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MORE RESULTS ON ALPHA-CALCULUS

Clément Frappier
Department of Mathematics and Industrial Engineering
Ecole Polytechnique of Montreal
C.P. 6079, Succ. Centre-ville, Montreal
Quebec H3C 3A7, CANADA
e-mail: clement.frappier@polymtl.ca

Abstract: The alpha-calculus was introduced in three earlier papers. Many of the results and concepts presented there are revisited. The main section contains precisions concerning a generalized Fourier Transform Theory. We also present more explicit formulas and study the theory as a function of the parameter α.

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1. Introduction

The basic presentation of the alpha-calculus is made in [3, 4, 5]. In this paper we give complementary results, particularly related to [5].

The α-derivative is defined by

$$b_\alpha\big(f(z);z\big)=\sum_{k=1}^{\infty}\frac{\big((-1)^k-1\big)}{k!}B_{k,\alpha}f^{(k)}(z). \tag{1}$$

The generalized Bernoulli polynomials $B_{k,\alpha}(x)$, with $B_{k,\alpha}:=B_{k,\alpha}(0)$, are introduced through the generating function

$$\frac{\exp\big((x-\frac{1}{2})z\big)}{g_\alpha\big(\frac{iz}{2}\big)}=\sum_{k=0}^{\infty}B_{k,\alpha}(x)\frac{z^k}{k!}, \tag{2}$$

where $g_\alpha(z):=2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}$. The α-derivative of order n is $b_\alpha^{(n)}\big(f(z);z\big)$

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$:= b_\alpha\big(b_\alpha^{(n-1)}\big(f(z);z\big);z\big)$, $n \geq 1$, with $b_\alpha^{(0)}\big(f(z);z\big) := f(z)$. We have

$$b_\alpha^{(n)}\big(f(z);z\big) = \sum_{k=n}^{\infty} \frac{d_{k,\alpha}^{(n)}}{k!} f^{(k)}(z), \tag{3}$$

where $d_{k,\alpha}^{(1)} = \big((-1)^k - 1\big)B_{k,\alpha}$ and, for $n > 1$,

$$d_{k,\alpha}^{(n)} = \sum_{\ell=0}^{k} \tbinom{k}{\ell} \big((-1)^{k-\ell} - 1\big) B_{k-\ell,\alpha} d_{\ell,\alpha}^{(n-1)}.$$

Let B_τ denote the class of all entire functions of exponential type τ, bounded on the real axis. If f is a polynomial or if $f \in B_\tau$ with $\tau < \tau(\alpha)$ then we have a representation of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{b_\alpha^{(n)}(f;z_0)}{n!} \phi_{n,\alpha}(z - z_0), \tag{4}$$

where $\phi_{0,\alpha}(z) \equiv 1$ and, for $n \geq 1$,

$$\phi_{n,\alpha}(z) = z^n - \sum_{k=0}^{n-1} \frac{d_{n,\alpha}^{(k)}}{k!} \phi_{k,\alpha}(z). \tag{5}$$

One way to define the corresponding α-integral is

$$_\alpha\!\int_{z_0}^{z} f(t)\,dt = \sum_{p=1}^{\infty} \frac{\psi_{p,\alpha}}{p!} \int_{z_0}^{z} b_\alpha^{(p-1)}\big(f(t);t\big)\,dt, \tag{6}$$

where $\psi_{p,\alpha} := \phi'_{p,\alpha}(0)$. Other representations are possible in certain circumstances (see Section 2).

Immediate consequences of the previous definitions are the formulas

$$_\alpha\!\int_{0}^{1} B_{m,\alpha}(x)\,dx = 0 \tag{7}$$

and

$$b_\alpha^{(n)}\big(f(z^m);z=0\big) = \sum_{k=0}^{\infty} \frac{d_{km,\alpha}^{(n)}}{k!} f^{(k)}(0) \tag{8}$$

for $m = 1, 2, 3, \ldots$.

A summation theorem for Bernoulli polynomials is [6, p. 1107]

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \tag{9}$$

for $m = 1, 2, 3, \dots$ and $n = 0, 1, 2, \dots$. Applying the operator b_α on both sides of (9), and using the relation [4, p. 341]

$$b_\alpha\big(B_n(x); x\big) = nB_{n-1,\alpha}(x), \tag{10}$$

we obtain

$$b_\alpha\big(B_n(mx); x\big) = nm^{n-1} \sum_{k=0}^{m-1} B_{n-1,\alpha}\left(x + \frac{k}{m}\right). \tag{11}$$

Formula (9) is the case $\alpha = \frac{1}{2}$ of (11) since $B_n'(x) = nB_{n-1}(x)$. The limiting case $\alpha \to \infty$ readily leads us to the classic formula

$$\sum_{k=0}^{m-1} (x+k)^{n-1} = \frac{B_n(x+m) - B_n(x)}{n}. \tag{12}$$

2. Representations of the Alpha-Integrals

In this section we give different kinds of representations for the α-integrals. It follows from the definition (6) and known inequalities that the α-integral is well defined, on a finite interval, for all polynomials and elements of B_τ if τ is small enough. Other functions, and distributions, can be considered. The restrictions to be imposed should be clear in each context.

2.1. A Representation with Double Integrals

We use the following result, which is of independent interest.

Lemma 2.1. *We have the formula*

$$\sum_{k=0}^{\infty} f^{(k)}(x) g^{(k)}(y) \frac{z^k}{k!} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-iuv) f(u+x) g(ivz+y)\, du\, dv. \tag{13}$$

Proof. Working in the sense of distributions, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-iuv)f(u+x)g(ivz+y)\,du\,dv$$
$$=\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{g^{(k)}(y)}{k!}z^k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-iuv)f(u+x)(iv)^k\,du\,dv$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{g^{(k)}(y)}{k!}z^k\int_{-\infty}^{\infty}\delta^{(k)}(u)f(u+x)\,du=\sum_{k=0}^{\infty}f^{(k)}(x)g^{(k)}(y)\frac{z^k}{k!}. \quad \square$$

The choice $g(y)=\exp(y)$ in (13) leads us to Taylor's formula.

Corollary 2.1. *Let $f\in B_\tau$, $\tau<2\pi$, be square integrable. We have the representation*

$$_\alpha\int_a^b f(t)\,dt$$
$$=\frac{1}{2\pi}\int_{-\tau}^{\tau}\int_{-\infty}^{\infty}\exp(-iuv)f(u)g_\alpha\left(\frac{v}{2}\right)\frac{\exp(ibv)-\exp(iav)}{\exp(\frac{iv}{2})-\exp(-\frac{iv}{2})}\,du\,dv. \quad (14)$$

Proof. In Formula (13) we choose

$$g(y)=g_\alpha\left(\frac{y}{2}\right)\frac{\exp(iby)-\exp(iay)}{\exp(\frac{iy}{2})-\exp(-\frac{iy}{2})}$$

and $x=0$, $y=0$, $z=-i$. It follows from [5, Formula (39)] that $g^{(k)}(0)={}_\alpha\int_a^b (it)^k\,dt$, whence

$$\sum_{k=0}^{\infty}f^{(k)}(0)g^{(k)}(0)\frac{(-i)^k}{k!}={}_\alpha\int_a^b\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}t^k\,dt={}_\alpha\int_a^b f(t)\,dt.$$

If $f\in B_\tau$ is square integrable then its Fourier transform is zero outside the interval $[-\tau,\tau]$. The integral with respect to v, in (13), is reduced to an integral over the interval $(-\tau,\tau)$, and we then see that the double integral has a sense for $\tau<2\pi$. $\square$

Remark 2.1. If b has the form $b=a+\ell$, where ℓ is an integer, then the term $\exp(\frac{iv}{2})-\exp(-\frac{iv}{2})$ cancels out in (14). In that case, the conditions on f

can be relaxed. If, for example, we take $b = a + \frac{1}{2}$ and replace a by $a - \frac{1}{2}$ then we obtain Formula (44) of [5] by using the evaluation [6, p. 764]

$$\int_{-\infty}^{\infty} g_\alpha\left(\frac{v}{2}\right) \exp\bigl(i(a-u)v\bigr)\, dv = \begin{cases} 4\sqrt{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}\bigl(1-4(u-a)^2\bigr)^{\alpha-\frac{1}{2}}, & |u-a| < \frac{1}{2}, \\ 0, & |u-a| > \frac{1}{2}. \end{cases}$$

We also note that the choice $g(y) = \left(\frac{\exp(\frac{y}{2})-\exp(-\frac{y}{2})}{g_\alpha(\frac{iy}{2})}\right)^n$, with $y = 0$ and $z = 1$, gives the representation

$$b_\alpha^{(n)}\bigl(f(x);x\bigr) = \frac{1}{2\pi}\int_{-\tau}^{\tau}\int_{-\infty}^{\infty} \exp(-iuv) f(u+x) \left(\frac{\exp(\frac{iv}{2}) - \exp(-\frac{iv}{2})}{g_\alpha(\frac{v}{2})}\right)^n du\, dv \tag{15}$$

if τ is small enough. That follows from (2) and the relation

$$d_{k,\alpha}^{(n)} = \frac{d^k}{dy^k}\left(\frac{\exp(\frac{y}{2}) - \exp(-\frac{y}{2})}{g_\alpha(\frac{iy}{2})}\right)^n\Bigg|_{y=0}.$$

2.2. Representations in the Finite Case

Here we give two representations of the α-integral over a finite interval (a, b).

Theorem 2.1. *We have the formula*

$$_\alpha\!\int_a^b f(t)\, dt = \sum_{n=0}^{\infty}\sum_{\ell=0}^{[\frac{n}{2}]} \frac{f^{(n)}(0)\Gamma(\alpha+1)\bigl(B_{n-2\ell+1}(b+\frac{1}{2}) - B_{n-2\ell+1}(a+\frac{1}{2})\bigr)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell+1)!} \tag{16}$$

if the double series is convergent.

Proof. We have the recurrence relation [2, p. 291]

$$\sum_{\ell=0}^{[\frac{n}{2}]} \frac{n!\Gamma(\alpha+1)B_{n-2\ell,\alpha}(x)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell)!} = \left(x - \frac{1}{2}\right)^n. \tag{17}$$

Replacing x by $(t+\frac{1}{2})$ and using the relation (10) in the form

$$ {}_{\alpha}\!\int_{z_0}^{z} B_{m,\alpha}(t)\,dt = \frac{B_{m+1}(z)-B_{m+1}(z_0)}{m+1}, $$

we obtain

$$ {}_{\alpha}\!\int_a^b t^n\,dt = \sum_{\ell=0}^{[\frac{n}{2}]} \frac{n!\Gamma(\alpha+1)\big(B_{n-2\ell+1}(b+\frac{1}{2})-B_{n-2\ell+1}(a+\frac{1}{2})\big)}{2^{4\ell}\ell!\Gamma(\alpha+\ell+1)(n-2\ell+1)!}. \tag{18} $$

The result follows since f has the representation $f(t)=\sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}$. □

Formula (43) of [5] follows from (16); we simply use the relation

$$ B_m(z+1)-B_m(z) = mz^{m-1} $$

and permute the order of summation, assuming of course that the series are absolutely convergent. An important simplification occurs if α is infinite or if $\alpha=-\frac{1}{2}$.

