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THE SACCHERI QUADRILATERAL, TRANSLATIONS AND TESSELLATIONS IN THE HYPERBOLIC PLANE

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Abstract: We describe the construction of the Saccheri quadrilateral in the hyperbolic plane using the notion of translation according to a hyperbolic line. We developed some software, in *Mathematica*, oriented to draw the hyperbolic isometries, so much on the Poincaré's half-plane and disk. With this software we have created the graphics are given in this paper. Also, the Saccheri quadrilaterals that tile the hyperbolic plane is described, and an explicit example of these tessellations is given.

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1. Introduction

The construction of a Saccheri quadrilateral in the hyperbolic plane is a problem requiring the use of certain isometries. However, in the literature on this matter, there are no explicit algorithms based on the use of the hyperbolic rule and compass. The introduction of computer resources in the scientific field has provided different computational means for the constructive resolution of some hyperbolic geometric problems.

We refer the reader to [3] [4], which presents an electronic tool whose computational support is *Mathematica* software. This tool consists of modules that allow us to draw different hyperbolic constructions in Poincare's models for the hyperbolic plane, usually denoted by H^2 and D^2. Such constructions include reflections, rotations, translations, glide reflections, and the orbits of a point. These isometries and geometric loci act on the hyperbolic plane; and if a Euclidean element appears in some representation of this plane, we shall note it in an explicit way.

The aim of this paper is to describe the construction of the Saccheri quadrilateral using the notion of translation according to a line. The method is implemented using the computer software indicated, with which we created the different graphics shown. Finally, the Saccheri quadrilateral that tiles the hyperbolic plane is described, and an explicit example is given.

We begin by distinguishing the two connected components in H^2 that determine the complement of a geodesic l - using the notion of normal vector of it in H^2 - at a point (x, y), defined in one of the following ways:

(a) $\overrightarrow{N} = \left(\frac{x-k_1}{r}, \frac{y}{r}\right)$, if the geodesic is of the type $x(t) = r\cos t + k_1$; $y(t) = r\sin t$, with $t \in (0, \pi)$;

(b) $\overrightarrow{N} = (1, 0)$, if the geodesic is of the type $x(t) = k_2$; $y(t) = t$, with $t \in \mathbb{R}^+$.

These connected components are called the positive and negative half-planes of l, and they are distinguished in the following way: a geodesic through a point A in l, tangent to the normal vector $\overrightarrow{N}$ of l at point A, goes from the negative half-plane to the positive half-plane when it is crossed according to the orientation of the normal vector $\overrightarrow{N}$.

At the same time, the angle determined by two geodesics at a common point of H^2 is equal to the angle determined by their normal vectors at this point, and this angle is considered oriented.

The concept of orientation is inherent and essential for the development of Euclidean geometry. Likewise, when we try to solve different hyperbolic constructive problems, we rely on the formal notion of this concept. For instance,

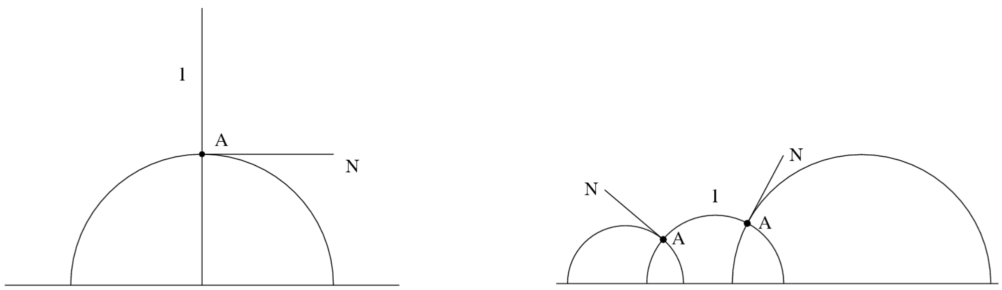

Figure 1: The normal vector and the half-planes of some geodesics

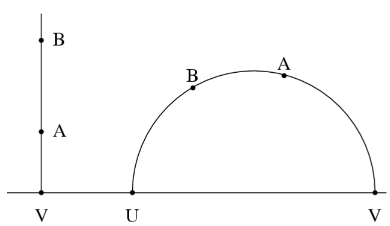

Figure 2: The orientation from A to B is positive

if there are rotations, the positively and negatively oriented angles must be determined; and work with translations calls for the determination of the positive and negative orientation for the translation of a point along a line.

