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STUDY OF TWO DIMENSIONAL MAGNETOTHERMOELASTIC PROBLEMS IN ROTATING MEDIUM BY EIGENFUNCTION EXPANSION METHOD

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Abstract: In this paper the linear theory of the generalized Thermoelasticity has been employed to study the disturbances in an infinite rotating elastic medium containing instantaneous heat source and permeated by primary uniform magnetic field. It is assumed that the medium under consideration is homogeneous, orthotropic, electrically, as well as thermally conducting. The fundamental equations of the two dimensional problem of generalized thermoelasticity with one relaxation parameter including heat source in an infinite rotating medium and under the influence of magnetic field have been deduced in the form of a vector-matrix differential equation in the Laplace-Fourier transform domain and have been solved by eigenvalue approach. The results obtained in the present analysis have been compared to those available in the existing literature. The graphs have been drawn to show the effect of rotation and relaxation.

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1. Introduction

The governing equations for displacement and temperature fields in the linear dynamical theory of classical thermo-elasticity consist of the coupled partial differential equation of motion and the Fourier's law of heat conduction equation. The equation for displacement field is governed by a wave type hyperbolic equation, whereas, that for the temperature field is a parabolic diffusion type equation. This amounts to the remark that the classical thermoelasticity predicts a finite speed for predominantly elastic disturbances but an infinite speed for predominantly thermal disturbances, which are coupled together. This means that a part of every solution of the equations extends to infinity, vide, Lord and Shulman [7]. Experimental investigations by Ackerman et al [26], [1], [2], von Gutfeld and Nethercot [20], Taylor et al [35], Jackson and Walker [22] and many others, conducted on different solids have shown that heat pulses do not propagate with infinite speeds. In order to overcome this paradox, efforts were made to modify classical thermoelasticity, on different grounds, for obtaining a wave type heat conduction equation, vide, Kaliski [23], Norwood and Warren [29], Green and Lindsay [19], Suhubi [34], Lebon [24], Dhaliwal and Singh [17] and many others. A comprehensive list on this generalization up to 1986 can be available in the work of Chandrasekharaiah [9].

It is now well known that there are two generalizations of the classical theory of thermoelasticity. The first proposed by Lord and Shulman [26] (L-S theory) which involves one relaxation time for aa thermo-elastic process, and the second by Green and Lindsay [19] (G-L theory), which takes into account two parameters on relaxation times. Owing to the mathematical complexities encountered in coupled thermo-elasticity, mainly due to inertia and coupling terms in governing equations, these types of problems are mostly confined to one dimensional problems, vide, Noda et al [28], Furukawa et al [18], Anwar et al [5], Chandrasekharaiah et al [11], [10], Roy Choudhuri [30], Mukhopadhyay and Bera [27] and many others. However, an attempt has been made to investigate a two dimensional problem in generalized thermoelasticity under the influence of magnetic field with the help of eigenvalue approach in the present analysis.

The non-uniform temperature distribution in elastic solid due to heat conduction gives rise to many interesting problems in industrial practices, which have been studied within the framework of the dynamic couple theory of thermoelasticity. Owing to the mathematical complexities involved in the theory as mentioned above, closed form solutions of the problems are very hard to achieve. A survey of some of the relevant thermoelastic studies is given by Dhaliwal and Rokne [16], Hetnarski [21], Das and Bhakta [12] and others.

It appears that from the review of available literature little attention has been paid to study the propagation of plane magneto-thermoelastic waves in a rotating medium in two dimensional problems. Since most large bodies like the earth, the moon and other planets have an angular velocity, it appears more realistic to study the propagation of plane thermoelastic or magneto-thermoelastic waves in a rotating medium with relaxation.

However, in the present paper following one parameter L-S theory, the authors have considered a problem of rotation in two dimensions, in generalized theory of magneto-thermo-elasticity by taking into account of both the dynamic effect and the influence of coupling terms. Although this is a two dimensional problem under the influence of instantaneous heat sources distributed over a plane area in an infinite, orthotropic elastic medium, it involves two displacement components due to rotation unlike that in the one dimensional case where the number of displacement components increase due to rotation as has been shown in the work of Sinha and Bera [32]. Baksi et al [7], Baksi and Bera [6] have solved thermoelastic problems in rotating medium in two and three dimensions by eigen function expansion method. The eigenvalue approach proposed by Das et al [13], [15] has been applied for the solution of the problem. One very interesting paper in this line on nonlinear magnetothermoelasticity in two dimensions by Librescu et al [25] may be worthy of mentioning.

The solution of the present problem has been achieved in closed form in Laplace-Fourier transform domain to determine deformations, stresses, temperature and magnetic field. Finally, numerical inversions in space-time domain have been obtained and some of the results have been shown graphically.

2. Formulation of the Problem: Basic Equations

We consider a homogeneous, orthotropic elastic solid, both thermally and electrically conducting, with uniform reference temperature T_0 and permeated by a primary magnetic field in a rotating medium. The medium is rotating uniformly with respect to an inertial frame and the constant rotating vector in an x, y, z rectangular Cartesian frame rotating with the medium is $\vec{\Omega} = \Omega\, \hat{n}$. The unit vector $\hat{n}$ will denote the axis of rotation (according to the right handed rule throughout) and Ω is the magnitude of $\vec{\Omega}$. In the present case the unit vector is in the direction of x-axis. The infinite solid is also subjected to instantaneous point heat source which acts on the line $y = 0$, $z = 0$. The problem is to study the subsequent distribution of temperature, deformation, strain, stress, etc., as well as the interactions between the fields.

Since we are looking for time-varying dynamic solutions, the time-independent part $\vec{\Omega}\times(\vec{\Omega}\times\vec{x})$ of the centripetal acceleration as well as body forces will be neglected. Thus our dynamic displacement $\vec{u}$ is actually measured from a steady-state deformed position, the deformation of which, however, is assumed small. Furthermore, a rotating medium can be thought of a type of transversely isotropic medium as pointed out by Schoenberg [31]. Hence the displacement equations of motion in the rotating frame of reference have two additional terms:

(i) The centripetal acceleration $\vec{\Omega}\times(\vec{\Omega}\times\vec{u})$ due to time-varying motion only;

(ii) The Coriolis acceleration $2\vec{\Omega}\times\dot{\vec{u}}$

The medium is supposed to be initially unstrained and unstressed. Using a fixed rectangular Cartesian co-ordinate system x, y, z, we state the basic field equations of linear electro- magneto- thermoelasticity with thermal relaxation as follows:

(a) The principle of balance of linear momentum in a rotating medium leads to the equations of motion

$$\tau_{ij,j}+(\vec{j}\times\vec{B})_i=\rho[\ddot{u}_i+\{\vec{\Omega}\times(\vec{\Omega}\times\vec{u})\}_i+\{2\vec{\Omega}\times\dot{\vec{u}}\}_i \quad (i,j=1,2,3). \tag{1}$$

(b) The balance of the angular momentum principle implies $\tau_{ij}=\tau_{ji}$, since no body couples and couple stresses are included.