Corollary 2.2. *We have*

$$ {}_{\infty}\!\int_a^b f(t)\,dt = \sum_{j=0}^{\infty}\big(f(a+\tfrac{1}{2}+j)-f(b+\tfrac{1}{2}+j)\big) \tag{19} $$

and

$$ {}_{-\frac{1}{2}}\!\int_a^b f(t)\,dt = \frac{1}{2}\big(f(a)-f(b)\big)+\sum_{j=1}^{\infty}\big(f(a+j)-f(b+j)\big) \tag{20} $$

if the series converge.

Proof. Letting $\alpha\to\infty$ in (16) we obtain

$$ {}_{\infty}\!\int_a^b f(t)\,dt = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{(n+1)!}\big(B_{n+1}(b+\tfrac{1}{2})-B_{n+1}(a+\tfrac{1}{2})\big), $$

and (19) follows with the help of an Euler–MacLaurin formula:

$$ \sum_{n=1}^{\infty}\frac{B_n(u)}{n!}f^{(n-1)}(0) = \int_0^{\infty} f(t)\,dt-\sum_{j=0}^{\infty} f(u+j) \tag{21} $$

when $\lim_{N\to\infty} f^{(k)}(N)=0$, $k=0,1,2,\ldots$.

Formula (20) is obtained similarly, using (21) and the relation $\Gamma(\ell+\frac{1}{2})=\frac{\sqrt{\pi}(2\ell)!}{2^{2\ell}\ell!}$. □

If a is replaced by $(a-\frac{1}{2})$ and b is replaced by $(a+\frac{1}{2})$ in (19), (20) then the series are telescoping and we get formulas (66), (67) of [5].

From now on we denote by $\{t\}$ the fractional part of the real number t, i.e., $\{t\}=t-[t]$.

Theorem 2.2. *Let* $\mathrm{Re}(\alpha)>-\frac{1}{2}$. *Assume that the* α*-integrals*

$$_\alpha\!\int_a^\infty f(t)\,dt \quad \text{and} \quad {}_\alpha\!\int_b^\infty f(t)\,dt$$

exist. We then have

$$_\alpha\!\int_a^b f(t)\,dt = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\infty \big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}}\big(f(t+a)-f(t+b)\big)\,dt. \tag{22}$$

Proof. As a particular case of formula (44) in [5], we have

$$_\alpha\!\int_0^1 f(t)\,dt = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 \big(t(1-t)\big)^{\alpha-\frac{1}{2}} f(t)\,dt. \tag{23}$$

If we denote by R the right-hand member of (22) then (see the next subsection for the definition of the improper α-integrals):

$$\begin{aligned} R &= \sum_{k=0}^{\infty}\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_k^{k+1}\big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}}\big(f(t+a)-f(t+b)\big)\,dt \\ &= \sum_{k=0}^{\infty}\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^{1}\big(t(1-t)\big)^{\alpha-\frac{1}{2}}\big(f(t+k+a)-f(t+k+b)\big)\,dt. \end{aligned}$$

Using formula (23) we obtain

$$\begin{aligned} R &= \sum_{k=0}^{\infty} {}_\alpha\!\int_0^1 \big(f(t+k+a)-f(t+k+b)\big)\,dt \\ &= \sum_{k=0}^{\infty} {}_\alpha\!\int_k^{k+1} \big(f(t+a)-f(t+b)\big)\,dt = {}_\alpha\!\int_0^\infty \big(f(t+a)-f(t+b)\big)\,dt, \end{aligned}$$

whence

$$R = {}_\alpha\!\int_a^\infty f(t)\,dt - {}_\alpha\!\int_b^\infty f(t)\,dt = {}_\alpha\!\int_a^b f(t)\,dt. \qquad \square$$

Formula (22) has a special character since we must use the values of $f(t)$ for $a < t < \infty$ in order to evaluate the α-integral over the finite interval (a, b). If we know the values of $f(t)$ for $-\infty < t < b$ then the corresponding formula is

$$
\begin{aligned}
{}_{\alpha}\!\int_a^b f(t)\,dt \\
&= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^{\infty}\big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}}\big(f(b-t)-f(a-t)\big)\,dt. \quad (24)
\end{aligned}
$$

The limiting case $b \to \infty$ in (22) yields the following result.

Corollary 2.3. *For* $\mathrm{Re}(\alpha) > -\frac{1}{2}$, *we have the representation*

$$
{}_{\alpha}\!\int_a^{\infty} f(t)\,dt = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_a^{\infty}\big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}} f(t)\,dt. \quad (25)
$$

2.3. Improper Alpha-Integrals

We consider improper α-integrals over infinite intervals. The definition presented in [5, Section 4], namely

$$
{}_{\alpha}\!\int_{-\infty}^{\infty} f(t)\,dt = \sum_{p=1}^{\infty}\frac{\psi_{p,\alpha}}{p!}\int_{-\infty}^{\infty}\frac{\partial_{\alpha}^{p-1}f(t)}{\partial_{\alpha}t^{p-1}}\,dt, \quad (26)
$$

is not the most practical one. It is more useful to define the improper α-integral over $(-\infty, \infty)$ with the help of the additivity property. But there the value depends in general on the way chosen to represent the interval $(-\infty, \infty)$ as $\bigcup_{k=-\infty}^{\infty}(B+k, B+k+1)$, where B is a real number. We put

$$
{}_{\alpha}\!\int_{-\infty}^{\infty} f(t)\,dt(B) := \sum_{k=-\infty}^{\infty}{}_{\alpha}\!\int_{B+k}^{B+k+1} f(t)\,dt, \quad (27)
$$

where the α-integrals ${}_{\alpha}\!\int_{B+k}^{B+k+1} f(t)\,dt$ can be evaluated with formula (44) of [5]:

$$
{}_{\alpha}\!\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} f(t)\,dt = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{a-\frac{1}{2}}^{a+\frac{1}{2}}\big(1-4(t-a)^2\big)^{\alpha-\frac{1}{2}} f(t)\,dt. \quad (28)
$$

We use the ordinary notation for $B=0$, i.e.,

$$ {}_{\alpha}\!\int_{-\infty}^{\infty} f(t)\,dt(0) =: {}_{\alpha}\!\int_{-\infty}^{\infty} f(t)\,dt = \sum_{k=-\infty}^{\infty} {}_{\alpha}\!\int_{k}^{k+1} f(t)\,dt. \tag{29} $$

The calculations done in [5] correspond essentially to the case $B=0$. See the forthcoming Section 3 for more detailed results.

Note that even the changes of variable $t \mapsto t+t_0$, $t \mapsto -t$ will affect the value in (27). A simple example where the value depends on B is $f(t)=\exp(-ct^2)$, $c>0$; using formula (67) of [5], we obtain

$$ \begin{aligned} {}_{-\frac{1}{2}}\!\int_{-\infty}^{\infty} \exp(-ct^2)\,dt(-\tfrac{1}{2}) &= \sum_{k=-\infty}^{\infty} {}_{-\frac{1}{2}}\!\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \exp(-ct^2)\,dt \\ &= \sum_{k=-\infty}^{\infty} \exp\bigl(-c(k+\tfrac{1}{2})^2\bigr), \end{aligned} $$

while

$$ \begin{aligned} {}_{-\frac{1}{2}}\!\int_{-\infty}^{\infty} \exp(-ct^2)\,dt &= \sum_{k=-\infty}^{\infty} {}_{-\frac{1}{2}}\!\int_{k}^{k+1} \exp(-ct^2)\,dt \\ &= \sum_{k=-\infty}^{\infty} \exp(-ck^2). \end{aligned} $$

There is no difficulty to define the improper α-integral over a semi-infinite interval (a,∞). If we choose a definition using the additivity property then the only choice is

$$ {}_{\alpha}\!\int_{a}^{\infty} f(t)\,dt = \sum_{k=0}^{\infty} {}_{\alpha}\!\int_{a+k}^{a+k+1} f(t)\,dt. \tag{30} $$

In the finite case (a,b) we can put, as was done in Subsection 2.2,

$$ {}_{\alpha}\!\int_{a}^{b} f(t)\,dt := {}_{\alpha}\!\int_{a}^{\infty} f(t)\,dt - {}_{\alpha}\!\int_{b}^{\infty} f(t)\,dt, \tag{31} $$

or

$$ {}_{\alpha}\!\int_{a}^{b} f(t)\,dt := {}_{\alpha}\!\int_{-\infty}^{b} f(t)\,dt - {}_{\alpha}\!\int_{-\infty}^{a} f(t)\,dt, \tag{32} $$

depending on the context.

We have the following result.

Theorem 2.3. *Let $f \in B_\tau$ satisfies a growth condition of the form*

$$f(x) = O(|x|^{-\delta}), \quad \delta > 1, \text{ as } x \to \pm\infty.$$

If $\tau \le 2\pi$ then, for any real number B,

$$_\alpha\!\int_{-\infty}^{\infty} f(t)\,dt(B) = \int_{-\infty}^{\infty} f(t)\,dt. \tag{33}$$

Proof. We write (27) in the form

$$_\alpha\!\int_{-\infty}^{\infty} f(t)\,dt(B) = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 \bigl(t(1-t)\bigr)^{\alpha-\frac{1}{2}} \sum_{k=-\infty}^{\infty} f(t+B+k)\,dt. \tag{34}$$

The permutation of the integral with the series is easily justified with our assumptions. Moreover we have, if $\tau \le 2\pi$, the quadrature formula [1]

$$\int_{-\infty}^{\infty} f(x)\,dx = \sum_{k=-\infty}^{\infty} f(A+k) \tag{35}$$

(for all real A) whence, from (34),

$$\begin{aligned} _\alpha\!\int_{-\infty}^{\infty} f(t)\,dt(B) &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 \bigl(t(1-t)\bigr)^{\alpha-\frac{1}{2}}\,dt \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)\,dx. \end{aligned}$$

□

2.4. Alpha-Integrals of Derivatives

The α-integral of $f^{(k)}(t)$, $k = 0, 1, 2, \ldots$, is expressible in terms of other α-integrals.