For this reason, a formal definition, both clear and inequivocal, of orientation is essential for elaborating the necessary computer tools for the constructive resolution of different hyperbolic geometric problems, such as the determination of a Saccheri quadrilateral. Consequently, if A, B and C are three points in the hyperbolic plane, the *orientation* of the angle $\angle ABC$ refers to its Euclidean orientation. If l is the line determined by A and B one of two situations may result. Firstly, if l is of the type $x(t) = r\cos t + k_1$; $y(t) = r\sin t$, with $t \in (0, \pi)$, and if we consider C to be the center of the Euclidean half-circumference corresponding with the line l, the *orientation* from A to B is *positive* if the angle $\angle ACB$ is positive. Secondly, if l is of the type $x(t) = k_2$; $y(t) = t$, with $t \in \mathbb{R}^+$, then the *orientation* from $A(k,p)$ to $B(k,q)$ is *positive* if $q > p$. In both cases the orientation is *negative* if the orientation from A to B is not positive.

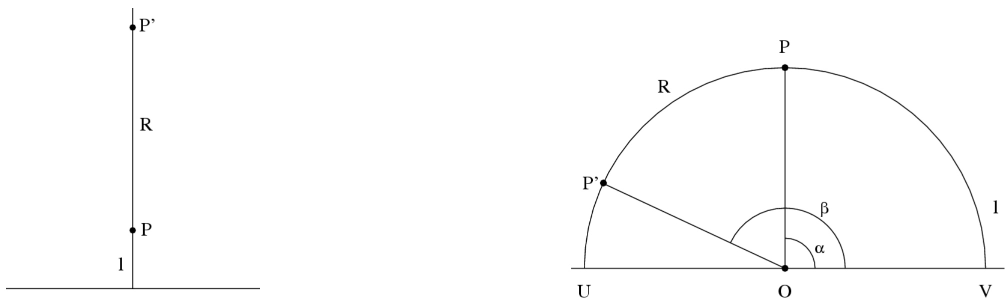

Figure 3: Translation along a line in H^2

2. Translation Along a Line

Given a point $P(a, b)$ in a line l, and a real number $R > 0$, *the translated point P along l to a given distance R and according to a considered orientation* constitutes the point P' of l, such that the orientation from P to P' is the given orientation, and $d(P, P') = R$.

We will distinguish two cases for effecting a translation on H^2. Firstly, if the line l is associated with the equation $x = a$; then the coordinates of P' are (a, be^R), if the orientation is positive, or $\left(a, \frac{b}{e^R}\right)$if it is negative.

Secondly, if l is the Euclidean half-circumference with center $O(k, 0)$ and radius r, then the point P' would have coordinates $(r\cos\beta + k, r\sin\beta)$, where β is the measure of the angle $\angle VOP'$, $\beta = 2\arctan(e^R \tan\frac{\alpha}{2})$, if the given orientation is positive; or $\beta = 2\arctan(e^{-R}\tan\frac{\alpha}{2})$, if it is negative; α is the measure of the angle $\angle VOP$, $\alpha = \arccos\frac{a-k}{r}$ ·

Let us consider that P is a point of D^2, l the line that contains P, and $R > 0$ a real number. Then, the translated point P' of P along l at a distance R and with a given orientation is obtained translating $\widetilde{P} = f_c^{-1}(P)$ to a distance R along the respective line l in H^2, according to the given orientation. This gives us $\widetilde{P'}$, and its transformate P' can be calculated by means of f_c, where f_c is the Cayley transformation [4].