(c) The variation of the magnetic and electric fields (expressed in S.I. units) are given by Maxwell's equations (in absence of the displacement current and the charge density)

$$\operatorname{curl}\vec{H}=\vec{j},\quad \operatorname{curl}\vec{E}=-\frac{\partial\vec{B}}{\partial t},\quad \operatorname{Div}\vec{B}=0,\quad \vec{B}=\mu_e\vec{H}. \tag{2}$$

The modified Ohm's law in a rotating medium is

$$\vec{j}=\sigma[\vec{E}+(\frac{\partial\vec{u}}{\partial t}+\vec{\Omega}\times\vec{u})\times\vec{B}]\,, \tag{3}$$

where $\vec{H}$= the total magnetic field vector $=(H_x,H_y,H_z)$, $\vec{B}$= magnetic inductance vector $=(B_x,B_y,B_z)$, $\vec{E}$=electric field vector=(E_x,E_y,E_z), μ_e= magnetic permeability of the medium, σ=electric conductivity of the medium, ρ= constant mass density, τ_{ij}= component of stress tensor $(i,j=1,2,3)$, $\vec{u}=$ displacement vector $=(u,v,w)$ and $\vec{\Omega}$= rotation vector $=(\Omega_x,\Omega_y,\Omega_z)$. If all the components of stress and displacements etc. are functions of y, z and t and if we take $\vec{H}=(H_x,0,0)$, $\vec{E}=(E_x,E_y,E_z)$, $\vec{u}=(0,v,w)$ and $\vec{\Omega}=(\Omega,0,0)$, we get from equations (2) and (3), after eliminating E_y and E_z and neglecting second order differentiation of H_x,

$$-\mu_e \frac{\partial H_x}{\partial t} = \mu_e H_x \Big(\frac{\partial^2 v}{\partial y \partial t} + \frac{\partial^2 w}{\partial z \partial t} \Big). \tag{4}$$

Equation (4) has been linearized by setting $H_x = H_0 + h_x$, where h_x denotes the change in the basic magnetic field H_0 (called the perturbed field) and then the product terms with h_x have been neglected. After linerarization, equation (4) becomes

$$h_x = -H_0 [\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}]. \tag{5}$$

The thermal stresses in an orthotropic infinite elastic solid subject to plane strain [4] in two dimensions are

$$\begin{aligned}
\tau_{11} &= A_{12} \frac{\partial v}{\partial y} + A_{13} \frac{\partial w}{\partial z} - \beta_1 (1 + \alpha \frac{\partial}{\partial t}) T\,, \\
\tau_{22} &= A_{22} \frac{\partial v}{\partial y} + A_{23} \frac{\partial w}{\partial z} - \beta_2 (1 + \alpha \frac{\partial}{\partial t}) T\,, \\
\tau_{33} &= A_{23} \frac{\partial v}{\partial y} + A_{33} \frac{\partial w}{\partial z} - \beta_3 (1 + \alpha \frac{\partial}{\partial t}) T\,, \\
\tau_{23} &= A_{44} (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\,,
\end{aligned} \tag{6}$$

where A_{ij} are the elastic moduli of the orthotropic material and β_1, β_2 and β_3 are the stress temperature coefficient.

From equations (1), (5) and (6), after neglecting higher order of small quantities, we get

$$\begin{aligned}
&(A_{22} + \mu_e H_0^2) \frac{\partial^2 v}{\partial y^2} + A_{44} \frac{\partial^2 v}{\partial z^2} + (A_{23} + A_{44} + \mu_e H_0^2) \frac{\partial^2 w}{\partial z \partial y} \\
&\qquad = \rho [\frac{\partial^2 v}{\partial t^2} - \Omega^2 v - 2\Omega \dot{w}] + \beta_2 \frac{\partial}{\partial y} (1 + \alpha \frac{\partial}{\partial t}) T\,,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&A_{44} \frac{\partial^2 w}{\partial y^2} + (A_{33} + \mu_e H_0^2) \frac{\partial^2 w}{\partial z^2} + (A_{23} + A_{44} + \mu_e H_0^2) \frac{\partial^2 v}{\partial y \partial z} \\
&\qquad = \rho [\frac{\partial^2 w}{\partial t^2} - \Omega^2 w + 2\Omega \dot{v}] + \beta_3 \frac{\partial}{\partial z} (1 + \alpha \frac{\partial}{\partial t}) T\,.
\end{aligned} \tag{8}$$

The temperature field $T(y,z,t)$ is assumed to satisfy the general heat conduction equation

$$k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} = \rho c_v [\frac{\partial T}{\partial t} + \alpha_0 \frac{\partial^2 T}{\partial t^2}] + T_0 [\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}][\beta_2 \frac{\partial v}{\partial y} + \beta_3 \frac{\partial w}{\partial z}] - (1 + \tau \frac{\partial}{\partial t}) Q(y,z,t) \, . \quad (9)$$

From these general equations (7)-(9), we can classify the problem into three classes below:

(i) If $\alpha = 0$, $\alpha_0 = 0$, and $\tau = 0$, then the problem reduces to the problem of classical thermoelasticity (CTE).

(ii) If $\alpha = 0$, $\alpha_0 = \tau \neq 0$, then the problem reduces to the problem of extended thermoelasticity (ETE).

(iii) If $\alpha \neq \alpha_0 \neq 0$, but $\tau = 0$, then the problem reduces to the problem of temperature rate thermoelasticity (TRDTE). It may be noted that α, α_0 and τ have the same unit "second".

We now assume that the heat sources are instantaneous and act on the line $y = 0$, $z = 0$. Then we can write

$$Q(y,z,t) = q_0 \delta(y) \delta(z) \delta(t), \quad (10)$$

where q_0 is the strength of the heat source and $\delta(t)$ is a Dirac delta function of t.

