International Journal of Pure and Applied Mathematics
Volume 23 No. 4 2005, 575-583

SOLUTION OF SEXTIC EQUATIONS

Yoshihiro Mochimaru
Department of International Development Engineering
Graduate School of Science and Engineering
Tokyo Institute of Technology
2-12-1, O-okayama, Meguru-ku, Tokyo, 152-8550, JAPAN
e-mail: ymochima@o.cc.titech.ac.jp

Abstract: Analytical solution schemes of sextic equations are discussed regarding the Lagrangean formula in a canonical type of the equation, hypergeometric series solution in a trinomial equation, and possibility of a reciprocal solution. All the more, sufficient conditions of convergence criteria are given for the solutions in a Lagrangean formula to sextic equations in a canonical form.

AMS Subject Classification: 11D41
Key Words: sextic equation, series solution, reciprocal solution

1. Introduction

The sextic equation $x^6 + A^*x^5 + B^*x^4 + C^*x^3 + D^*x^2 + E^*x + F^* = 0$ can be reduced by two successive Tschirnhausian transforms $-y = p_1 + q_1x + x^2$ and $-z = p_2 + q_2y + ry^2 + sy^3 + y^4$ to

$$z^6 + Dz^2 + Ez + F = 0, \tag{1}$$

which can be easily reduced at most to a 2-parameter sextic equation [3]. All the more, if any, some (not necessarily all) root(s) of the implicit equation of the form

$$y = 1 + t\phi\,(y) \tag{2}$$

is given by the Lagrangean formula [2] as

Received: August 18, 2005

$$f(y) = f(1) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{dy^{n-1}} \left[\phi^n(y) f'(y)\right]_{y=1}, \tag{3}$$

where f is a suitably chosen function, supposing the right hand side of equation (3) converges, and $\phi(y), f(y)$ are of class C^∞.

2. Convergence Criteria for the Equation $y = 1 + ay^P + by^Q$

2.1. General

In this paper the indices P and Q are assumed to be real. Consider a Lagrangean series corresponding to $f(y) \equiv y$:

$$\begin{aligned} y &= 1 + \sum_{n=1}^{\infty} \left[\sum_{\alpha+\beta=n} \frac{a^\alpha b^\beta}{\alpha!\beta!} \left\{ \frac{d^{n-1}}{dy^{n-1}} y^{P\alpha+Q\beta} \right\}_{y=1} \right] \\ &= 1 + \sum_{n=1}^{\infty} \left[\sum_{\alpha+\beta=n} \frac{a^\alpha b^\beta}{\alpha!\beta!} \prod_{k=0}^{n-2} (P\alpha + Q\beta - k) \right]. \end{aligned} \tag{4}$$

In equation (4), as usual $\prod_{k=0}^{-1} () \equiv 1$.

2.2. In Case of $0 < \frac{1}{2}(P+Q) < 1; P > Q$

As long as n is sufficiently large,

$$\left| \prod_{k=0}^{n-2} (P\alpha + Q\beta - k) \right|$$

$$\begin{aligned} &\leq \left\{ \prod_{k=0}^{[n(P+Q)/2]} (Pn - k) \right\} \left\{ \prod_{k=[n(P+Q)/2]+1}^{n-2} (k - Qn) \right\} \\ &\leq \exp\left[\frac{1}{2} \ln Pn + \frac{1}{2} \ln\left\{ \frac{n}{2}(P-Q) + 1 \right\} + \int_0^{n(P+Q)/2} \ln(Pn - x)\, dx \right] \\ &\quad \times \exp\left[\frac{1}{2} \ln(n - 2 - Qn) + \frac{1}{2} \ln\left\{ \frac{n}{2}(P-Q) + 1 \right\} \right. \end{aligned}$$

$$
\begin{aligned}
& \left. + \int_{(P+Q)n/2}^{n-2} \ln(x - Qn)\, dx \right] \\
\leq \;& \exp\left[n \ln n - n \right. \\
& \left. + n \left\{ P \ln P + (1-Q) \ln(1-Q) - (P-Q) \ln \frac{P-Q}{2} \right\} + O(1) \right], \quad (5)
\end{aligned}
$$

where the symbol [] only at lower or upper bound for product notation stands for Gauss' symbol, whereas

$$\sum_{\alpha+\beta=n} \frac{|a|^{\alpha} |b|^{\beta}}{\alpha! \beta!} = \frac{1}{n!} (|a| + |b|)^n, \quad (6)$$

$$n! \approx n^n \mathrm{e}^{-n} \sqrt{2\pi n} (n \to +\infty),$$

$$\rho \equiv \frac{\left\{ \frac{1}{2}(P-Q) \right\}^{P-Q}}{P^P (1-Q)^{1-Q}}. \quad (7)$$

Thus, if $|a| + |b| < \rho$, then the Lagrangean series (4) converges.