3. Translation According to a Line

Given a point Q, a line l which does not contain Q, a real number $R > 0$, and a certain orientation, *the translated point of Q according to the line l to a distance R, and with the considered orientation,* constitutes the point Q' such that the following conditions are satisfied:

Figure 4: Translation along a line in D^2

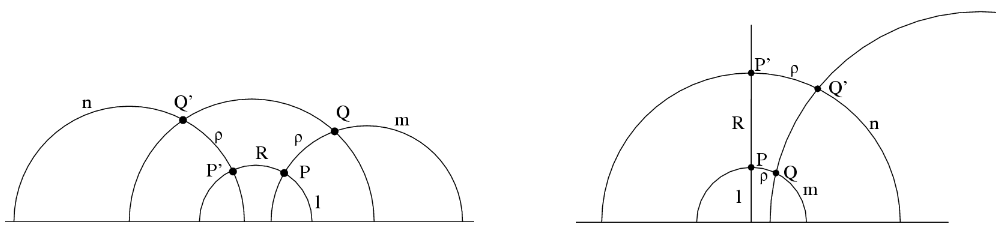

Figure 5: Translation according a line in H^2

— $d(Q, l) = d(Q', l)$;

— if P and P' are the respective orthogonal projections of Q and Q' along l, then $d(P, P') = R$;

— Q and Q' are in the same half-plane of line l;

— the orientation from P to P' is the given orientation.

It is important to note the following remarks. Firstly, to accomplish the translation, measure R is taken on line l. Secondly, the quadrilateral $\mathcal{C}(PP'Q'Q)$ is the Saccheri quadrilateral [2].

To construct the translation in H^2, line m is drawn through point Q perpendicular to l. Let $P(a, b)$ be the intersection of l and m, and ρ the distance from Q to P. Also, let $P'(c, d)$ be the translated point of P along l to distance R with the given orientation, and n the line perpendicular to l which contains P'. Finally, the translated point Q' of Q is obtained with a translation of point P' to distance ρ in the line n, such that Q and Q' are in the same half-plane of l.

For D^2, we write $\widetilde{Q} = f_c^{-1}(Q) \in H^2$ and $\widetilde{l} = f_c^{-1}(l)$ to express the line on H^2. We translate $\widetilde{Q}$ a distance R according to $\widetilde{l}$ with the considered orientation to obtain $\widetilde{Q'}$. The image of $\widetilde{Q'}$ by f_c determines, on D^2, the translated point Q'

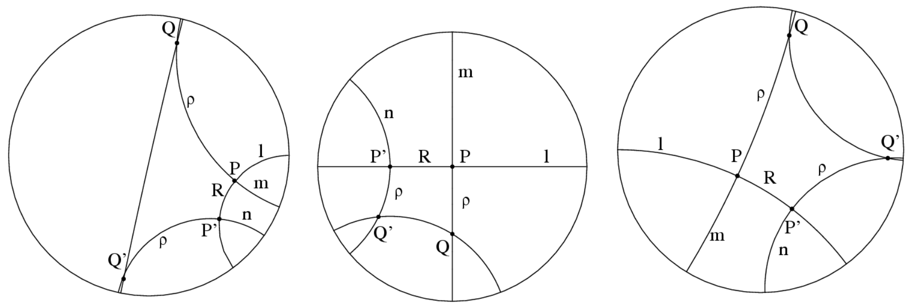

Figure 6: Translation according a line in D^2

of Q according to l to distance R on l, with the given orientation (see Figure 6).

4. Properties of the Translation

In this section we state the relationship between the lengths of the sides of the Saccheri quadrilateral, and finish by demonstrating an important property on ultraparallel lines.