3. Method of Solution: Formulation of a Vector Matrix Differential Equation

We apply the Laplace-Fourier double integral transform of the form

$$\bar{T}(y,z,p) = \int_0^\infty T(y,z,t) \exp(-pt) dt$$

and

$$\bar{T_1}(\xi,z,p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{T}(y,z,p) \exp(i\xi y) dy$$

(where p and ξ are transform parameters) to the equations (7) to (9) and we obtain

$$[-\xi^2 (A_{22} + \mu_e H_0^2) + \rho(p^2 - \Omega^2)] \bar{v_1} + A_{44} \frac{d^2 \bar{v_1}}{dz^2}$$

$$- i\xi(A_{23} + A_{44} + \mu_e H_0^2)\frac{d\bar{w}_1}{dz} + 2\rho\Omega p\bar{w}_1 + i\xi\acute{\beta}_2\bar{T}_1 = 0\,, \quad (11)$$

$$[-\xi^2 A_{44} + \rho(p^2 - \Omega^2)]\bar{w}_1 + (A_{33} + \mu_e H_0^2)\frac{d^2\bar{w}_1}{dz^2} - i\xi(A_{23} + A_{44} + \mu_e H_0^2)\frac{d\bar{v}_1}{dz} - 2\rho\Omega p\bar{v}_1 - \acute{\beta}_3\frac{d\bar{T}_1}{dz} = 0\,, \quad (12)$$

$$-k_y\xi^2\bar{T}_1 + k_z\frac{d^2\bar{T}_1}{dz^2} = \rho\acute{c}_v p\bar{T}_1 + \acute{T}_0 p[-i\xi\beta_2\bar{v}_1 + \beta_3\frac{d\bar{w}_1}{dz}] - \acute{q}_0\frac{\delta(z)}{\sqrt{2\pi}}, \quad (13)$$

where we have used

$$\int_0^\infty \delta(t)\exp(-pt)dt = 1 \text{ and } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \delta(y)\exp(i\xi y)dy = \frac{1}{\sqrt{2\pi}}\,,$$

and $\acute{c}_v = c_v(1 + \alpha_0 p)$, $\acute{T}_0 = T_0(1 + \tau p)$, $\acute{q}_0 = q_0(1 + \tau p)$, $\acute{\beta}_i = \beta_i(1 + \alpha p)$, i=2,3.

We also assume that at time $t = 0$, the body is at rest in an undeformed and unstressed state and is maintained at reference temperature. Then the following initial conditions hold: $v(y, z, 0) = \frac{\partial v(y,z,0)}{\partial t} = 0$, $w(y, z, 0) = \frac{\partial w(y,z,0)}{\partial t} = 0$, $T(y, z, 0) = \frac{\partial T(y,z,0)}{\partial t} = 0$. Moreover, T, v and w must be bounded at infinity so as to satisfy the regularity condition at infinity. For this we have assumed that T, v and w as well as their derivatives with respect to y vanish at infinity. This assumption has been adopted from the discussions available in the book of Sneddon [33] to have a tractable solution. Equations (11), (12) and (13) can be written in form of a vector matrix differential equation as follows:

$$\frac{d\bar{V}}{dz} = M\bar{V} + f(z), \quad (14)$$

where $\bar{V} = [\bar{v_1}, \bar{w_1}, \bar{T_1}, \acute{\bar{v_1}}, \acute{\bar{w_1}}, \acute{\bar{T_1}}]^T$ and $f(z) = \left(0, 0, 0, 0, 0, -\acute{q}_0\frac{\delta(z)}{\sqrt{2\pi}}\right)$. The matrix M is

$$\mathbf{M} = \begin{pmatrix} \bar{0} & \bar{I} \\ P & Q \end{pmatrix}, \quad (15)$$

where

$$\mathbf{P} = \begin{pmatrix} C_{41} & C_{42} & C_{43} \\ C_{51} & C_{52} & 0 \\ C_{61} & 0 & C_{63} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & C_{45} & 0 \\ C_{54} & 0 & C_{56} \\ 0 & C_{65} & 0 \end{pmatrix},$$

$\bar{0}=$ null matrix and $\bar{I}=$ unit matrix, each of order three and

$$c_{41}=\frac{\xi^2(A_{22}+\mu_e H_0^2)+\rho(p^2-\Omega^2)}{A_{44}},\quad c_{42}=\frac{-2\rho\Omega p}{A_{44}},\quad c_{43}=\frac{-i\xi\acute{\beta_2}}{A_{44}},$$
$$c_{45}=\frac{i\xi(A_{23}+A_{44}+\mu_e H_0^2)}{A_{44}},\quad c_{51}=\frac{2\rho\Omega p}{A_{33}},\quad c_{52}=\frac{\xi^2 A_{44}+\rho(p^2-\Omega^2)}{A_{33}+\mu_e H_0^2},$$
$$c_{54}=\frac{i\xi(A_{23}+A_{44}+\mu_e H_0^2)}{A_{33}+\mu_e H_0^2},\quad c_{56}=\frac{\acute{\beta_3}}{A_{33}+\mu_e H_0^2},\tag{16}$$
$$c_{61}=\frac{-i\xi\beta_2\acute{T_0}p}{k_z},\quad c_{63}=\frac{k_y\xi^2+\rho\acute{c_v}p}{k_z},\quad c_{65}=\frac{\beta_3\acute{T_0}p}{k_z}.$$

4. Solution of the Vector Matrix Differential Equation

The characteristic equation of the matrix M takes the form

$$\begin{aligned}\lambda^6-\lambda^4(c_{41}+c_{52}+c_{63}+c_{45}c_{54}+c_{56}c_{65})-\lambda^3(c_{45}c_{51}+c_{42}c_{54})\\ +\lambda^2(c_{52}c_{63}+c_{41}c_{52}+c_{41}c_{63}-c_{43}c_{61}-c_{42}c_{51}+c_{45}(c_{54}c_{63}-c_{56}c_{61})\\ +c_{65}(c_{41}c_{56}-c_{43}c_{54}))+\lambda(c_{42}(c_{54}c_{63}-c_{56}c_{61})+c_{51}(c_{45}c_{63}-c_{43}c_{65}))\\ +c_{63}(c_{42}c_{51}-c_{41}c_{52})+c_{43}c_{52}c_{61}-c_{41}c_{52}c_{63}=0\,.\end{aligned}\tag{17}$$