Lemma 2.2. *Let F be k times differentiable. For all complex numbers a, b, c, we have*

$$\bigl(F(a+bx+cx^2)\bigr)^{(k)} = \sum_{s=0}^{[\frac{k}{2}]} \frac{k!(b+2cx)^{k-2s}c^s}{s!(k-2s)!} F^{(k-s)}(a+bx+cx^2). \tag{36}$$

Formula (36) can be proved by mathematical induction. It is also a consequence of Faa di Bruno Formula (see (105)). We use it to prove the following result.

Theorem 2.4. *Let* $\operatorname{Re}(\alpha) > \frac{(2k-1)}{2}$*, where* k *is a positive integer. We have the formula*

$$
{}_{\alpha}\!\int_0^{\infty} f^{(k)}(t)\,dt = \sum_{s=0}^{[\frac{k}{2}]} \frac{(-1)^{k-s}k!2^{2(k-s)}\Gamma(\alpha+1)}{s!(k-2s)!\Gamma\big(\alpha+1-(k-s)\big)} \times {}_{\alpha-(k-s)}\!\int_0^{\infty} (1-2\{t\})^{k-2s} f(t)\,dt. \quad (37)
$$

Proof. The idea is easily understood in the case $k=1$. Using formula (23), we have

$$
\begin{aligned}
{}_{\alpha}\!\int_0^{\infty} f'(t)\,dt &= \sum_{k=0}^{\infty} {}_{\alpha}\!\int_k^{k+1} f'(t)\,dt \\
&= \sum_{k=0}^{\infty} \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 \big(t(1-t)\big)^{\alpha-\frac{1}{2}} f'(t+k)\,dt.
\end{aligned}
$$

An integration by parts yields, for $\operatorname{Re}(\alpha) > \frac{1}{2}$,

$$
\begin{aligned}
{}_{\alpha}\!\int_0^{\infty} f'(t)\,dt &= -\frac{2^{2\alpha}\Gamma(\alpha+1)(\alpha-\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \sum_{k=0}^{\infty} \int_0^1 \big(t(1-t)\big)^{\alpha-\frac{3}{2}} (1-2t) f(t+k)\,dt \\
&= -4\alpha \frac{2^{2(\alpha-1)}\Gamma\big((\alpha-1)+1\big)}{\sqrt{\pi}\Gamma\big((\alpha-1)+\frac{1}{2}\big)} \\
&\quad \times \sum_{k=0}^{\infty} \int_0^1 \big(\{t\}(1-\{t\})\big)^{(\alpha-1)-\frac{1}{2}} (1-2\{t\}) f(t+k)\,dt,
\end{aligned}
$$

whence

$$
{}_{\alpha}\!\int_0^{\infty} f'(t)\,dt = -4\alpha \frac{2^{2(\alpha-1)}\Gamma\big((\alpha-1)+1\big)}{\sqrt{\pi}\Gamma\big((\alpha-1)+\frac{1}{2}\big)} \times \int_0^{\infty} \big(\{t\}(1-\{t\})\big)^{(\alpha-1)-\frac{1}{2}} (1-2\{t\}) f(t)\,dt.
$$

It follows from (25) that

$$
{}_{\alpha}\!\int_0^{\infty} f'(t)\,dt = -4\alpha \; {}_{\alpha-1}\!\int_0^{\infty} (1-2\{t\}) f(t)\,dt, \quad (38)
$$

which is the case $k=1$ of (37). The proof can now be completed by mathematical induction, using exactly the same idea. ∎

Example 2.1. Let $f(t)=\exp(-ct)$, $c>0$. Replacing α by $(\alpha+1)$ in (38) we obtain, for $\mathrm{Re}(\alpha)>-\frac{1}{2}$,

$$ {}_{\alpha}\!\int_{0}^{\infty}(1-2\{t\})\exp(-ct)\,dt=\frac{c\,g_{\alpha+1}(\frac{ic}{2})}{4(\alpha+1)\big(\exp(\frac{c}{2})-\exp(-\frac{c}{2})\big)}. \tag{39} $$

3. Alpha Fourier Transforms

In this section we develop the theory initiated in [5]. The α-Fourier transform of the function $f(t)$ is defined by

$$ \mathcal{F}_{\alpha}\big(f(t)\big)=F_{\alpha}(w):=\frac{1}{\sqrt{2\pi}}\,{}_{\alpha}\!\int_{-\infty}^{\infty}f(t)\exp(iwt)\,dt. \tag{40} $$

If the right-hand member of (40) is an improper α-integral over $(-\infty,\infty)$ then we use the definition (27). In that case, the result may depend on the real number B. We use the notation $\mathcal{F}_{\alpha}\big(B;f(t)\big)$ or $F_{\alpha}(B;w)$ to indicate that dependence. For $B=0$ we use the ordinary notation, i.e., $\mathcal{F}_{\alpha}\big(0;f(t)\big)=:\mathcal{F}_{\alpha}\big(f(t)\big)$ or $F_{\alpha}(0;w)=:F_{\alpha}(w)$. We note that

$$ \mathcal{F}_{\alpha}\big(B;f(t+b)\big)=\exp(-iwb)\mathcal{F}_{\alpha}\big(B+b;f(t)\big), \tag{41} $$

which is the precise version of formula (73) in [5]. The formula (74) of [5] is true for any B:

$$ \mathcal{F}_{\alpha}\big(B;(it)^{n}f(t)\big)=\frac{\partial^{n}}{\partial w^{n}}F_{\alpha}(B;w). \tag{42} $$

We also have

$$ \mathcal{F}_{\alpha}\big(B;\exp(ict)f(t)\big)=F_{\alpha}(B;w+c). \tag{43} $$

We now review the main results presented in [5]. The generalized Fourier's Theorem, Parseval's Equality and Poisson's Formula are respectively

$$ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_{\alpha}(w)\exp(-iwt)\,dw={}_{\alpha}\!\int_{-\infty}^{\infty}f(u)\delta(u-t)\,du, \tag{44} $$

$$ \int_{-\infty}^{\infty}|F_{\alpha}(w)|^{2}\,dw={}_{\alpha}\!\int_{-\infty}^{\infty}{}_{\alpha}\!\int_{-\infty}^{\infty}\big(f(t)\big)^{*}f(u)\delta(u-t)\,du\,dt \tag{45} $$

and

$$ \frac{\sqrt{2\pi}}{L}\sum_{k=-\infty}^{\infty}F_{\alpha}\left(\frac{2k\pi}{L}\right)\exp\left(-\frac{2k\pi it}{L}\right)={}_{\alpha}\!\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}f(u)\delta(u-t+kL)\,du. \tag{46} $$

3.1. The Finite Case

Consider a truncated function of the form

$$f(t) = \begin{cases} g(t), & a < t < b, \\ \frac{1}{2} g(t^+), & t = a, \\ \frac{1}{2} g(t^-), & t = b, \\ 0, & \text{otherwise}, \end{cases} \tag{47}$$

where $g(t)$ is analytic in an appropriate region.

Let $\alpha \geq \frac{1}{2}$. Assuming that (22) is applicable, we have

$$\begin{aligned} &{}_\alpha\!\int_a^b g(u)\delta(u-t)\,dt \\ &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \int_0^\infty \big(\{u\}(1-\{u\})\big)^{\alpha-\frac{1}{2}} \big(g(u+a)\delta(u+a-t) \\ &\qquad\qquad - g(u+b)\delta(u+b-t)\big)\,du. \end{aligned} \tag{48}$$

It follows that

$${}_\alpha\!\int_a^b g(u)\delta(u-t)\,du = 0 \quad \text{if } t < a, \tag{49}$$

$${}_\alpha\!\int_a^b g(u)\delta(u-t)\,du = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}} g(t) \quad \text{if } a < t < b, \tag{50}$$

and

$$\begin{aligned} {}_\alpha\!\int_a^b g(u)\delta(u-t)\,du = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})} \Big(&\big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}} \\ &- \big(\{t-b\}(1-\{t-b\})\big)^{\alpha-\frac{1}{2}} \Big) g(t) \quad \text{if } t > b\,. \end{aligned} \tag{51}$$

Formula (51) is somewhat surprising. Its right-hand member is zero when $\alpha = \frac{1}{2}$ or when $b = a + \ell$, where ℓ is an integer. The value will not be zero in general, even though the impulse $u = t$ is not in the interval (a, b).

Example 3.1. Let $g(t) = \exp(ct)$, $c < 0$. We have

$$F_\alpha(w) = \frac{1}{\sqrt{2\pi}} g_\alpha\!\left(\frac{i}{2}(c+iw)\right) \left(\frac{\exp\big((c+iw)b\big) - \exp\big((c+iw)a\big)}{\exp\big(\frac{1}{2}(c+iw)\big) - \exp\big(-\frac{1}{2}(c+iw)\big)} \right) \tag{52}$$

for $-\infty < w < \infty$. It follows from (44) and (49), (50), (51) that the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\alpha\left(\frac{i}{2}(c+iw)\right)\left(\frac{\exp\big((c+iw)b\big)-\exp\big((c+iw)a\big)}{\exp\big(\frac{1}{2}(c+iw)\big)-\exp\big(-\frac{1}{2}(c+iw)\big)}\right)\exp(-iwt)\,dw$$

is equal to 0 if $t<a$, to $\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}}\exp(ct)$ if $a<t<b$, and to

$$\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\Big(\big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}}-\big(\{t-b\}(1-\{t-b\})\big)^{\alpha-\frac{1}{2}}\Big)\exp(ct)$$

if $t>b$. The preceding calculation can be done directly with a geometric series and a known integral evaluation.

The following Parseval's Equality is more general than formula (97) of [5].

Theorem 3.1. *For $\alpha \geq \frac{1}{2}$ we have the equality*

$$\begin{aligned}\int_{-\infty}^{\infty}|F_\alpha(w)|^2\,dw &= \left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\right)^2\int_0^\infty\big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}}\Big(\big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}}|g(t+a)|^2\\ &\qquad -\big(\{t+b-a\}(1-\{t+b-a\})\big)^{\alpha-\frac{1}{2}}|g(t+b)|^2\Big)\,dt, \qquad (53)\end{aligned}$$

where $F_\alpha(w)$ is the α-Fourier transform of the truncated function (47).

Formula (53) is a consequence of (45), (50) and (22). In like manner we obtain the following generalized Poisson's formula, which contains formula (103) of [5].

