f(0,0)| + |-3f(0,0)| \leq 10\varepsilon \,, \qquad (11)
\end{aligned}
$$

for all $x, z \in \mathbb{R}$.

According to a theorem of F. Skof [17] (or P.W. Cholewa [3]), there exists a unique quadratic function $q : \mathbb{R} \to \mathbb{R}$ such that

$$
|f(x,0) - q(x)| \leq 5\varepsilon \qquad (12)
$$

for any $x \in \mathbb{R}$.

On the other hand, it follows from (7), (8) and (11) that

$$
\begin{aligned}
&|f(0,x+z) + f(0,x-z) - 2f(0,x) - 2f(0,z)| \\
&\leq |f(0,x+z) - f(x+z,0) + f(0,0)| \\
&+ |f(0,x-z) - f(x-z,0) + f(0,0)| + |-2[f(0,x) - f(x,0) + f(0,0)]| \\
&+ |-2[f(0,z) - f(z,0) + f(0,0)]| \\
&+ |f(x+z,0) + f(x-z,0) - 2f(x,0) - 2f(z,0)| + |2f(0,0)| \leq 18\varepsilon \,,
\end{aligned}
$$

for all $x, z \in \mathbb{R}$.

Due to a theorem of F. Skof [17], there exists a unique quadratic function $q_1 : \mathbb{R} \to \mathbb{R}$ such that

$$
|f(0,x) - q_1(x)| \leq 9\varepsilon \qquad (13)
$$

for each $x \in \mathbb{R}$.

By using (7), (8), (12) and (13), we have

$$
\begin{aligned}
&|q_1(x) - q(x)| \leq |q_1(x) - f(0,x)| + |f(x,0) - q(x)| \\
&+ |f(0,x) - f(x,0) + f(0,0)| + |-f(0,0)| \leq 16\varepsilon \,, \qquad (14)
\end{aligned}
$$

for all $x \in \mathbb{R}$. Since q and q_1 are quadratic functions, they satisfy

$$
q(2^n x) = 4^n q(x) \quad \text{and} \quad q_1(2^n x) = 4^n q_1(x)
$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

If $q(x) \neq q_1(x)$ for some x, then it follows from the last equalities and (14) that

$$
|q_1(2^n x) - q(2^n x)| = 4^n |q_1(x) - q(x)| \leq 16\varepsilon
$$

for all $n \in \mathbb{N}$, a contradiction. Hence, we have $q(x) = q_1(x)$ for every $x \in \mathbb{R}$.

In view of (7), (8) and (12), we get

$$|f(0,x) - q(x)| \le |f(0,x) - f(x,0) + f(0,0)| \\ + |f(x,0) - q(x)| + |-f(0,0)| \le 7\varepsilon\,,$$

for every $x \in \mathbb{R}$.

Finally, it follows from (8), (9) and (12) that

$$|f(x,y) - q(x+y)| \le |-f(x+y,0) + f(x,y) + f(0,0) + f(0,x) - f(x,0)| \\ + |f(x+y,0) - q(x+y)| + |f(x,0) - f(0,x) - f(0,0)| \le 7\varepsilon\,,$$

for all $x, y \in \mathbb{R}$. □

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