The coefficients of λ^3 and λ become zero on simplification. The roots of the characteristic equation (17), which are also the eigenvalues of the matrix M, are of the form $\lambda=\pm\lambda_1,\lambda=\pm\lambda_2,\lambda=\pm\lambda_3$. The right eigenvector $\bar{X}=[X_1,X_2,X_3,X_4,X_5,X_6]^T$ corresponding to eigenvalue λ can be calculated as:

$$\bar{\mathbf{X}}=\begin{bmatrix}\lambda^2(c_{45}c_{56}+c_{43})+\lambda c_{42}c_{56}-c_{43}c_{52}\\ \lambda^3c_{56}+\lambda(c_{43}c_{54}-c_{41}c_{56})+c_{43}c_{51}\\ \lambda^4-\lambda^2(c_{41}+c_{52}+c_{45}c_{54})-\lambda(c_{42}c_{54}+c_{45}c_{51})+c_{41}c_{52}-c_{42}c_{51}\\ \lambda(\lambda^2(c_{45}c_{56}+c_{43})+\lambda c_{42}c_{56}-c_{43}c_{52})\\ \lambda(\lambda^3c_{56}+\lambda(c_{43}c_{54}-c_{41}c_{56})+c_{43}c_{51})\\ \lambda(\lambda^4-\lambda^2(c_{41}+c_{52}+c_{45}c_{54})-\lambda(c_{42}c_{54}+c_{45}c_{51})+c_{41}c_{52}-c_{42}c_{51})\end{bmatrix}.$$

This result completely agrees with that of Das et al [14] for $\Omega=0$ and $\vec{H}=\vec{0}$. Moreover, if $\vec{H}=\vec{0}$ and $\Omega\neq 0$, the results of the present analysis completely agree with those of [7].

We can easily calculate the eigenvector $\bar{X}$ corresponding to the eigenvalue $\lambda = \lambda_i$. For our further reference we shall use the following notations:

$$\begin{aligned}&\bar{X}_1 = [\bar{X}]_{\lambda=\lambda_1}, \bar{X}_2 = [\bar{X}]_{\lambda=-\lambda_1}, \quad \bar{X}_3 = [\bar{X}]_{\lambda=\lambda_2}, \bar{X}_4 = [\bar{X}]_{\lambda=-\lambda_2},\\ &\bar{X}_5 = [\bar{X}]_{\lambda=\lambda_3}, \bar{X}_6 = [\bar{X}]_{\lambda=-\lambda_3}.\end{aligned} \tag{18}$$

The left eigenvector $\bar{Y} = [Y_1, Y_2, Y_3, Y_4, Y_5, Y_6]$ corresponding the eigenvalue λ can be calculated as:

$$\mathbf{\bar{Y}} = \begin{bmatrix} \lambda^3 c_{61} + \lambda^2 c_{51} c_{65} + \lambda[c_{41} c_{54} c_{65} - c_{61}(c_{52} + c_{45} c_{54})] - c_{42} c_{54} c_{61} \\ c_{52}[\lambda^2 c_{65} + c_{54} c_{61} - c_{41} c_{65}] + \lambda c_{42}(c_{61} + c_{54} c_{65}) \\ \lambda^5 - \lambda^3(c_{41} + c_{52} + c_{45} c_{54} + c_{56} c_{65}) + \lambda^2(c_{42} c_{54} + c_{45} c_{51}) \\ +\lambda[c_{41}(c_{52} + c_{56} c_{65}) - c_{42} c_{51} - c_{45} c_{56} c_{61} - c_{42} c_{56} c_{61}] \\ \lambda^2(c_{61} + c_{54} c_{65}) + \lambda c_{51} c_{65} - c_{52} c_{61} \\ \lambda^3 c_{56} + \lambda(c_{45} c_{61} - c_{41} c_{65}) + c_{42} c_{61} \\ \lambda^4 - \lambda^2(c_{41} + c_{52} + c_{45} c_{54} + c_{56} c_{65}) - \lambda(c_{42} c_{54} + c_{45} c_{51}) + c_{41} c_{52} \end{bmatrix}.$$

This agrees with Das et al [14] for Ω=0 and $\vec{H}$=0. For, simplicity, we shall, henceforth, denote them as:

$$\begin{aligned}&\bar{Y}_1 = [\bar{Y}]_{\lambda=\lambda_1}, \bar{Y}_2 = [\bar{Y}]_{\lambda=-\lambda_1}, \quad \bar{Y}_3 = [\bar{Y}]_{\lambda=\lambda_2}, \bar{Y}_4 = [\bar{Y}]_{\lambda=-\lambda_2},\\ &\bar{Y}_5 = [\bar{Y}]_{\lambda=\lambda_3}, \bar{Y}_6 = [\bar{Y}]_{\lambda=-\lambda_3}.\end{aligned} \tag{19}$$

Assuming the regularity condition at infinity, the solution of equation (14) can be written as (see Appendix I):

$$\bar{V}(z,p) = a_2(z)\bar{X}_2 \exp(-\lambda_1 z) + a_4(z)\bar{X}_4 \exp(-\lambda_2 z) + a_6(z)\bar{X}_6 \exp(-\lambda_3 z), \quad z > 0\,, \tag{20}$$

where

$$a_2(z) = -\frac{1}{\bar{Y}_2\bar{X}_2}\int_{-\infty}^{z}[\lambda_1^4 - \lambda_1^2(c_{41} + c_{52} + c_{45}c_{54}) - \lambda_1(c_{42}c_{54} + c_{45}c_{51}) + c_{41}c_{52}]\acute{q}_0 \frac{\delta(s)\exp(-\lambda_1 s)}{\sqrt{2\pi}k_z} ds, \quad z > 0\,. \tag{21}$$

Integrating this expression, we obtain finally

$$a_2(z) = -\frac{1}{\bar{Y}_2\bar{X}_2}[\lambda_1^4 - \lambda_1^2(c_{41} + c_{52} + c_{45}c_{54}) + \lambda_1(c_{42}c_{54} + c_{45}c_{51}) + c_{41}c_{52}]\frac{\acute{q}_0}{\sqrt{2\pi}k_z}, \quad z > 0\,. \tag{22}$$

Similarly, $a_4(z)$ and $a_6(z)$ can be obtained as

$$a_4(z) = -\frac{1}{\bar{Y}_4\bar{X}_4}[\lambda_2^4 - \lambda_2^2(c_{41} + c_{52} + c_{45}c_{54}) + \lambda_2(c_{42}c_{54} + c_{45}c_{51}) + c_{41}c_{52}]\frac{\acute{q_0}}{\sqrt{2\pi}k_z}, \quad z > 0\,, \tag{23}$$

$$a_6(z) = -\frac{1}{\bar{Y}_6\bar{X}_6}[\lambda_3^4 - \lambda_3^2(c_{41} + c_{52} + c_{45}c_{54}) + \lambda_3(c_{42}c_{54} + c_{45}c_{51}) + c_{41}c_{52}]\frac{\acute{q_0}}{\sqrt{2\pi}k_z}, \quad z > 0\,. \tag{24}$$

It may be mentioned that $\bar{Y}_i\bar{X}_i \neq 0$, see Appendix I.