Theorem 3.2. *Let $\alpha > \frac{1}{2}$. If $F_\alpha(w)$ is the α-Fourier transform of the truncated function (47) then*

$$\begin{aligned}&\frac{\sqrt{2\pi}}{L}\sum_{k=-\infty}^{\infty}F_\alpha\left(\frac{2k\pi}{L}\right)\exp\left(-\frac{2k\pi i t}{L}\right)\\ &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{\substack{k=-\infty\\ a<t-kL<b}}^{\infty}\big(\{t-kL-a\}(1-\{t-kL-a\})\big)^{\alpha-\frac{1}{2}}g(t-kL)\\ &\qquad+\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{\substack{k=-\infty\\ t-kL>b}}^{\infty}\Big(\big(\{t-kL-a\}(1-\{t-kL-a\})\big)^{\alpha-\frac{1}{2}}\end{aligned}$$

$$- \Big(\{t-kL-b\}(1-\{t-kL-b\})\Big)^{\alpha-\frac{1}{2}}\Big)g(t-kL). \quad (54)$$

The former formulas are typical examples of the kind of results obtained with the method of alpha-calculus.

We conclude this subsection with a representation of the α-Fourier transform over an interval (a,b). We need the evaluation

$$\int_0^1 \big(t(1-t)\big)^{\alpha-\frac{1}{2}}(t-\tfrac{1}{2})^k \exp(iwt)\,dt = \frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{2^{2\alpha}\Gamma(\alpha+1)(2i)^k}\exp\Big(\frac{iw}{2}\Big)g_\alpha^{(k)}\Big(\frac{w}{2}\Big), \quad (55)$$

$k=0,1,2,\dots$, which is a consequence of the representation [6, p. 962]

$$\frac{J_\alpha(z)}{z^\alpha} = \frac{1}{\sqrt{\pi}2^\alpha\Gamma(\alpha+\frac{1}{2})}\int_{-1}^1 (1-u^2)^{\alpha-\frac{1}{2}}\exp(izu)\,du. \quad (56)$$

Theorem 3.3. *For a truncated function of the form (47), we have*

$$F_\alpha(w) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^\infty\sum_{\ell=0}^\infty \frac{g_\alpha^{(k)}(\frac{w}{2})}{k!(2i)^k}\Big(g^{(k)}(a+\ell+\tfrac{1}{2})\exp\big(iw(a+\ell+\tfrac{1}{2})\big) - g^{(k)}(b+\ell+\tfrac{1}{2})\exp\big(iw(b+\ell+\tfrac{1}{2})\big)\Big). \quad (57)$$

Proof. Using (22) and the additivity property, we obtain

$$\begin{aligned} F_\alpha(w) &= \frac{1}{\sqrt{2\pi}}\,{}_\alpha\!\int_a^b g(t)\exp(iwt)\,dt \\ &= \frac{1}{\sqrt{2\pi}}\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{\ell=0}^\infty\int_0^1\big(t(1-t)\big)^{\alpha-\frac{1}{2}}\Big(g(t+a+\ell) \\ &\quad\times\exp\big(iw(t+a+\ell)\big) - g(t+b+\ell)\exp\big(iw(t+b+\ell)\big)\Big)\,dt. \end{aligned} \quad (58)$$

From (58) and the expansions

$$g(t+a+\ell) = \sum_{k=0}^\infty \frac{g^{(k)}(a+\ell+\frac{1}{2})}{k!}(t-\tfrac{1}{2})^k,$$

$$g(t+b+\ell) = \sum_{k=0}^\infty \frac{g^{(k)}(b+\ell+\frac{1}{2})}{k!}(t-\tfrac{1}{2})^k,$$

we get

$$F_\alpha(w) = \frac{1}{\sqrt{2\pi}} \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!}\Big(g^{(k)}(a+\ell+\tfrac{1}{2}) \times \exp\big(iw(a+\ell)\big) - g^{(k)}(b+\ell+\tfrac{1}{2})\exp\big(iw(b+\ell)\big)\Big) \times \int_0^1 \big(t(1-t)\big)^{\alpha-\frac{1}{2}}(t-\tfrac{1}{2})^k \exp(iwt)\,dt. \tag{59}$$

The result follows with the help of (55). □

Corollary 3.1. *For a truncated function of the form (47), where the interval (a,b) is replaced by $(a-\frac{1}{2}, a+\frac{1}{2})$, we have the representation*

$$F_\alpha(w) = \frac{\exp(iwa)}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{g_\alpha^{(k)}(\frac{w}{2})g^{(k)}(a)}{k!(2i)^k}. \tag{60}$$

We also observe that formula (16) is obtainable from (57), where $w=0$. We can for example use (21) and the evaluation

$$g_\alpha^{(2\ell)}(0) = \frac{(-1)^\ell(2\ell)!\Gamma(\alpha+1)}{2^{2\ell}\ell!\Gamma(\alpha+\ell+1)},$$

$\ell = 0,1,2,\dots$. Another possibility is to use the same evaluation and the formula (43) of [5].

3.2. The Infinite Case

We adopt the definition (27) and study the α-Fourier transform of a function f over the interval $(-\infty,\infty)$. We assume that f is analytic and some growth conditions must be imposed on f in order to ensure the convergences.

Theorem 3.4. *We have the representation*

$$F_\alpha(B;w) = \sum_{k=-\infty}^{\infty} (-1)^k g_\alpha(k\pi)\exp(-2k\pi iB)F(w+2k\pi), \tag{61}$$

where F is the ordinary Fourier transform of f.

Proof. We have, in view of (23),

$$\begin{aligned}F_\alpha(B;w) &= \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}{}_\alpha\!\int_{B+k}^{B+k+1} f(t)\exp(iwt)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\,{}_\alpha\!\int_0^1 \sum_{k=-\infty}^{\infty} f(t+B+k)\exp\big(iw(t+B+k)\big)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}\int_0^1 \big(t(1-t)\big)^{\alpha-\frac12}\sum_{k=-\infty}^{\infty} f(t+B+k)\\ &\qquad\qquad\times\exp\big(iw(t+B+k)\big)\,dt. \qquad (62)\end{aligned}$$

We now use Poisson's formula in the form

$$\sqrt{2\pi}\sum_{k=-\infty}^{\infty} F(A+2k\pi)\exp(-2k\pi it) \;=\; \sum_{k=-\infty}^{\infty} f(t+k)\exp\big(iA(t+k)\big). \qquad (63)$$

Substituting in (62), we obtain

$$\begin{aligned}F_\alpha(B;w) &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}\int_0^1 \big(t(1-t)\big)^{\alpha-\frac12}\sum_{k=-\infty}^{\infty} F(w+2k\pi)\\ &\qquad\times\exp\big(-2k\pi i(t+B)\big)\,dt\\ &= \sum_{k=-\infty}^{\infty} F(w+2k\pi)\exp(-2k\pi iB)\\ &\qquad\times\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac12)}\int_0^1 \big(t(1-t)\big)^{\alpha-\frac12}\exp(-2k\pi it)\,dt\\ &= \sum_{k=-\infty}^{\infty}(-1)^k g_\alpha(k\pi)\exp(-2k\pi iB)F(w+2k\pi),\end{aligned}$$

where the last step uses a particular case of (55). □

The value $w=0$ in (61) yields the representation

$${}_\alpha\!\int_{-\infty}^{\infty} f(t)\,dt(B) = \sqrt{2\pi}\sum_{k=-\infty}^{\infty}(-1)^k g_\alpha(k\pi)\exp(-2k\pi iB)F(2k\pi). \qquad (64)$$

Formula (64) implies (33) since, under the assumptions of Theorem 2.3, only the value $k=0$ remains in its right-hand member.

Let us examine more closely formulas (44), (45) and (46). We have, using (23),

$$
\begin{aligned}
{}_{\alpha}\!\int_{-\infty}^{\infty} f(u)\delta(u-t)\,du(B) &= \sum_{k=-\infty}^{\infty} {}_{\alpha}\!\int_{B+k}^{B+k+1} f(u)\delta(u-t)\,du \\
&= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \sum_{k=-\infty}^{\infty} \int_0^1 \big(u(1-u)\big)^{\alpha-\frac{1}{2}} f(u+B+k)\delta(u+B+k-t)\,du \\
&= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \sum_{\substack{k=-\infty \\ 0<t-B-k<1}}^{\infty} \Big((t-B-k)\big(1-(t-B-k)\big)\Big)^{\alpha-\frac{1}{2}} f(t),
\end{aligned}
$$

whence

$$
{}_{\alpha}\!\int_{-\infty}^{\infty} f(u)\delta(u-t)\,du(B) = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \big(\{t-B\}(1-\{t-B\})\big)^{\alpha-\frac{1}{2}} f(t). \quad (65)
$$

From (65) we now see that (44) is equivalent to the relation

$$
\mathcal{F}_{\alpha}\big(B; f(t)\big) = \mathcal{F}\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \big(\{t-B\}(1-\{t-B\})\big)^{\alpha-\frac{1}{2}} f(t)\right). \quad (66)
$$

Example 3.2. The application of (61) to the function $f(t)=\exp(-ct^2)$, $c>0$, readily gives

$$
\mathcal{F}_{\alpha}\big(B; \exp(-ct^2)\big) = \frac{1}{\sqrt{2c}} \sum_{k=-\infty}^{\infty} (-1)^k g_{\alpha}(k\pi) \exp(-2k\pi i B) \exp\left(-\frac{1}{4c}(w+2k\pi)^2\right). \quad (67)
$$

In particular,

$$
{}_{\alpha}\!\int_{-\infty}^{\infty} \exp(-ct^2)\,dt(B) = \sqrt{\frac{\pi}{c}} \sum_{k=-\infty}^{\infty} (-1)^k g_{\alpha}(k\pi) \exp\left(-\frac{\pi^2}{c}k^2 - 2k\pi i B\right). \quad (68)
$$

The preceding calculations, in conjunction with (44), give the relation

$$\sum_{k=-\infty}^{\infty}(-1)^k g_\alpha(k\pi)\exp(2k\pi it)=\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(t(1-t)\big)^{\alpha-\frac{1}{2}} \tag{69}$$

for $0\le t\le 1$ and $\alpha\ge\frac{1}{2}$.

It is clear from (66) that we have the following generalized versions of Parseval's equality and Poisson's formula.