2.3. In Case of $\frac{1}{2}(P+Q) \geq 1, P > Q$

If $n \geq 3$,

$$
\begin{aligned}
& \left| \prod_{k=0}^{n-2} (P\alpha + Q\beta - k) \right| \\
\leq \;& \exp\left\{ \sum_{k=0}^{n-2} \ln(Pn - k) \right\} \\
< \;& \exp\left[\frac{1}{2} \ln(-Qn) + \frac{1}{2} \ln(n - 2 - Qn) + \int_0^{n-2} \ln(x - Qn)\, dx \right] \\
= \;& \exp\left[n \ln n - \ln n - n \right. \\
& \left. + n \left\{ (1-P) \ln(P-1) + P \ln P \right\} + O(1) \right], \quad (8)
\end{aligned}
$$

$$\rho \equiv (P-1)^{P-1} \Big/ P^P. \quad (9)$$

Thus, if $|a| + |b| \leq \rho$, then the Lagrangean series (4) converges.

2.4. In Case of $\frac{1}{2}(P+Q) \leq 0, P > Q$

If $n \geq 3$,

$$\begin{aligned} & \left| \prod_{k=0}^{n-2} (P\alpha + Q\beta - k) \right| \\ \leq \quad & \exp\left\{ \sum_{k=0}^{n-2} \ln(k - Qn) \right\} \\ \leq \quad & \exp\left[n\ln n - \ln n - n \right. \\ & \left. + n\left\{ (1-Q)\ln(1-Q) + Q\ln(-Q) \right\} + O(1) \right], \end{aligned} \tag{10}$$

$$\rho \equiv \frac{(-Q)^{-Q}}{(1-Q)^{1-Q}}. \tag{11}$$

In equation (10), one middle expression is omitted. Thus, if $|a| + |b| \leq \rho$, then the Lagrangean series (4) converges.

3. Convergence Criteria for the Equation $y = 1 + ay^P (a \neq 0)$

3.1. In Case of P Being Rational and $P > 1$

Let $P = N/M(M > 0, M, N :$ integer, $(M, N) = 1$). The ratio of the M-th adjacent term to the n-th term in equation (4) becomes

$$\begin{aligned} & \frac{\left\{ a^{n+M}/(n+M)! \right\} (Pn+N) \cdots \left(Pn + N - \overline{n+M-2} \right)}{\left\{ a^n/n! \right\} Pn (Pn-1) \cdots \left(Pn - \overline{n-2} \right)} \\ = \quad & a^M \frac{n!}{(n+M)!} \frac{(Pn+N) \cdots (Pn+1)}{\left\{ Pn - n + 2 + (N-M) - 1 \right\} \cdots (Pn - n + 2)} \\ \to \quad & a^M \frac{P^N}{(P-1)^{N-M}} (n \to +\infty). \end{aligned} \tag{12}$$

Thus the radius of convergence of equation (4) for a is $(P-1)^{P-1} P^{-P}$.

3.2. In Case of $P = 1$

Obviously the radius of convergence $= 1$.

3.3. In Case of P Being Rational and $0 < P < 1$

Let $P = N/M(M > 0, M, N :$ integer, $(M, N) = 1)$. If $\mathrm{mod}(n, M) \neq 0$, then the ratio of the M-th adjacent term to the n-th term in equation (4) becomes

$$\frac{\left\{a^{n+M}/(n+M)!\right\}(Pn+N)\cdots\left(Pn+N-\overline{n+M-2}\right)}{\left\{a^n/n!\right\}Pn\,(Pn-1)\cdots\left(Pn-\overline{n-2}\right)}$$
$$= \frac{a^M n!}{(n+M)!}(Pn+N)\cdots(Pn+1)(Pn-n+1)\cdots\left(Pn-n+2-\overline{M-N}\right)$$
$$\to a^M P^N (P-1)^{M-N} \; (n \to +\infty), \quad (13)$$

Thus the radius of convergence $= P^{-P}(1-P)^{P-1}$.