Writing (a_2, a_4, a_6) as (A_1, A_2, A_3) the deformations $\bar{v_1}(\xi, z, p)$, $\bar{w_1}(\xi, z, p)$ and temperature $\bar{T_1}(\xi, z, p)$ in the transformed domain can be compactly written as

$$\bar{v_1}(\xi, z, p) = \sum_{i=1}^{3} A_i[\lambda_i^2(c_{43} + c_{45}c_{56}) - \lambda_i c_{42}c_{56} - c_{43}c_{52}]\exp(-\lambda_i z), \tag{25}$$

$$\bar{w_1}(\xi, z, p) = \sum_{i=1}^{3} A_i[-\lambda_i^3 c_{56} - \lambda_i(c_{43}c_{54} - c_{41}c_{56}) + c_{43}c_{51}]\exp(-\lambda_i z), \tag{26}$$

$$\bar{T_1}(\xi, z, p) = \sum_{i=1}^{3} A_i[\lambda_i^4 - \lambda_i^2(c_{41} + c_{52} + c_{45}c_{54}) + \lambda_i(c_{42}c_{54} + c_{45}c_{51}) + (c_{41}c_{52} - c_{42}c_{51})]\exp(-\lambda_i z). \tag{27}$$

Using equations (6), (5) and (25)-(27), stresses and basic magnetic field in the Laplace-Fourier transform domain can be written as

$$\begin{aligned}(\bar{\tau_{11}})_1 = {} & -i\xi A_{12}\sum_{i=1}^{3} A_i[\lambda_i^2(c_{43} + c_{45}c_{56}) - \lambda_i c_{42}c_{56} - c_{43}c_{52}]\exp(-\lambda_i z) \\ & + A_{13}\sum_{i=1}^{3} A_i[\lambda_i^4 c_{56} + \lambda_i^2(c_{43}c_{54} - c_{41}c_{56}) - \lambda_i c_{43}c_{51}]\exp(-\lambda_i z) \\ & - \acute{\beta_1}\sum_{i=1}^{3} A_i[\lambda_i^4 - \lambda_i^2(c_{41} + c_{52} + c_{45}c_{54}) + \lambda_i(c_{42}c_{54} + c_{45}c_{51}) + (c_{41}c_{52} - c_{42}c_{51})] \\ & \times \exp(-\lambda_i z),\end{aligned} \tag{28}$$

$$(\bar{\tau_{22}})_1 = -i\xi A_{22}\sum_{i=1}^{3} A_i[\lambda_i^2(c_{43}+c_{45}c_{56}) - \lambda_i c_{42}c_{56} - c_{43}c_{52}]\exp(-\lambda_i z)$$
$$+ A_{23}\sum_{i=1}^{3} A_i[\lambda_i^4 c_{56} + \lambda_i^2(c_{43}c_{54} - c_{41}c_{56}) - \lambda_i c_{43}c_{51}]\exp(-\lambda_i z)$$
$$- \acute{\beta_2}\sum_{i=1}^{3} A_i[\lambda_i^4 - \lambda_i^2(c_{41}+c_{52}+c_{45}c_{54}) + \lambda_i(c_{42}c_{54}+c_{45}c_{51})$$
$$+ (c_{41}c_{52} - c_{42}c_{51})]\exp(-\lambda_i z), \quad (29)$$

$$(\bar{\tau_{33}})_1 = -i\xi A_{23}\sum_{i=1}^{3} A_i[\lambda_i^2(c_{43}+c_{45}c_{56}) - \lambda_i c_{42}c_{56} - c_{43}c_{52}]\exp(-\lambda_i z)$$
$$+ A_{33}\sum_{i=1}^{4} A_i[\lambda_i^4 c_{56} + \lambda_i^2(c_{43}c_{54} - c_{41}c_{56}) - \lambda_i c_{43}c_{51}]\exp(-\lambda_i z)$$
$$- \acute{\beta_3}\sum_{i=1}^{3} A_i[\lambda_i^4 - \lambda_i^2(c_{41}+c_{52}+c_{45}c_{54}) + \lambda_i(c_{42}c_{54}+c_{45}c_{51})$$
$$+ (c_{41}c_{52} - c_{42}c_{51})]\exp(-\lambda_i z), \quad (30)$$

$$(\bar{\tau_{23}})_1 = A_{44}\sum_{i=1}^{3} A_i[-\lambda_i^3(c_{43}+c_{45}c_{56}) + \lambda_i^2 c_{42}c_{56} + \lambda_i c_{43}c_{51}]\exp(-\lambda_i z),$$
$$- i\xi A_{44}\sum_{i=1}^{3} A_i[-\lambda_i^3 c_{56} - \lambda_i(c_{43}c_{54} - c_{41}c_{56}) + c_{43}c_{51}]$$
$$\times \exp(-\lambda_i z), \quad (31)$$

$$(\bar{h_x})_1 = -H_0[-i\xi\sum_{i=1}^{3} A_i[\lambda_i^2(c_{43}+c_{45}c_{56}) - \lambda_i c_{42}c_{56} - c_{43}c_{52}]\exp(-\lambda_i z)]$$
$$+ \sum_{i=1}^{3} A_i[\lambda_i^4 c_{56} + \lambda_i^2(c_{43}c_{54} - c_{41}c_{56}) - \lambda_i c_{43}c_{51}]\exp(-\lambda_i z). \quad (32)$$