Theorem 3.5. *We have the equality*

$$\int_{-\infty}^{\infty}|F_\alpha(B;w)|^2\,dw \\ =\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\right)^2\int_{-\infty}^{\infty}\big(\{t-B\}(1-\{t-B\})\big)^{2\alpha-1}|f(t)|^2\,dt. \tag{70}$$

Theorem 3.6. *We have the formula*

$$\frac{\sqrt{2\pi}}{L}\sum_{k=-\infty}^{\infty}F_\alpha\Big(B;\frac{2k\pi}{L}\Big)\exp\Big(-\frac{2k\pi it}{L}\Big) \\ =\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{k=-\infty}^{\infty}\big(\{t-B-kL\}(1-\{t-B-kL\})\big)^{\alpha-\frac{1}{2}}f(t-kL). \tag{71}$$

Example 3.3. Let us apply (70) to the function $f(t)=\exp(-ct^2)$, $c>0$. Using (67) we obtain a formula which can be written, after elementary transformations, as

$$\sqrt{c}\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\right)^2\int_0^1\big(t(1-t)\big)^{2\alpha-1}\theta(c,t+B)\,dt=1+4\sum_{k=1}^{\infty}(-1)^k g_\alpha(k\pi) \\ \times\exp\Big(-\frac{\pi}{c}k^2\Big)\cos(2k\pi B)+2\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{k-\ell}g_\alpha(k\pi)g_\alpha(\ell\pi) \\ \times\exp\Big(-\frac{\pi}{c}(k-\ell)^2\Big)\cos\big(2(k-\ell)\pi B\big)+2\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(-1)^{k+\ell}g_\alpha(k\pi)g_\alpha(\ell\pi) \\ \times\exp\Big(-\frac{\pi}{c}(k+\ell)^2\Big)\cos\big(2(k+\ell)\pi B\big), \tag{72}$$

where $(x>0)$

$$\theta(x,t):=\sum_{k=-\infty}^{\infty}\exp\big(-\pi x(k+t)^2\big) \tag{73}$$

is Jacobi theta function.

In a similar manner, the formula (71) gives, in particular:

$$\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}(-1)^k g_\alpha(k\pi)\exp\Big(-\frac{\pi}{c}(k-\ell)^2-2\ell\pi i t\Big) = \sqrt{c}\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(t(1-t)\big)^{\alpha-\frac{1}{2}}\theta(c,t) \quad (74)$$

for $0 \le t \le 1$ and $\alpha \ge \frac{1}{2}$.

3.3. The Semi-Infinite Case

Here we study α-Fourier transforms of functions defined over intervals of the form (a,∞), where a is a real number. Consider the truncated function

$$f(t) = \begin{cases} g(t), & a < t < \infty, \\ \frac{1}{2}g(t^+), & t = a, \\ 0, & -\infty < t < a, \end{cases} \quad (75)$$

where $g(t)$ satisfies certain conditions.

The analogue of formula (57) is

$$F_\alpha(w) = \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{g_\alpha^{(k)}(\frac{w}{2})g^{(k)}(a+j+\frac{1}{2})}{k!(2i)^k}\exp\big(iw(a+j+\tfrac{1}{2})\big). \quad (76)$$

That is in fact its limiting case $b \to \infty$.

In order to obtain the precise version of (44), we evaluate

$$\begin{aligned} {}_\alpha\!\int_a^\infty g(u)\delta(u-t)\,du &= \sum_{k=0}^{\infty} {}_\alpha\!\int_0^1 g(u+a+k)\delta(u+a+k-t)\,du \\ &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{k=0}^{\infty}\int_0^1 \big(u(1-u)\big)^{\alpha-\frac{1}{2}}g(u+a+k)\delta(u+a+k-t)\,du \\ &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{\substack{k=0 \\ 0<t-a-k<1}}^{\infty}\Big((t-a-k)\big(1-(t-a-k)\big)\Big)^{\alpha-\frac{1}{2}}g(t), \end{aligned}$$

whence

$${}_\alpha\!\int_a^\infty g(u)\delta(u-t)\,du = g_{a,\alpha}(t)g(t), \quad (77)$$

where ($\alpha > \frac{1}{2}$)

$$g_{a,\alpha}(t) := \begin{cases} \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(\{t-a\}(1-\{t-a\})\big)^{\alpha-\frac{1}{2}}, & a < t < \infty \\ 0, & -\infty < t \le a. \end{cases} \tag{78}$$

We thus have

$$F_\alpha(w) = \mathcal{F}\big(g_{a,\alpha}(t)f(t)\big). \tag{79}$$

Remark 3.1. If we let $\alpha \to \infty$ in (77) then we obtain

$$\lim_{\alpha\to\infty} \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\big(\{t\}(1-\{t\})\big)^{\alpha-\frac{1}{2}} = \sum_{j=0}^{\infty} \delta(j+\tfrac{1}{2}-t), \tag{80}$$

for all positive t.

Here is the corresponding version of (45).

Theorem 3.7. *For a truncated function of the form (75), we have the equality*

$$\int_{-\infty}^{\infty} |F_\alpha(w)|^2\, dw \\ = \left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\right)^2 \int_a^\infty \big(\{t-a\}(1-\{t-a\})\big)^{2\alpha-1}|g(t)|^2\, dt. \tag{81}$$

Example 3.4. The choice $f(t) = \exp(ct)$, $c < 0$, leads us to the evaluation

$$\int_{-\infty}^{\infty} \left|\frac{g_\alpha\big(\frac{i}{2}(c+iw)\big)}{\exp\big(\frac{1}{2}(c+iw)\big) - \exp\big(-\frac{1}{2}(c+iw)\big)}\right|^2 dw \\ = \frac{2\sqrt{\pi}\big(\Gamma(\alpha+1)\big)^2\Gamma(2\alpha)}{\big(\Gamma(\alpha+\frac{1}{2})\big)^2\Gamma(2\alpha+\frac{1}{2})}\,\frac{g_{2\alpha-\frac{1}{2}}(ic)}{\sinh(-c)}. \tag{82}$$

The integral in (82) is convergent for $\alpha > 0$ (in fact for $\mathrm{Re}(\alpha) > 0$). Compare with the evaluation (98) of [5].

Remark 3.2. The limiting case $\alpha \to \infty$ of (81) gives the equality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\sum_{j=0}^{\infty} g(j+\tfrac{1}{2})\exp\big(iw(j+\tfrac{1}{2})\big)\right|^2 dw \\ = \int_0^\infty \left|\sum_{j=0}^{\infty}\delta(j+\tfrac{1}{2}-t)g(t)\right|^2 dt. \tag{83}$$

As far as (46) is concerned, we have the following result.

Theorem 3.8. *For a truncated function of the form (75), we have*

$$\frac{\sqrt{2\pi}}{L}\sum_{k=-\infty}^{\infty} F_\alpha\left(\frac{2k\pi}{L}\right)\exp\left(-\frac{2k\pi it}{L}\right) = \sum_{k=-\infty}^{\infty} g_{a,\alpha}(t-kL)g(t-kL), \quad (84)$$

where $g_{a,\alpha}$ is defined by (78) and $\alpha > \frac{1}{2}$.

Corollary 3.2. *For any positive integer q we have, for $0 \le t \le 1$ and $\alpha > \frac{1}{2}$,*

$$\begin{aligned}\sqrt{2\pi}\sum_{k=-\infty}^{\infty} F_\alpha(2\pi qk)\exp(-2k\pi it) &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\times\frac{1}{q}\sum_{s=0}^{\infty}\sum_{r=0}^{q-1}\left(\frac{(t+r)}{q}\left(1-\frac{(t+r)}{q}\right)\right)^{\alpha-\frac{1}{2}} \\ &\qquad g\left(\frac{(t+r)}{q}+s\right). \quad (85)\end{aligned}$$

Proof. In (84) we take $a=0$, $L=\frac{1}{q}$ and we replace t by $\frac{t}{q}$. We obtain

$$\begin{aligned}\sqrt{2\pi}q\sum_{k=-\infty}^{\infty} F_\alpha(2\pi qk)\exp(-2k\pi it) &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{k=-\infty}^{[t]}\left(\left\{\frac{t-k}{q}\right\}\left(1-\left\{\frac{t-k}{q}\right\}\right)\right)^{\alpha-\frac{1}{2}} g\left(\frac{t-k}{q}\right) \\ &= \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{k=0}^{\infty}\left(\left\{\frac{t+k}{q}\right\}\left(1-\left\{\frac{t+k}{q}\right\}\right)\right)^{\alpha-\frac{1}{2}} g\left(\frac{t+k}{q}\right) \quad (86)\end{aligned}$$

for $0 \le t \le 1$. Each integer $k \ge 0$ can be written uniquely in the form $k = sq+r$, where $0 \le r < q$, $s = 0,1,2,\ldots$. The result follows by substitution in (86), noticing that $0 \le t+r \le q$. ☐

A consequence of (85) is (see (22), where $a=0$ and $b\to\infty$):

$$\lim_{q\to\infty}\sum_{k=-\infty}^{\infty} F_\alpha(2\pi qk) = \frac{1}{\sqrt{2\pi}}\ {}_\alpha\!\int_0^\infty g(u)\,du. \quad (87)$$

It is interesting to examine more closely what happens when $q \to \infty$ in (85). In order to see that we first apply Corollary 3.2 to a function of the form $g(t) = \exp(-cqt)h(t)$, where $c \geq 0$, taking into account (43). We then use the resulting formula (76) and substitute in the left-hand member of (85) with $a = 0$. We obtain the formula

$$\begin{aligned} q^{\alpha+\frac{1}{2}} \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} & \frac{(-1)^{qk} g_\alpha^{(\ell)}\big(q(k\pi + \frac{ic}{2})\big)}{\ell!(2i)^\ell} \\ & \times h^{(\ell)}(j+\tfrac{1}{2}) \exp\big(-qc(j+\tfrac{1}{2})\big) \exp(-2k\pi i t) \\ = \frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} & \sum_{s=0}^{\infty} \sum_{r=0}^{q-1} \bigg((t+r)\Big(1 - \frac{(t+r)}{q}\Big)\bigg)^{\alpha-\frac{1}{2}} \\ & \times \exp\big(-c(t+r) - qcs\big) h\bigg(\frac{(t+r)}{q} + s\bigg). \end{aligned} \tag{88}$$

We now use the asymptotic expansion (obtainable from [6, p. 972]):

$$g_\alpha^{(\ell)}(z) \sim 2^\alpha \Gamma(\alpha+1) \sqrt{\frac{2}{\pi}} \frac{\cos(z - \frac{\alpha\pi}{2} - \frac{\pi}{4} + \frac{\ell\pi}{2})}{z^{\alpha+\frac{1}{2}}}, \tag{89}$$

as $|z| \to \infty$, $|\arg(z)| < \pi$. Letting $q \to \infty$ in (88) we obtain (without specifying a particular function h), for $c > 0$,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} & \frac{\exp(-2k\pi i t)}{(2k\pi + ic)^{\alpha+\frac{1}{2}}} \\ & = \frac{\exp\big(-\frac{\pi i}{2}(\alpha+\frac{1}{2})\big)}{\Gamma(\alpha+\frac{1}{2})} \sum_{r=0}^{\infty} (t+r)^{\alpha-\frac{1}{2}} \exp\big(-c(t+r)\big) \end{aligned} \tag{90}$$

for $0 \leq t \leq 1$ and $\alpha > \frac{1}{2}$. If $c = 0$ then we get the formula

$$\begin{aligned} 2^{\alpha+1}\Gamma(\alpha+1)\sqrt{\frac{2}{\pi}} & \Bigg(\sum_{k=1}^{\infty} \frac{\cos(2k\pi t - \frac{\alpha\pi}{2} - \frac{\pi}{4})}{(k\pi)^{\alpha+\frac{1}{2}}} \frac{h(0)}{2} \\ & + \cos\Big(\frac{\alpha\pi}{2} + \frac{\pi}{4}\Big) \sum_{k=1}^{\infty} \frac{\cos(2k\pi t)}{(k\pi)^{\alpha+\frac{1}{2}}} \sum_{j=1}^{\infty} h(j) \Bigg) = \lim_{q\to\infty} q^{\alpha+\frac{1}{2}} \\ \times \Bigg(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \frac{1}{q} & \sum_{s=0}^{\infty} \sum_{r=0}^{q-1} \bigg(\frac{(t+r)}{q} \Big(1 - \frac{(t+r)}{q}\Big)\bigg)^{\alpha-\frac{1}{2}} h\bigg(\frac{(t+r)}{q} + s\bigg) \end{aligned}$$

$$-\sum_{\ell=0}^{\infty}\sum_{j=0}^{\infty}\frac{g_\alpha^{(\ell)}(0)h^{(\ell)}(j+\frac{1}{2})}{\ell!(2i)^\ell}\Bigg) \tag{91}$$

for $0 \le t \le 1$ and $\alpha > \frac{1}{2}$.