Theorem 4.1. *Let Q' be the translated point of Q according to the line l, which does not contain Q, to distance R, with a given orientation, and $x = d(Q,Q')$ and $\rho = d(Q,l) = d(Q',l)$. It holds that*

$$\cosh(x) = 1 + (\cosh(R) - 1)\cosh^2(\rho).$$

Proof. Let P and P' be the orthogonal projections of Q and Q', respectively; by considering the triangle $\mathcal{T}(P,Q',P')$, we use c to denote length of the segment $\widehat{PQ'}$, α the angle $\angle P'PQ'$, β the angle $\angle PQ'P'$ and $\gamma = \dfrac{\pi}{2}$ the angle $\angle PP'Q'$.

By the so-called First Law of Cosines [5], we have that

$$\frac{\cosh(R)\cosh(\rho) - \cosh(c)}{\sinh(R)\sinh(\rho)} = 0; \tag{1}$$

hence

$$\cosh(c) = \cosh(R)\cosh(\rho). \tag{2}$$

In a similar way, by the Law of Sines [5], we obtain that

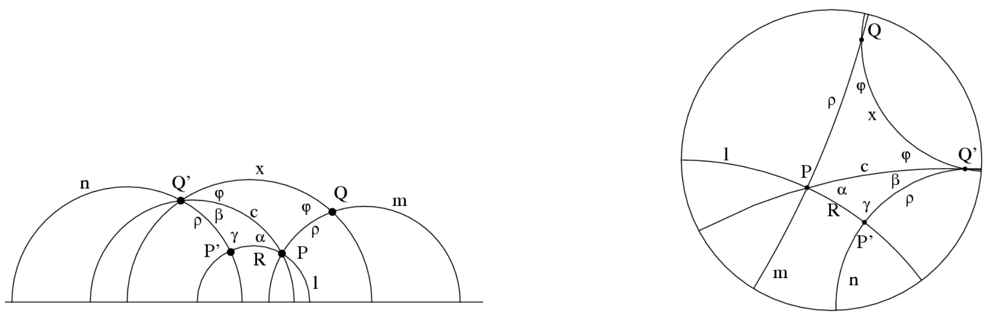

Figure 7: Translation according a line and Saccheri quadrilateral

$$\frac{\sinh(c)}{\sin(\gamma)} = \frac{\sinh(\rho)}{\sin(\alpha)};$$

and thus it follows that

$$\sin(\alpha) = \frac{\sinh(\rho)}{\sinh(c)}. \tag{3}$$

Applying (1) once again in the triangle $\mathcal{T}(P, Q, Q')$, it holds that

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha) = \frac{\cosh(c)\cosh(\rho) - \cosh(x)}{\sinh(c)\sinh(\rho)}.$$

Taking into account (2) and (3) we obtain that

$$\sinh(\rho) = \frac{\cosh(R)\cosh^2(\rho) - \cosh(x)}{\sinh(\rho)}.$$

Therefore,

$$\cosh(x) = 1 + (\cosh(R) - 1)\cosh^2(\rho).$$ □

Corollary 4.1. *Under the assumptions in the above theorem, we have that*

$$\cosh(x) > \cosh(R).$$

Proof. Since $\cosh(x) > 1$, we have that

$$1 + \cosh(R)\cosh^2(\rho) > \cosh^2(\rho),$$

and therefore

$$\frac{1 + \cosh(R)\cosh^2(\rho)}{\cosh(R)} > \frac{\cosh^2(\rho)}{\cosh(R)},$$

which is equivalent to

$$\frac{1 + \cosh(R)\cosh^2(\rho) - \cosh^2(\rho)}{\cosh(R)} > 1.$$

Hence

$$\cosh(x) > \cosh(R).$$ □

We also obtain the following metric property for ultraparallel lines.