We now write down from equations (27) to (32), the expressions of the temperature, the stresses and magnetic field from the Laplace-Fourier transform domain to the Laplace transform domain as

$$[\bar{T}, \bar{\tau_{11}}, \bar{\tau_{22}}, \bar{\tau_{33}}, \bar{h_x}](y, z, p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [(\bar{T})_1, (\bar{\tau_{11}})_1, (\bar{\tau_{22}})_1, (\bar{\tau_{33}})_1, (\bar{h_x})_1] \cos(\xi y) d\xi\,, \quad (33)$$

and

$$\bar{\tau_{23}}(y, z, p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (\bar{\tau_{23}})_1 \sin(\xi y) d\xi. \quad (34)$$

5. Numerical Solution

The Laplace-Fourier inversion of the expressions for temperature and stresses in the space-time domain are very complex and we prefer to develop an efficient computer software for the purpose of inversion of these double integral transforms. As such, for the inversion of Laplace transform we follow the method of Bellman [8], vide Appendix II and choose seven values of the time t=t_i, $i = 1(1)7$, at which the stresses and temperature are to be determined, where t_i's are the roots of the shifted Legendre polynomial of degree seven, [see Appendix II]. Simultaneous calculations for the inversion of the Fourier transform were done by evaluating the infinite integrals (33) and (34) numerically by seven-point Gaussian quadrature formula for several prescribed values of y and z. The following data (in SI units) of Cobalt (considered as orthotropic) have been used,vide Dhaliwal and Singh [17].

$$\begin{aligned}
&A_{12} = 1.65 \times 10^{11}\text{Nm}^{-2}, \quad A_{13} = 1.027 \times 10^{11}\text{Nm}^{-2},\\
&A_{22} = 3.071 \times 10^{11}\text{Nm}^{-2}, \quad A_{23} = 1.027 \times 10^{11}\text{Nm}^{-2},\\
&A_{33} = 3.581 \times 10^{11}\text{Nm}^{-2}, \quad A_{44} = 1.510 \times 10^{11}\text{Nm}^{-2},\\
&\beta_1 = 7.04 \times 10^{6}\text{Nm}^{-2}\text{deg}^{-1}, \quad \beta_2 = 7.04 \times 10^{11}\text{Nm}^{-2}\text{deg}^{-1},\\
&\beta_3 = 6.90 \times 10^{11}\text{Nm}^{-2}\text{deg}^{-1}, \quad ky = 69\text{Wm}^{-1}\text{deg}^{-1}, \quad kz = 69\text{Wm}^{-1}\text{deg}^{-1},\\
&\rho = 8.836 \times 10^{3}\text{kgm}^{-3}, \quad c_v = 4.27 \times 10^{2}\text{J(kg)}^{-1}\text{deg}^{-1},\\
&T_0 = 298^0\text{K}, \quad H_0 = 0.38\text{Am}^{-1}, \quad \mu_e = 1\text{Hm}^{-1}.
\end{aligned}$$

We now present our results in the form of graphs (Figure 1 - Figure 6) to compare with the cases CTE, ETE, TRDTE for the stresses when time variable $t = 0.025775, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612,$ and 2.04612 are labelled in the abscissa and for particular values of the space variables

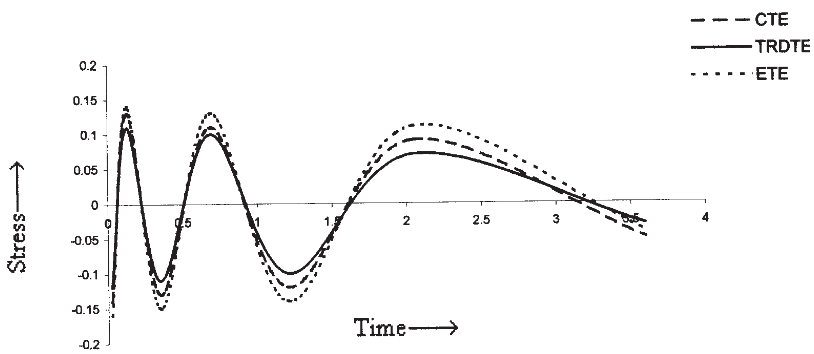

Figure 1: Distribution of normal stress τ_{11} (for $y = 1$, $z = 1$, and $\Omega = 10^5$) versus time

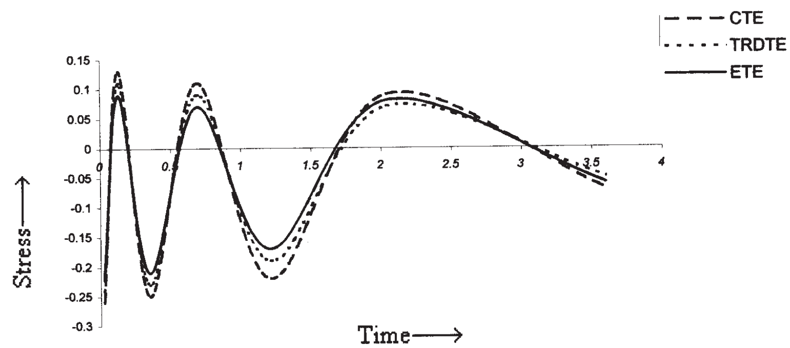

Figure 2: Distribution of normal stress τ_{22} (for $y = 1$, $z = 1$, and $\Omega = 10^5$) versus time

$y = 1$meter and $z = 1$meter. The numerical value of rotation parameter is taken as 10^5radians/second. The material constants α, α_0 and thermal relaxation parameter τ (all expressed in seconds) are taken as follows:

(i) CTE $\alpha = 0$, $\alpha_0 = 0$, $\tau = 0$,

(ii) ETE $\alpha = 0$, $\alpha_0 = 10^{-5}$, $\tau = 10^{-5}$,

(iii) TRDTE $\alpha = 10^{-5}$, $\alpha_0 = 10^{-7}$, $\tau = 0$.