Example 3.5. The application of (85) to $g(t) = \exp(ct)$, $c < 0$, $t = 0$, gives

$$q\exp\left(\frac{c}{2}\right)\sum_{k=-\infty}^{\infty}(-1)^{qk}g_\alpha\left(qk\pi-\frac{ic}{2}\right)$$
$$=\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{r=1}^{q}\left(\frac{r}{q}\left(1-\frac{r}{q}\right)\right)^{\alpha-\frac{1}{2}}\exp\left(\frac{cr}{q}\right). \tag{92}$$

The preceding formula is in fact valid for all real c. In particular, for $q = 2$ we obtain

$$\sum_{k=-\infty}^{\infty}g_\alpha(2k\pi-ic)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \tag{93}$$

for all real c and $\alpha > \frac{1}{2}$.

The same example in (91) where $t = 0$ (or directly from (92)) gives, for $\alpha > \frac{1}{2}$,

$$\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+1)}{\pi^{\alpha+1}}\cos\left(\frac{\alpha\pi}{2}+\frac{\pi}{4}\right)\cosh\left(\frac{c}{2}\right)\exp\left(\frac{c}{2}\right)\zeta(\alpha+\tfrac{1}{2})$$
$$=\lim_{q\to\infty}q^{\alpha+\frac{1}{2}}\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{r}{q}\left(1-\frac{r}{q}\right)\right)^{\alpha-\frac{1}{2}}\exp\left(\frac{cr}{q}\right)\right.$$
$$\left.-\exp\left(\frac{c}{2}\right)g_\alpha\left(\frac{ic}{2}\right)\right), \tag{94}$$

where ζ is Riemann zeta function. The limiting case $c \to 0$ in (94) leads us to the relation

$$\frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+1)}{\pi^{\alpha+1}}\cos\left(\frac{\alpha\pi}{2}+\frac{\pi}{4}\right)\zeta(\alpha+\tfrac{1}{2})$$
$$=\lim_{q\to\infty}q^{\alpha+\frac{1}{2}}\left(\frac{2^{2\alpha}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\frac{1}{q}\sum_{r=1}^{q-1}\left(\frac{r}{q}\left(1-\frac{r}{q}\right)\right)^{\alpha-\frac{1}{2}}-1\right). \tag{95}$$

Dividing both sides of (95) by $(\alpha - \frac{5}{2})$ and letting $\alpha \to \frac{5}{2}$, we get

$$\frac{15}{\pi^2}\zeta(3) = \lim_{q\to\infty} q^3 \left(\frac{47}{30} + \frac{30}{q} \sum_{r=1}^{q} \left(\frac{r}{q}\left(1 - \frac{r}{q}\right)\right)^2 \ln\left(\frac{r}{q}\left(1 - \frac{r}{q}\right)\right)\right). \tag{96}$$

The preceding formulas can be written in another manner if we use Sonin's formula [7, p. 31], namely

$$\sum_{y<r\le x} f(r) = \int_y^x f(t)\,dt + \int_y^x \{t\} f'(t)\,dt - \{x\}f(x) + \{y\}f(y). \tag{97}$$

As an illustration, (95) becomes

$$\begin{aligned} &\frac{\Gamma(\alpha - \frac{1}{2})}{2^{\alpha-\frac{3}{2}}\pi^{\alpha+\frac{1}{2}}} \cos\left(\frac{\alpha\pi}{2} + \frac{\pi}{4}\right)\zeta(\alpha + \tfrac{1}{2}) \\ &\qquad = \lim_{q\to\infty} \int_0^q \{t\}\left(1 - \frac{2t}{q}\right)\left(t\left(1 - \frac{t}{q}\right)\right)^{\alpha-\frac{3}{2}} dt \end{aligned} \tag{98}$$

for $\alpha > \frac{1}{2}$.

4. The Alpha-Calculus of the Second Kind

We consider quantities of the form

$$Q_{\alpha,m} := \frac{\partial^{m-1}}{\partial\alpha^{m-1}} Q_\alpha\,, \tag{99}$$

where Q_α is a quantity of α-calculus. In [5, Section 5] several relations of the second kind are given for basic sequences of α-calculus.

We begin here with a few simple formulas related to Subsections 5.1 and 5.2 of [5]. We note that

$$d^{(m)}_{2n+m,\frac{1}{2},2} = \frac{m(2n+m)!}{(2n+1)!} d^{(1)}_{2n+1,\frac{1}{2},2}, \tag{100}$$

which follows at once from formula (119) in [5]. It is also readily seen, from formula (62) of [4], that

$$\psi_{2n+1,\frac{1}{2},2} = -d_{2n+1,\frac{1}{2},2}. \tag{101}$$

We thus have

$$b^{(m)}_{\frac{1}{2},2}\big(f(z);z\big) = -m\sum_{n=1}^{\infty}\frac{\psi_{2n+1,\frac{1}{2},2}}{(2n+1)!}f^{(2n+m)}(z) = \frac{m}{36}f^{(m+2)}(z)+\cdots \tag{102}$$

for $m = 1,2,3,\ldots$. It may be noticed here that [5, formula (141)] $\psi_{2n+1,\frac{1}{2},2} = 2B_{2n+1,\frac{1}{2},2}$, $n = 1,2,3,\ldots$.

In the spirit of formula (146) of [5], we mention the following consequence of (22):

$$ {}_{\frac{1}{2},2}\!\int_a^b f(t)\,dt = 2\int_a^b f(t)\,dt + \int_0^{\infty}\ln\big(\{t\}(1-\{t\})\big)\big(f(t+a)-f(t+b)\big)\,dt. \tag{103}$$

For the remaining of the section we concentrate the study on the kind of results presented in Subsection 5.3 of [5]. The next lemmas will be useful.

Lemma 4.1. *For all positive integers n and s, we have*

$$\frac{\partial^{2s}}{\partial z^{2s}}\left(\frac{1}{g_\alpha(\frac{iz}{2})}\right)^n\Bigg|_{z=0} = \frac{(2s)!}{s!2^{4s}}\sum_{r=1}^{s}\sum_{\pi(s,r)}(-1)^r\frac{(n+r-1)!}{(n-1)!}c(k_1,\ldots,k_s) \times\prod_{\nu=1}^{s}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}\right)^{k_\nu}, \tag{104}$$

where $c(k_1,\ldots,k_s) = \frac{s!}{k_1!\cdots k_s!(1!)^{k_1}\cdots(s!)^{k_s}}$ and $\pi(s,r)$ means that the summation is extended over the nonnegative integers $k_1,\ldots,k_s$ such that $k_1+2k_2+\cdots+sk_s = s$ and $k_1+k_2+\cdots+k_s = r$.

Proof. The relation (104) is a direct consequence of Faa di Bruno Formula [6, p. 24]

$$\Big(F\big(G(z)\big)\Big)^{(N)} = \sum_{r=1}^{N}\sum_{\pi(N,r)}c(k_1,\ldots,k_N)\prod_{\mu=1}^{N}\big(G^{(\mu)}(z)\big)^{k_\mu}F^{(r)}\big(G(z)\big), \tag{105}$$

and the evaluation

$$g_\alpha^{(\mu)}(0) = \begin{cases}\frac{(-1)^\nu(2\nu)!\Gamma(\alpha+1)}{2^{2\nu}\nu!\Gamma(\alpha+\nu+1)}, & \mu = 2\nu,\ \nu\ge 0,\\ 0, & \mu = 2\nu+1,\ \nu\ge 0.\end{cases} \tag{106}$$

□

Lemma 4.2. *The sequences* $d_{k,\alpha}^{(n)}$ *can be written as*

$$d_{k,\alpha}^{(n)} = \sum_{s=0}^{[\frac{k-n}{2}]} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{k}{2s} \left(-\frac{(n-2j)}{2}\right)^{k-2s} \times \frac{\partial^{2s}}{\partial z^{2s}} \left(\frac{1}{g_\alpha(\frac{iz}{2})}\right)^n \Bigg|_{z=0}. \qquad (107)$$

Proof. That is an immediate consequence of Leibniz' formula

$$\big(F(z)G(z)\big)^{(N)} = \sum_{r=0}^{N} \binom{N}{r} F^{(r)}(z) G^{(N-r)}(z)$$

and the relation

$$d_{k,\alpha}^{(n)} = \frac{d^k}{dz^k} \left(\frac{\exp(\frac{z}{2}) - \exp(-\frac{z}{2})}{g_\alpha(\frac{iz}{2})}\right)^n \Bigg|_{z=0}. \qquad \square$$

Lemma 4.3. *Let* $r_i := \sum_{\mu=i}^{s} k_\mu, \quad 1 \le i \le s$. *We have the partial fractions decomposition*

$$\frac{1}{(\alpha+1)^{r_1}(\alpha+2)^{r_2}\cdots(\alpha+s)^{r_s}} = \sum_{i=1}^{s} \sum_{\ell=1}^{r_i} \frac{c_{\ell,i}}{(\alpha+i)^\ell}, \qquad (108)$$

with

$$\sum_{i=1}^{s} c_{1,i} = 0 \qquad (109)$$

for $s \ge 2$, *and*

$$c_{r_\nu,\nu} = \prod_{\substack{j=1 \\ j \ne \nu}}^{s} \frac{1}{(j-\nu)^{r_j}} \qquad (110)$$

for $1 \le \nu \le s$.