Corollary 4.2. *The minimum distance between two ultraparallel lines is attained at the unique common perpendicular.*

Proof. By the previous corollary, $\cosh(x) > \cosh(R)$, so $x > R$, since the function $\cosh(z)$ is strictly increasing for $z > 0$. □

5. Construction of the Saccheri Quadrilateral

The construction of the Saccheri quadrilateral can be solved by means of the translation of a point according a line which does not contain it.

Theorem 5.1. *Given two positive real numbers R and ρ, there exists — unique up to congruences— a Saccheri quadrilateral whose base has length R, and whose sides intersecting the base have length ρ. The length of the side opposite the base is to real number x such that*

$$\cosh(x) = 1 + (\cosh(R) - 1)\cosh^2(\rho).$$

Proof. This quadrilateral is constructed fixing an arbitrary line l and a point in the hyperbolic plane at a distance ρ of l. Then the translated point of Q, Q' is constructed along line l to R units, with any orientation, as indicated in Section 3. The quadrilateral $\mathcal{C}(PP'Q'Q)$ is the Saccheri quadrilateral determined by lengths R and ρ.

Any other Saccheri quadrilateral having a base with length R such that its sides coincide with the base of length ρ is congruent with this one, by means of the hyperbolic rigid movement which takes the base of this quadrilateral along segment $\widehat{PP'}$with the appropiate orientation.

The length of the side opposite the base is determined in Theorem 4.1. □

Because of their construction, and their uniqueness up to congruences, Saccheri quadrilaterals have two equal and acute angles opposite the base [2].

6. The Saccheri Quadrilaterals that Tile the Hyperbolic Plane

In order to tile the hyperbolic plane by means of polygons with k sides and vertex $P_1, P_2, ..., P_k$, making use of reflections on their sides, determined by the segments $\overset{\frown}{P_{i-1}P_i}(P_0 = P_k)$, the inner angles are necessarily of the type $\frac{\pi}{n_i}$ for all $i = 1, 2, ..., k$ with $n_i \geq 2$, $n_i \in \mathbb{N}$ and $\sum_{i=1}^{k} \frac{\pi}{n_i} < (k-2)\pi$ [3]. We are concerned with the determination of tiling Saccheri quadrilaterals. For this purpose we will check whether their acute angles can be taken as $\frac{\pi}{n}$ with $n > 2$, $n \in \mathbb{N}$. First of all, let us study the acute angles in a Saccheri quadrilateral.

Lemma 6.1. *Let $\mathcal{C}(PP'Q'Q)$ be a Saccheri quadrilaleral, with base R, sides ρ intersecting the base, and acute angles φ. Then, the following assertions hold:*

$$\sin(\varphi) = \frac{\cosh(\rho)\sinh(R)}{\sinh(x)},$$
$$\cos(\varphi) = \frac{\cosh(\rho)(\cosh(x) - \cosh(R))}{\sinh(x)\sinh(\rho)},$$

where $x = \operatorname{argcosh}\left(1 + (\cosh(R) - 1)\cosh^2(\rho)\right)$.

Proof. Applying the Law of Sines in the triangle $\mathcal{T}(P, Q, Q')$, the First Law of Cosines in $\mathcal{T}(P, Q', P')$, and (2), we have that

$$\sin(\varphi) = \frac{\cosh(\rho)\sinh(R)}{\sinh(x)}.$$

By the First Law of Cosines in the triangle $\mathcal{T}(P, Q, Q')$ and (2) we have that

$$\cos(\varphi) = \frac{\cosh(\rho)(\cosh(x) - \cosh(R))}{\sinh(x)\sinh(\rho)}. \qquad \square$$

Lemma 6.2. *For all $R > 0$, with $0 < \varphi < \frac{\pi}{2}$, there exists a unique $\rho > 0$ such that*

$$\sin(\varphi) = \frac{\cosh(\rho)\sinh(R)}{\sinh(x)},$$

with $x = \operatorname{argcosh}\left(1 + (\cosh(R) - 1)\cosh^2(\rho)\right)$.