The behaviour of the stresses, etc., for $t \to 0$ can be estimated from the initial value theorem

$$\lim_{t \to 0} \phi(t) = \lim_{p \to \infty} p\bar{\phi}(p)\,.$$

We have computed the stresses and perturbed magnetic field taking $y = z =$ 1meter and $\Omega = 10^5$ radians/second for different values of time t as mentioned

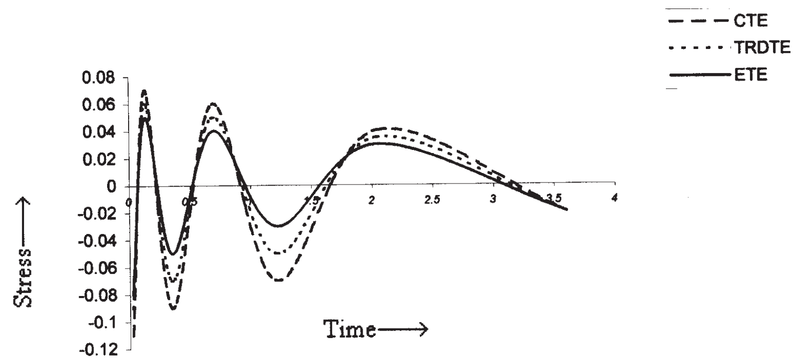

Figure 3: Distribution of normal stress τ_{33} (for $y = 1$, $z = 1$, and $\Omega = 10^5$) versus time

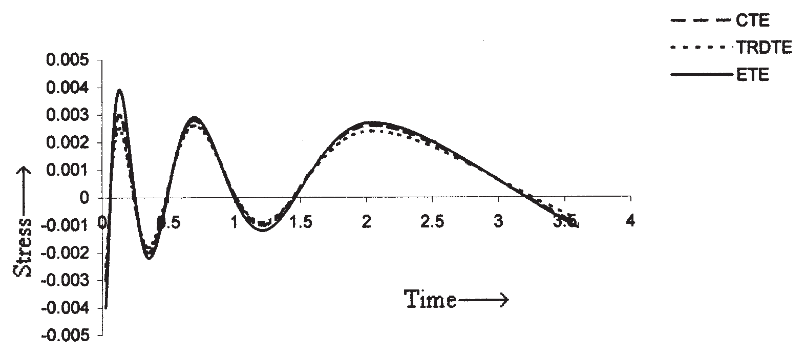

Figure 4: Distribution of normal stress τ_{23} (for $y = 1$, $z = 1$, and $\Omega = 10^5$) versus time

earlier. They are presented graphically with the help of a software developed for cubic spline formalism.

It is observed that:

(i) The characteristics of the stresses $\tau_{11}, \tau_{22}, \tau_{33}$ and τ_{23} (Figs.1-4) for the material under consideration for fixed values of y, z and Ω, are almost the same with respect to wave propagation with respect to time in all the cases of CTE, TRDTE and ETE. The amplitudes for the stresses are higher initially and gradually decrease with time.

(ii) The amplitude of the magnetic field starts from the positive value whereas the amplitude of the stress starts from the negative value. This is a significant point to be noted.

(iii) For fixed values of y, z and Ω, the amplitudes of $\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}$ and

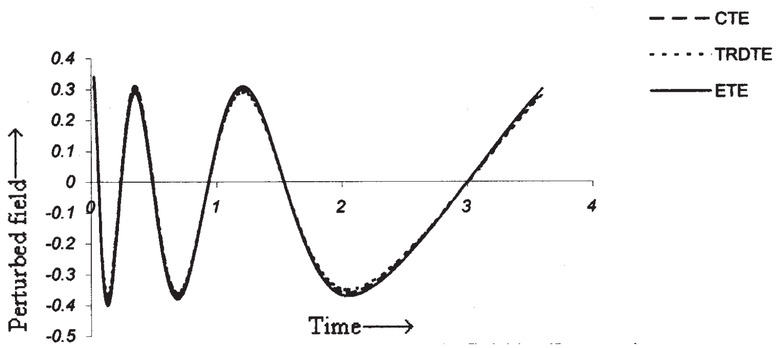

Figure 5: Distribution of normal stress h_x (for $y = 1$, $z = 1$, and $\Omega = 10^5$) versus time

h_x (Figsure 1 - Figure 5) decrease with greater wave-length as t increases in all the above three cases.

(iv) The variation of amplitudes are shown in the Figure 6. The amplitude of τ_{11} is minimum and it proceeds almost along the t-axis. The maximum amplitude is found to be for the stress τ_{22}. The shearing stress τ_{23} has the amplitude in between the stresses τ_{22} and τ_{33}. These are obtained in the case of CTE.

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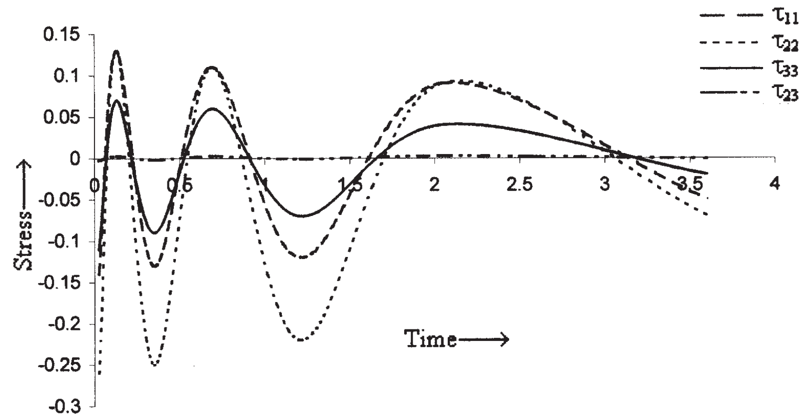

Figure 6: Comparison of stresses (τ_{11}, τ_{22}, τ_{33}, τ_{23}) in CTE versus time

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Appendix I: Solution of the Vector Matrix Differential Equation

Let us consider a vector-matrix differential equation

$$\frac{d\vec{V}}{dx} = M\vec{V} + \vec{f}(x)\,, \tag{35}$$

with the condition

$$\vec{V}(x_0) = \vec{C} \tag{36}$$

where M is an $n \times n$ constant real matrix, $\vec{C}$ is a given constant real n vector and $\vec{f}(x)$ is a real n vector function.

Let

$$\vec{V} = \vec{X}e^{\lambda x} \tag{37}$$

be a solution of the homogeneous equation

$$\frac{d\vec{V}}{dx} = M\vec{V}\,, \tag{38}$$

where λ is a scalar and $\vec{X}$ is an n-vector independent of x. Substituting (37) in (38) we get, $(M\vec{X} - \lambda\vec{X})e^{\lambda x} = \theta \Rightarrow M\vec{X} - \lambda\vec{X} = \theta \Rightarrow M\vec{X} = \lambda\vec{X}$.