Proof. Formula (108) is simply a Heaviside expansion. We obtain (109) by multiplying both sides of (108) by α and by letting $\alpha \to \infty$. The relation (110) is obtained by multiplying both members of (108) by $(\alpha+\nu)^{r_\nu}$ and by taking the limit as $\alpha \to -\nu$. $\square$

We now present some asymptotic results. As a consequence of the formula (8.515) of [6, p. 988], and the representation

$$g_\alpha(z) = 1 - \sum_{k=1}^{\infty} \frac{J_k(z) z^k}{2^k (k-1)!(\alpha+k)}, \tag{111}$$

we get

$$\lim_{\alpha\to\infty} \alpha^m g_{\alpha,m}(z) = (-1)^m (m-1)! \frac{z^2}{4}. \tag{112}$$

In what follows Δ represents the operator of finite differences defined by $\Delta^0 f(z) = f(z)$, $\Delta f(z) = f(z+1) - f(z)$ and, for $n \geq 1$, $\Delta^{n+1} f(z) = \Delta\big(\Delta^n f(z)\big)$.

Theorem 4.1. *For $m = 2, 3, \ldots$, $n = 0, 1, \ldots$, we have*

$$\lim_{\alpha\to\infty} \alpha^m b_{\alpha,m}^{(n)}\big(f(z); z\big) = \frac{n}{16} (-1)^m (m-1)! \Delta^n f''\left(z - \frac{n}{2}\right). \tag{113}$$

In particular,

$$\lim_{\alpha\to\infty} \alpha^m d_{k,\alpha,m}^{(n)} = \frac{n}{16} (-1)^m (m-1)! k(k-1) \Delta^n \left(\zeta - \frac{n}{2}\right)^{k-2} \Bigg|_{\zeta=0} \tag{114}$$

for $k = 2, 3, \ldots$.

Proof. Formulas (113) and (114) are in fact equivalent. The application of (113) to $f(z) = z^k$ gives (114) since [3, p. 287] $d_{k,\alpha}^{(n)} = b_\alpha^{(n)}(z^k; z=0)$. Formula (113) is a consequence of (114) in view of the definition (3). We prove (114).

Lemma 4.1 and Lemma 4.2 give the explicit representation

$$\begin{aligned} \lim_{\alpha\to\infty} \alpha^m d_{k,\alpha,m}^{(n)} = & \sum_{s=1}^{[\frac{k-n}{2}]} \sum_{j=0}^{n} \sum_{r=1}^{s} \sum_{\pi(s,r)} (-1)^{n-j} \binom{n}{j} \binom{k}{2s} \left(-\frac{(n-2j)}{2}\right)^{k-2s} \\ & \frac{(2s)!(-1)^r (n+r-1)!}{s! 2^{4s} (n-1)!} c(k_1, \ldots, k_s) \\ & \times \lim_{\alpha\to\infty} \alpha^m \frac{\partial^{m-1}}{\partial\alpha^{m-1}} \prod_{\nu=1}^{s} \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)}\right)^{k_\nu}. \end{aligned} \tag{115}$$

We compute the derivatives with respect to α with the help of (108); we obtain

$$\lim_{\alpha\to\infty} \alpha^m \frac{\partial^{m-1}}{\partial\alpha^{m-1}} \prod_{\nu=1}^{s} \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} \right)^{k_\nu}$$
$$= \lim_{\alpha\to\infty} \sum_{i=1}^{s} \sum_{\ell=1}^{r_i} \frac{(-1)^{m-1}(\ell+m-2)!\alpha^m c_{\ell,i}}{(\ell-1)!(\alpha+i)^{\ell+m-1}} = \sum_{i=1}^{s} (-1)^{m-1}(m-1)! c_{1,i}. \quad (116)$$

Substituting in (115) and taking into account (109), we get

$$\lim_{\alpha\to\infty} \alpha^m d^{(n)}_{k,\alpha,m}$$
$$= -\frac{n}{8} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \binom{k}{2} \left(-\frac{(n-2j)}{2} \right)^{k-2s} (-1)^{m-1}(m-1)!. \quad (117)$$

The result follows at once since

$$\Delta^n F(z) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} F(z+j). \quad (118)$$

The choice $f(z) = B_{N+1}(z)$, $n = 1$, in (113), readily gives us (see formula (75) of [4]) the Proposition 5.5 of [5]. The choice $f(z) = \exp(cz)$ leads us to the asymptotic relation

$$\lim_{\alpha\to\infty} \alpha^m \frac{\partial^{m-1}}{\partial\alpha^{m-1}} \left(\frac{1}{g_\alpha(\frac{ic}{2})} \right)^n = \frac{n}{16} (-1)^m (m-1)!\, c^2 \quad (119)$$

for $m = 2, 3, \ldots$.

Remark 4.1. It is more difficult to obtain similar results for the polynomials $\phi_{n,\alpha}(z)$. The representation $\phi_{n,\alpha}(z) = \frac{\partial^n}{\partial\zeta^n} \exp\big(z w_\alpha(\zeta)\big)\big|_{\zeta=0}$ and the explicit evaluation $w_\infty(\zeta) = 2\ln\Big(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\Big)$ can be used to show that

$$\lim_{\alpha\to\infty} \alpha^2 \phi_{n,\alpha,2}(z)$$
$$= -\frac{nz}{2} \frac{\partial^{n-1}}{\partial\zeta^{n-1}} \frac{\big(\frac{\zeta+\sqrt{\zeta^2+4}}{2}\big)^{2z}}{\sqrt{\zeta^2+4}} \left(\ln\left(\frac{\zeta+\sqrt{\zeta^2+4}}{2} \right) \right)^2 \Bigg|_{\zeta=0}. \quad (120)$$

We now present some results concerning quantities of the form $\lim_{m\to\infty} Q_{\frac{\alpha}{m},m}$, where $Q_{\frac{\alpha}{m},m}$ means that we first evaluate $Q_{\alpha,m}$ and then replace α by $\frac{\alpha}{m}$. As a direct consequence of (111) we obtain

$$\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!} g_{\frac{\alpha}{m},m}(z) = \frac{z}{2} J_0'(z) \exp(-\alpha). \quad (121)$$

A more sophisticated result follows.

Theorem 4.2. *For each complex number z, we have*

$$\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!} m^{\frac{n}{2}+1} b^{(n)}_{\frac{\alpha}{m},m}\left(f\left(\frac{z}{\sqrt{m}}\right);z\right) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\binom{n-1+\ell}{n-1}}{2^{4\ell}(\ell-1)!} f^{(n+2\ell)}(0)\exp(-\alpha). \quad (122)$$

In particular,

$$\lim_{m\to\infty} \frac{(-1)^{m-1}}{(m-1)!m^{s-1}} d^{(n)}_{k,\frac{\alpha}{m},m} = \frac{(-1)^{s}\binom{n-1+s}{s}k!}{2^{4s}(s-1)!}\exp(-\alpha), \quad (123)$$

where $(k-n)$ is even and $s := \frac{(k-n)}{2}$.

Proof. Formula (123) is the case $f(z) = z^k$ of (122) but, as in the proof of Theorem 4.1, the two formulas are equivalent. We prove (123).

Lemmas 4.1, 4.2, and 4.3 permit us to write

$$\begin{aligned}\frac{(-1)^{m-1}}{(m-1)!m^{\frac{(k-n)}{2}-1}} d^{(n)}_{k,\frac{\alpha}{m},m} &= \sum_{s=0}^{\frac{k-n}{2}}\sum_{j=0}^{n}\sum_{r=1}^{s}\sum_{\pi(s,r)}(-1)^{n-j}\binom{n}{j}\binom{k}{2s}\left(-\frac{(n-2j)}{2}\right)^{k-2s} \\ &\quad \frac{(2s)!(-1)^{r}(n+r-1)!}{s!2^{4s}(n-1)!}c(k_1,\ldots,k_s) \\ &\quad \sum_{i=1}^{s}\sum_{\ell=1}^{r_i}\frac{c_{\ell,i}(\ell+m-2)!m^{\ell-\frac{(k-n)}{2}}}{(\frac{\alpha}{m}+i)^{\ell+m-1}(\ell-1)!(m-1)!m^{\ell-1}}. \quad (124)\end{aligned}$$

As $m\to\infty$ only the terms corresponding to $i=1$ remain. Moreover, we have $\ell \le r_1 \le r \le s \le \frac{k-n}{2}$; so we get $\lim_{m\to\infty} m^{\ell-\frac{(k-n)}{2}} = 0$ except if $\ell = r_1 = r = s = \frac{k-n}{2}$, with $k_2 = k_3 = \cdots = k_s = 0$ and $k_1 = s$. By (110) we then have $c_{r_1,1} = 1$. It follows from (124) that

$$\begin{aligned}\lim_{m\to\infty}\frac{(-1)^{m-1}}{(m-1)!m^{\frac{(k-n)}{2}-1}} d^{(n)}_{k,\frac{\alpha}{m},m} &= \frac{(-1)^{s}(2s)!(n+s-1)!\binom{k}{2s}\exp(-\alpha)}{s!2^{4s}(s-1)!(n-1)!} \\ &\quad \times\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(-\frac{(n-2j)}{2}\right)^{k-2s}, \quad (125)\end{aligned}$$

where $s = \frac{k-n}{2}$. We obtain the result from (125) since [6, p. 5]

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left(-\frac{(n-2j)}{2}\right)^{n} = n!. \tag{126}$$

The proof of the next result is similar to that of Theorem 4.1 and Theorem 4.2, except that we start with formula (71) of [2].

Theorem 4.3. *Let $s := [\frac{k}{2}]$, where k is a positive integer. We have*

$$\lim_{m\to\infty}\frac{(-1)^m}{(m-1)!m^{s-1}}B_{k,\frac{\alpha}{m},m}(x) = \frac{(-1)^{s-1}k!}{2^{4s}(s-1)!}(x-\tfrac{1}{2})^{k-2s}\exp(-\alpha). \tag{127}$$

If k is odd and $x = 0$ then (127) also follows from (123) where $n = 1$. The cases $k = 2, 3, 4, 5$ have been calculated directly at the end of Section 5 of [5].

Remark 4.2. As in Remark 4.1, it is more difficult to obtain similar results for the polynomials $\phi_{n,\alpha}(z)$. We mention the following empirical calculations. Let

$$A_n := \exp(\alpha)\lim_{m\to\infty}\frac{(-1)^{m-1}}{(m-1)!m^s}\phi_{n,\frac{\alpha}{m},m}(z),$$

where $s := \left[\frac{n-3}{2}\right]$. We have $A_3 = \frac{3}{8}z$, $A_4 = \frac{3}{2}z^2$, $A_5 = \frac{15}{16}z$, $A_6 = \frac{225}{32}z^2$, $A_7 = \frac{1575}{512}z$, $A_8 = \frac{2205}{64}z^2$, $A_9 = \frac{6615}{512}z$ and $A_{10} = \frac{99225}{512}z^2$.