Proof. We construct the function

$$F : (0, +\infty) \longrightarrow \mathbb{R}, \; F(\rho) = -\sin(\varphi) + \frac{\cosh(\rho)\sinh(R)}{\sinh(x)}.$$

Deriving and simplyfying we obtain that

$$F'(\rho) = -\left(\frac{\cosh^2(\rho)(\cosh(R)-1)}{2+\cosh^2(\rho)(\cosh(R)-1)}\right)^{3/2} \frac{\sinh(\rho)\sinh(R)}{\cosh^2(\rho)(\cosh(R)-1)} < 0,$$

since $x, R, \rho > 0.$ Therefore, F is strictly decreasing.

Meanwhile,

$$\lim_{\rho\to 0^+} F(\rho) = -\sin(\varphi) + 1 > 0,$$

since $-1 < -\operatorname{sen}(\varphi) < 0$. And also

$$\lim_{\rho\to +\infty} F(\rho) = -\sin(\varphi) < 0$$

because

$$\lim_{\rho\to +\infty} F(\rho) = -\sin(\varphi) + \lim_{\rho\to +\infty} \frac{\cosh(\rho)\sinh(R)}{\sinh(x)}$$

and, applying the L'Hopital Rule,

$$\lim_{\rho\to +\infty} \frac{\cosh(\rho)\sinh(R)}{\sinh(x)} = 0.$$

The existence and uniqueness of ρ follow by continuity and connectedness, and by the monotony of F, respectively. □

Remark 6.1. 1. $\varphi = 0$ corresponds to $\rho = +\infty$.

2. $\varphi = \frac{\pi}{2}$ corresponds to $\rho = 0$.

3. One can take $\rho \in \mathbb{R}$, in which case there are two solutions, one positive and one negative.

As a consequence of the previous lemmas we have the following result.

Theorem 6.2. *For every $R > 0$ and $0 < \varphi < \frac{\pi}{2}$, there exists a Saccheri quadrilateral, unique up to congruences, with base R and acute angles φ.*

Proof. Applying Lemma 6.2, we obtain a unique $\rho > 0$ satisfying

$$\sin(\varphi) = \frac{\cosh(\rho)\sinh(R)}{\sinh(x)}.$$

The Saccheri quadrilateral of base R and side ρ is the one required. By Lemma 6.1, the quadrilateral's acute angles are φ. □

Theorem 6.3. *For all $R > 0$ and $n \in \mathbb{N}$, $n > 2$, there exists a Saccheri quadrilateral with base R and acute angles $\varphi = \frac{\pi}{n}$, unique up to congruences, which tiles the hyperbolic plane.*

Proof. This theorem is a version of Theorem 6.2 for $\varphi = \frac{\pi}{n}$ with $n \in \mathbb{N}$, $n > 2$. These quadrilaterals tile the hyperbolic plane as they have angles $\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{\pi}{n}$, $\frac{\pi}{n}$ [3]. □

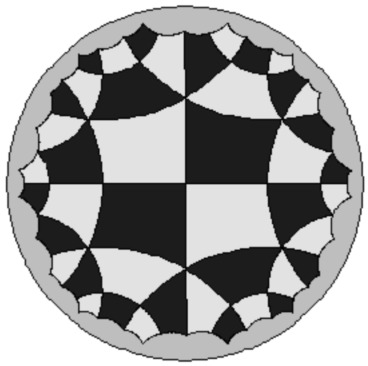

Figure 8: Tiling of Poincare's disk by a Saccheri quadrilateral

7. Graphic Example

As an example for illustrating this situation, we show a tiling of Poincare's disk by a Saccheri quadrilateral for $\varphi = \frac{\pi}{3}$, $R = 1$, $\rho \simeq 1.047$, with the aid of *Mathematica* software (see Figure 8).

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