This may be interpreted that λ is an eigenvalue of the matrix M and $\vec{X}$ is the corresponding right eigenvector. Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ be n distinct eigenvalues of the matrix M and $\vec{X}_1, \vec{X}_2, \vec{X}_3 \ldots \vec{X}_n$ be the corresponding right eigenvectors of the matrix A. Then the vectors $\vec{X}_1, \vec{X}_2, \vec{X}_3 \ldots \vec{X}_n$ are linearly independent and so they form a basis of the space Γ^n, where Γ denotes the field of complex numbers. We can find scalars $b_1, b_2, b_3, \ldots, b_n$ such that $\vec{C} = b_1\vec{X}_1 + b_2\vec{X}_2 + \cdots + b_n\vec{X}_n$. Let us choose, $c_i = b_i e^{-\lambda_i x_0}$ $(i = 1, 2, \ldots, n)$.

Let

$$\vec{u}(x) = \sum_{i=1}^{n} c_i \vec{X}_i e^{\lambda_i x}. \tag{39}$$

Thus $\vec{u}(x)$ is a solution of the differential equation (38) and

$$\vec{u}(x_0) = \sum_{i=1}^{n} c_i \vec{X}_i e^{\lambda_i x_0}, \quad \vec{X}_i = \sum_{i=1}^{n} b_i \vec{X}_i = \vec{C}.$$

Now, let

$$w(\vec{x}) = \sum_{i=1}^{n} a_i(x) \vec{X}_i e^{\lambda_i x} \tag{40}$$

be a solution of equation (35), where $a_1(x), a_2(x), .., a_n(x)$ are scalar function of x such that $a_i(x_0) = 0$. Differentiating (40) with respect to x we get

$$w(\vec{x}) = \sum_{i=1}^{n} a_i(\acute{x}) \vec{X}_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i \lambda_i \vec{X}_i e^{\lambda_i x}. \tag{41}$$

Substituting (40) and (41) in (35) we have

$$\sum_{i=1}^{n} \acute{a}_i \vec{X}_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x)[M\vec{X}_i - \lambda_i \vec{X}_i]e^{\lambda_i x} + \vec{f}(x) = \vec{f}(x)\,. \tag{42}$$

Multiplying (42) by $\vec{Y_j}e^{-\lambda_j x}$ (where $\vec{Y_1}, \vec{Y_2}, \ldots, \vec{Y_n}$ are left eigenvectors of M corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$) we get

$$\sum_{i=1}^{n} \acute{a}_i \vec{Y}_j \vec{X}_i e^{(\lambda_i - \lambda_j)x} = \vec{Y}_j \vec{f}(x) e^{-\lambda_j x} \quad \text{or} \quad \acute{a}_i \vec{Y}_j \vec{X}_j = \vec{Y}_i \vec{f}(x) e^{-\lambda_j x},$$

respectively

$$\vec{Y}_j \vec{X}_i = 0 \text{ for } i \neq j \acute{a}_j(x) = \frac{1}{\vec{Y}_j \vec{X}_j} \vec{Y}_j \vec{f}(x) e^{-\lambda_j x}$$

or,

$$a_j(x) = \int_{x_0}^{x} (\vec{Y}_j \vec{X}_j)^{-1} \vec{Y}_j \vec{f}(s) e^{-\lambda_j s} ds,$$

$a_j(x_0) = 0$ for $j = 1, 2, \ldots, n$.

Now let us take $\vec{V}(x) = \vec{u}(x) + \vec{w}(x)$.

Differentiating we get

$$\begin{aligned} \acute{\vec{V}}(x) = \acute{\vec{u}}(x) + \acute{\vec{w}}(x) = M\vec{u}(x) + M\vec{w}(x) + \vec{f}(x) = M[\vec{u}(x) + \vec{w}(x)] + \vec{f}(x) \\ = M\vec{V}(x) + \vec{f}(x)\,, \end{aligned}$$

and

$$\vec{V}(x0) = \vec{u}(x_0) + M\vec{w}(x_0) = \vec{C}\,.$$

Hence, $\vec{V}(x) = \vec{u}(x) + \vec{w}(x)$ is the unique solution of the differential equation (35) satisfying the condition (36).

Appendix II: Numerical Inversion of the Laplace Transform

Let the Laplace transform $F(p)$ of $u(t)$ be given by

$$F(p) = \int_0^\infty e^{-pt} u(t) dt\,, \quad p \geq 0\,. \tag{43}$$

We assume that $u(t)$ is sufficiently smooth to permit the approximate method we apply. Putting

$$x = e^{-t} \tag{44}$$

in (43), we get

$$F(p) = \int_0^1 x^{p-1} g(x) dx\,, \tag{45}$$

where $u(-\log x) = g(x)$. Applying the Gaussian quadrature formula in (45) we get

$$\sum_{i=1}^{N} W_i x_i^{p-1} g(x_i) = F(p)\,, \tag{46}$$

where x_i are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and W_i's are the corresponding coefficients. Thus x_i and W_i are known. Equation (46) can be written as

$$\begin{aligned}
&W_1 g(x_1) + W_2 g(x_2) + \cdots + W_N g(x_N) = F(1)\,,\\
&W_1 x_1 g(x_1) + W_2 x_2 g(x_2) + \cdots + W_N x_N g(x_N) = F(2)\,,\\
&\vdots\\
&W_1 x_1^{N-1} g(x_1) + W_2 x_2^{N-1} g(x_2) + \cdots + W_N x_N^{N-1} g(x_N) = F(N)\,.
\end{aligned}$$

Thus

$$\begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix} = \begin{pmatrix} W_1 & W_2 & W_N \\ W_1 x_1 & W_2 x_2 & W_N x_N \\ \vdots & \vdots & \vdots \\ W_1 x_1^{N-1} & W_2 x_2^{N-1} & W_N x_N^{N-1} \end{pmatrix}^{-1} \begin{pmatrix} F(1) \\ F(2) \\ \vdots \\ F(N) \end{pmatrix}$$

Hence, $g(x_1), g(x_2), \ldots, g(x_N)$ are known.

Now $U(-log x_1) = g(x_1)$, $U(-log x_2) = g(x_2), \ldots, U(-log x_N) = g(x_N)$.

For $N = 7$, we have the following results:

Roots of the shifted Legender Polynomial:

x_i	$u(-\log x_i) = g(x_i)$
$x_1 = -0.94910791$	3.671194951
$x_2 = -0.74153119$	2.046127431
$x_3 = -0.40584515$	1.213762484
$x_4 = 0$	0.69314718
$x_5 = 0.40584515$	0.352508528
$x_6 = 0.74153119$	0.138382000
$x_7 = 0.94910791$	0.025775394.