5. Other Observations

5.1. Another Definition of the Alpha-Derivative

It was noticed in [4, formula (93)] that the alpha-derivative of order n can also be defined by

$$b_\alpha^{(n)}\big(f(z);z\big) = \sum_{k=n}^{\infty}\frac{\delta_{k,\alpha}^{(n)}}{k!}\Delta^k f(z), \tag{128}$$

where

$$\delta_{k,\alpha}^{(n)} := \sum_{\ell=n}^{k} s(k,\ell)d_{\ell,\alpha}^{(n)}. \tag{129}$$

Here the $s(k,\ell)$ are the Stirling numbers of the first kind. Direct calculations show that $\delta_{n,\alpha}^{(n)} = n!$, $\delta_{n+1,\alpha}^{(n)} = -\frac{n(n+1)!}{2}$, $\delta_{n+2,\alpha}^{(n)} = \frac{n(n+2)!\big((2n+4)\alpha+(2n+3)\big)}{16(\alpha+1)}$ and $\delta_{n+3,\alpha}^{(n)} = -\frac{n(n+2)(n+3)!\big((2n+8)\alpha+(2n+5)\big)}{96(\alpha+1)}$.

Proposition 5.1. *We have the equality*

$$\delta_{k,\alpha}^{(n)} = \frac{d^k}{d\zeta^k}\left(\frac{\sqrt{\zeta}-\frac{1}{\sqrt{\zeta}}}{g_\alpha\left(\frac{i}{2}\ln(\zeta)\right)}\right)^n\Bigg|_{\zeta=1}. \tag{130}$$

Proof. We apply (128) to the function $f(z)=\exp(cz)$. We obtain

$$\left(\frac{\exp(\frac{c}{2})-\exp(-\frac{c}{2})}{g_\alpha(\frac{ic}{2})}\right)^n = \sum_{k=n}^{\infty}\frac{\delta_{k,\alpha}^{(n)}}{k!}\left(\exp(c)-1\right)^n, \tag{131}$$

and (130) follows after the change of variable $c=\ln(\zeta)$.

We can find that

$$\delta_{k,\frac{1}{2}}^{(n)} = n!s(k,n), \tag{132}$$

$$\delta_{k,\infty}^{(n)} = \frac{(-1)^{k+n}k!\Gamma(k-\frac{n}{2})}{(k-n)!\Gamma(\frac{n}{2})} \tag{133}$$

and

$$\delta_{k,-\frac{1}{2}}^{(n)} = \frac{(-1)^{k+n}(k-1)!k!}{(k-n)!(n-1)!2^{k-n}}, \tag{134}$$

for $k \geq n$. Formulas (128) and (133) give

$$\Delta^n f\left(z-\frac{n}{2}\right) = \sum_{k=n}^{\infty}\frac{(-1)^{k+n}\Gamma(k-\frac{n}{2})}{(k-n)!\Gamma(\frac{n}{2})}\Delta^k f(z). \tag{135}$$

5.2. Jacobi Theta Function

It is well-known that the Jacobi theta function (73) satisfies the functional equation

$$\theta(x,t) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}, itx\right)\exp(-\pi x t^2). \tag{136}$$

The latter formula is generally proved by choosing a Gaussian function in the traditional Poisson Formula. The most elementary way to obtain it is probably by expanding $\theta(x,t)$ in Fourier series, noticing that $\theta(x,t)$ is a periodic function for the variable t, with period one. We observe that the relation (136) can be obtained as the limiting case $\alpha\to\infty$ of [5, formula (103)], without specifying a particular function g. We take $a=0$, replace t by $\frac{t}{\sqrt{\alpha}}$ and L by $\frac{L}{\sqrt{\alpha}}$; taking into account (60), we obtain

$$\frac{2}{\sqrt{\pi}}g(0)\sum_{k=-\infty}^{\infty}\exp\bigl(-4(t-kL)^2\bigr)=\lim_{\alpha\to\infty}\frac{1}{L}\sum_{k=-\infty}^{\infty}\sum_{\ell=0}^{\infty}\frac{g^{(\ell)}(0)g_\alpha^{(\ell)}\bigl(\frac{k\pi}{L}\sqrt{\alpha}\bigr)}{\ell!(2i)^\ell}\times\exp\biggl(-\frac{2k\pi it}{L}\biggr). \quad (137)$$

The integral representation (56) gives

$$\lim_{\alpha\to\infty}g_\alpha^{(\ell)}(z\sqrt{\alpha})=\begin{cases}\exp\bigl(-\frac{z^2}{4}\bigr), & \ell=0,\\ 0, & \ell>0.\end{cases} \quad (138)$$

It follows from (137) and (138) that

$$\sum_{k=-\infty}^{\infty}\exp\bigl(-4(t-kL)^2\bigr)=\frac{\sqrt{\pi}}{2L}\sum_{k=-\infty}^{\infty}\exp\biggl(-\biggl(\frac{k\pi}{2L}+2it\biggr)^2-4t^2\biggr), \quad (139)$$

which is equivalent to (136).

5.3. The Discrete Cauchy–Riemann Equations

It is easy to obtain discrete relations between the real and imaginary part of an analytic function

$$f(z)=f(x+iy)=u(x,y)+iv(x,y). \quad (140)$$

For example we can first take $y=0$ in (140) and second take $x=0$ and replace y by $-ix$. We obtain

$$u(x,0)+iv(x,0)=u(0,-ix)+iv(0,-ix). \quad (141)$$

Another relation is

$$i\bigl(u(x,ix+y)+u(x,-ix+y)-2u(0,y)\bigr)=v(x,ix+y)-v(x,-ix+y), \quad (142)$$

which can be proved with a chain rule and the traditional Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\,. \quad (143)$$

However, it is less obvious how to obtain the equivalent of (143) in the context of finite differences calculus. The method of α-calculus permits to discover the following simple equation:

$$i\bigl(u(x+\tfrac{1}{2},iy)-u(x-\tfrac{1}{2},iy)\bigr)=v\bigl(x,i(y+\tfrac{1}{2})\bigr)-v\bigl(x,i(y-\tfrac{1}{2})\bigr). \quad (144)$$

The telescoping process applied to (144) gives, for $n = 1, 2, 3, \ldots$,

$$i\big(u(n+\tfrac{1}{2}, iy) - u(\tfrac{1}{2}, iy)\big) = \sum_{k=1}^{n}\Big(v\big(k, i(y+\tfrac{1}{2})\big) - v\big(k, i(y-\tfrac{1}{2})\big)\Big). \tag{145}$$

The foregoing formula has its counterpart in $(-\frac{1}{2})$-space. Using the corresponding Cauchy–Riemann equation [4, p. 362]

$$ib_{-\frac{1}{2}}\big(u(x, iy); x\big) = b_{-\frac{1}{2}}\big(v(x, iy); y\big) \tag{146}$$

and the identity [4, p. 336]

$$b_{-\frac{1}{2}}\big(f(z); z\big) = -2f(z) + 4\sum_{m=1}^{\infty}(-1)^{m-1}f(z+m), \tag{147}$$

we obtain the relation

$$\begin{aligned} &- iu(x, iy) + 2i\sum_{m=1}^{\infty}(-1)^{m-1}u(x+m, iy) \\ &\qquad = -v(x, iy) + 2\sum_{m=1}^{\infty}(-1)^{m-1}v\big(x, i(y+m)\big). \end{aligned} \tag{148}$$

The identity (148) holds under more restrictive conditions than for (145). The function $f(z)$ must be $(-\frac{1}{2})$-analytic and growth conditions ensure the absolute convergences.

6. Concluding Remarks

The following conjecture was made in [5, Section 7]: given $f \in B_\tau$, we have the inequality

$$\big|b_\alpha\big(f(x); x\big)\big| \le \left|\frac{2\sin(\frac{\tau}{2})}{g_\alpha(\frac{\tau}{2})}\right| \max_{-\infty<t<\infty}|f(t)| \tag{149}$$

for $-\infty < x < \infty$, provided that τ is small enough. The inequality (149) is known to be true when $\alpha = \frac{1}{2}$, and when $\alpha \to \infty$ if $\tau < \pi$. It is already an interesting problem to prove or disprove (149) for $\alpha = -\frac{1}{2}$.

Given a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$, of degree $\le n$, the Bernstein's inequality asserts that

$$|P'(z)| \le nM, \tag{150}$$

where $|z| \le 1$ and $M := \max_{|\zeta|=1} |P(\zeta)|$. An inequality of the form

$$\left| b_\alpha\big(P(z); z\big) \right| \le nM \tag{151}$$

is impossible in general; for the polynomial $P(z) = z^3$, we have

$$\max_{|z|=1} |b_\alpha(z^3; z)| = 3 + \frac{(2\alpha - 1)}{8(\alpha + 1)} > 3$$

if $\alpha > \frac{1}{2}$. The same example shows that the inequality

$$\left| b_\alpha\big(P(z); z\big) \right| \le n \max_{|\zeta|=1} |\phi_{n-1,\alpha}(\zeta)| M \tag{152}$$

is also impossible in general. The latter inequality is considered because

$$b_\alpha\big(\phi_{n,\alpha}(\zeta); \zeta\big) = n\phi_{n-1,\alpha}(\zeta)$$

and (150) is often written in the form

$$|P'(z)| \le \max_{|\zeta|=1} \left| b_{\frac{1}{2}}(\zeta^n; \zeta) \right| M.$$

In that context, empirical computations *seem to indicate that*

$$\max_{|z|=1} |\phi_{n,\alpha}(z)| = |\phi_{n,\alpha}(i)| \tag{153}$$

for $n = 0, 1, 2, \ldots$ and $\alpha \ge \frac{1}{2}$.

An interesting generalization of (150) *would be*

$$\left| b_\alpha\big(P(z); z\big) \right| \le \max_{|\zeta|=1} |b_\alpha(\zeta^n; \zeta)| M. \tag{154}$$

That inequality is obviously true for $n = 0, 1, 2$. It is valid for $n = 3$ if $\alpha \ge \frac{1}{2}$ since

$$\begin{aligned} \left| a_1 + 2a_2 z + a_3 \left(\frac{(2\alpha - 1)}{8(\alpha + 1)} + 3z^2 \right) \right| &\le |a_1 + 2a_2 z + 3a_3 z^2| + \frac{(2\alpha - 1)}{8(\alpha + 1)} |a_3| \\ &\le 3M + \frac{(2\alpha - 1)}{8(\alpha + 1)} M = \max_{|\zeta|=1} |b_\alpha(\zeta^3; \zeta)| M. \end{aligned}$$

A similar argument shows that (154) also holds for $n = 4$ if $\alpha \ge \frac{1}{2}$.

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