3.4. In Case of P Being Rational and $P < 0$

Let $P = -N/M(M > 0, M, N :$ integer, $(M, N) = 1)$. Then the ratio of the M-th adjacent term to the n-th term in equation (4) becomes

$$\frac{\left\{a^{n+M}/(n+M)!\right\}(Pn-N)\cdots\left(Pn-N-\overline{n+M-2}\right)}{\left\{a^n/n!\right\}Pn\,(Pn-1)\cdots\left(Pn-\overline{n-2}\right)}$$

$$= \frac{a^M n!}{(n+M)!}\frac{(Pn-n+2-1)\cdots(Pn-n+2-N-M)}{(Pn)\cdots(Pn-N+1)}$$
$$\to a^M \frac{(P-1)^{M+N}}{P^N}(n \to +\infty). \quad (14)$$

Thus the radius of convergence $= (-P)^{-P}(1-P)^{P-1}$.

3.5. In Case of $P = 0$

Obviously the radius of convergence $= +\infty$.

3.6. In Case of P Being Irrational and $0 < P < 1$

As long as n is sufficiently large,

$$\left|\prod_{k=0}^{n-2}(Pn-k)\right| < \left\{\prod_{k=0}^{[Pn]-1}(Pn-k)\right\}\left\{\prod_{k=[Pn]+2}^{n-2}(k-Pn)\right\}$$

$$< \exp\left[n\ln n - \ln n + n\left\{-1 + (1-P)\ln(1-P) + P\ln P\right\} + O(1)\right], \quad (15)$$

$$\rho \equiv (1-P)^{P-1}\,P^{-P}. \quad (16)$$

Thus if $|a| \le \rho$, then the Lagrangean series (4) converges.

3.7. In Case of P Being Irrational and $P > 1$

If $n \ge 3$,

$$\left|\prod_{k=0}^{n-2}(Pn-k)\right| = \exp\left\{\sum_{k=0}^{n-2}\ln(Pn-k)\right\}$$
$$< \exp\left[n\ln n - \ln n + n\left\{-1 + (1-P)\ln(P-1) + P\ln P\right\} + O(1)\right], \quad (17)$$

$$\rho \equiv (P-1)^{P-1}\,P^{-P}. \quad (18)$$

Thus if $|a| \le \rho$, then the Lagrangean series (4) converges.

3.8. In Case of P Being Irrational and $P < 0$

If $n \ge 3$,

$$\left|\prod_{k=0}^{n-2}(Pn-k)\right| = \exp\left\{\sum_{k=0}^{n-2}\ln(|P|n+k)\right\}$$
$$< \exp\left[n\ln n - \ln n + n\left\{-1 + (1-P)\ln(1-P) + P\ln|P|\right\} + O(1)\right], \quad (19)$$

$$\rho \equiv |P|^{-P}\,(1-P)^{P-1}. \quad (20)$$

Thus if $|a| \le \rho$, the the Lagrangean series (4) converges.

4. Constitution of the Roots of equation (1)

4.1. General

Application of the Lagrangean formula to a concrete equation depends on the form $f()$ and choice of $\phi()$. First, four examples are shown for $f(y) = y$.

4.2. In Case of $D \neq 0$

If $|E|/|D|^{5/4} + |F|/|D|^{3/2} \leq 2/(3\sqrt{3})$, then four of the six roots $z_n, 0 \leq n \leq 3$, of equation (1) are given by

$$z_n = \omega_n \left(-D\right)^{1/4} {y_n}^{1/4}, \tag{21}$$

$$y_n = 1 - {\omega_n}^3 E \left\{ \left(-D\right)^{1/4} \right\}^{-5} {y_n}^{-1/4} - {\omega_n}^2 F \left\{ \left(-D\right)^{1/4} \right\}^{-6} {y_n}^{-1/2}, \tag{22}$$

$$\omega_n \equiv \exp\left(n\pi i/2\right). \tag{23}$$

4.3. In Case of $E \neq 0$

If $|D|/|E|^{4/5} + |F|/|E|^{6/5} \leq 5/6^{6/5}$, then five of the six roots $z_n, 0 \leq n \leq 4$, of equation (1) are given by

$$z_n = \omega_n \left(-E\right)^{1/5} {y_n}^{1/5}, \tag{24}$$

$$y_n = 1 - \omega_n D \left\{ \left(-E\right)^{1/5} \right\}^{-4} {y_n}^{1/5} - {\omega_n}^4 F \left\{ \left(-E\right)^{1/5} \right\}^{-6} {y_n}^{-1/5}, \tag{25}$$

$$\omega_n \equiv \exp\left(2n\pi i/5\right). \tag{26}$$

4.4. In Case of $F \neq 0$

If $|D|/|F|^{2/3} + |E|/|F|^{5/6} \leq 3\left(8/3125\right)^{1/6}$, then the six roots $z_n, 0 \leq n \leq 5$, of equation (1) are given by

$$z_n = \omega_n \left(-F\right)^{1/6} {y_n}^{1/6}, \tag{27}$$

$$y_n = 1 - {\omega_n}^2 D \left\{ \left(-F\right)^{1/6} \right\}^{-4} {y_n}^{1/3} - \omega_n E \left\{ \left(-F\right)^{1/6} \right\}^{-5} {y_n}^{1/6}, \tag{28}$$

$$\omega_n \equiv \exp(n\pi i/3). \tag{29}$$

4.5. In Case of $DF \neq 0$

If $|E|/\sqrt{|DF|} + |F|^2/|D|^3 \leq 4/27$, then two roots $z_n, 0 \leq n \leq 1$, are given by

$$\frac{1}{z_n} = \omega_n \left(-\frac{D}{F}\right)^{1/2} {y_n}^{1/2}, \tag{30}$$

$$y_n = 1 + \omega_n \frac{E}{D} \left(-\frac{D}{F}\right)^{1/2} {y_n}^{1/2} + \frac{F^2}{D^3} {y_n}^{-2}, \tag{31}$$

$$\omega_n = \exp\left(n\pi i\right). \tag{32}$$

5. In Case of $f(y) = y \ln y - y + 1$

If the Lagrangean formula is applied to $y = 1 + ay^{1/3}$, then the radius of convergence for a is given by

$$1 \bigg/ \lim_{n \to +\infty} \left[\frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} \left(y^{n/3} \ln y \right)_{y=1} \right]^{1/n}$$

and actually approximately 0.80.

6. In Case of $f(y) = \mathrm{e}^{y-1}$

If the Lagrangean formula is applied to $y = 1 + ay^{1/3}$, then the radius of convergence for a is given by

$$1 \bigg/ \lim_{n \to +\infty} \left[\frac{1}{n!} \frac{d^{n-1}}{dy^{n-1}} \left(y^{n/3} \mathrm{e}^{y-1} \right)_{y=1} \right]^{1/n}$$

and actually approximately $1/0.523 = 1.912$.

7. Trinomial Cases and Reciprocal Solutions

In case of $D = 0$, equation (1) itself is a trinomial equation, and thus all roots are expressed in terms of hypergeometric functions [1]. In case of $E = 0$, equation (1) itself is a compound cubic equation, two roots of which, for example, are given (provided $D \neq 0$) by $z = \pm \left[-R \sinh \left\{ \frac{1}{3} \sinh^{-1} \left(\frac{4F}{R^3} \right) \right\} \right]^{1/2}$, $R \equiv \sqrt{4D/3}$, and especially in case of $|27F^2/(4D^3)| < 1$ the said two roots are expressed as $z = \pm \left[(-F/D) {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \frac{27F^2}{4D^3} \right) \right]^{1/2}$.

Possibility of reciprocal solutions of equation (1) and a sextic equation of t derived through equation (1) by setting $1/z = t$ exists if the coefficients D, E, and F satisfy the Sylvester's resultant of specific polynomials $= 0$, the condition to the former of which is of the type

$$\sum a_{klm} D^k E^l F^m = 0, 4k + 5l + 6m = 56, k, l, m : \text{non-negative integer},$$

where a_{klm}'s can all be integers (including zeros), and if irreducible (divided by a factor),

$$-6144D^{10} + 3456000D^7F^2 - 486000000D^4F^4 - 720000D^6E^2F$$

$$+742500000D^3E^2F^3 - 75000D^5E^4 - 421875000D^2E^4F^2 + 105468750DE^6F$$
$$-9765625E^8 = 0. \qquad (33)$$

8. Conclusions

Analytical solutions through Lagrangean formula to a canonical sextic equations are given with convergence criteria. In addition to it a possible case of reciprocal solution is shown.

References

[1] R. Birkeland, Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen, *Math. Zeitschrift*, **26** (1927), 566-578.

[2] E. Goursat, *A Course in Mathematical Analysis, 1*, Dover Pub. (1959), 404-406.

[3] Y. Mochimaru, New way for a two-parameter canonical form of sextic equations and its Solvable cases, *Int. J. Pure and Applied Math.*, **18** (2005), 215-224.