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# PERIODICITY THEOREMS FOR GRADED FRACTAL BUNDLES RELATED TO CLIFFORD STRUCTURES

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**Abstract:** Given generators $A_1^1, A_2^1, ..., A_{2p-1}^1$ of a Clifford algebra $Cl_{2p-1}(\mathbb{C})$, $p = 2, 3, ...$, we consider the sequence

$$\begin{gathered} A_\alpha^{q+1} = \begin{pmatrix} A_\alpha^q & 0 \\ 0 & -A_\alpha^q \end{pmatrix}, \quad \alpha = 1, 2, ..., 2p+2q-3; \\ A_{2p+2q-2}^{q+1} = \begin{pmatrix} 0 & I_{p,q} \\ I_{p,q} & 0 \end{pmatrix}, \quad A_{2p+2q-1}^{q+1} = \begin{pmatrix} 0 & iI_{p,q} \\ -iI_{p,q} & 0 \end{pmatrix} \end{gathered} \tag{1}$$

of generators of Clifford algebras $Cl_{2p+2q-1}(\mathbb{C})$, $q = 1, 2, ...$, and the sequence of the corresponding systems of closed squares $Q_{q+1}^\alpha$ of diameter 1, centred at the origin of $\mathbb{C}$, where $I_{p,q} = I_{2^{p+q-2}}$. Then we decompose each $Q_q^\alpha$ into the corresponding $4^{p+q-2}$ equal squares with sides parallel to the sides of the original square. In the case of

$$A_\alpha^q = \left(a_{\alpha j}^{qk}\right), \quad j, k = 1, 2, ..., 2^{p+q-2}, \tag{2}$$

we include to the object constructed all closed squares corresponding to the

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matrix elements equal $\alpha_{\alpha j}^{qk}$ whenever it is different from zero. We may say that we consider the *bundle* $(\Sigma_\alpha)$ *of* $a_{\alpha j}^{qk}$*-graded fractals*

$$\Sigma_\alpha = \left(Q_q^\alpha\right), \; q = 1, 2, ... \text{ for } \alpha < 2p; \\ q = [\frac{1}{2}\alpha] - p + 1, [\frac{1}{2}\alpha] - p + 2, ... \text{ for } \alpha \geq 2p, \tag{3}$$

where [ ] denotes the function "entier"; we endow them with the functions

$$g_q^\alpha \left(a_{\alpha j}^{qk}; z\right) = a_{\alpha j}^{qk} \text{ if } g_q^\alpha (z) = a_{\alpha j}^{qk}; \\ g_q^\alpha \left(a_{\alpha j}^{qk}; z\right) = 0 \text{ if } g_q^\alpha (z) \neq a_{\alpha j}^{qk}, \tag{4}$$

where $g_q^\alpha$ is the *gradating function*: $g_q^\alpha (z) = a_{\alpha j}^{qk}$ inside the square corresponding to the pair $(j, k)$. By (3), for $\alpha \geq 2p$ each $Q_q^\alpha$ is decomposed into $4^{p+q-1}$ equal squares. We obtain *periodicity theorems* for the sequences of the gradating functions $g_q^\alpha(z)$. They play a crucial role in the further theory and applications to dynamical systems on infinite-dimensional Clifford algebras, analysis of a complex variable (value distribution theory, cluster sets, prime ends, Picard's Theorems), and physical systems, especially alloys (binary, ternary, etc).

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## 1. Introduction

Given generators $A_1^1, A_2^1, ..., A_{2p-1}^1$ of a Clifford algebra $Cl_{2p-1}(\mathbb{C}), p = 2, 3, ...,$ consider the sequence of generators (1) of Clifford algebras $Cl_{2p+2q-1}(\mathbb{C})$, $q = 1, 2, ....$ Replace each of the matrices in question by a closed square $Q_q^\alpha$ of diameter 1, centred at the origin of $\mathbb{C}$. Then divide it into $4^{p+q-2}$ equal squares $Q_{qk}^{\alpha j}$ with sides parallel to the sides of the original square. In the case of (2), include to the object constructed all closed squares corresponding to the matrix elements equal $a_{\alpha j}^{qk}$ whenever it is different from zero. We may say that we consider a *graded* or coloured *fractal* (3) with the *gradating function* $g_q^\alpha$ equal $a_{\alpha j}^{qk}$ inside the square corresponding to the pair $(j, k)$. In (3) the symbol [ ] denotes the function "entier". We have to remember that, by (3), for $\alpha \geq 2p$ each $Q_q^\alpha$ is decomposed into $4^{p+q-1}$ equal squares. Let $\mathbf{a}_{\alpha j}^{qk}$ stand for the totality of matrix elements $a_{\alpha\iota}^{qk}$, $\iota$, $\kappa = 1, 2, ..., 2^{p+q-2}$, equal $a_{\alpha j}^{qk}$. Then our graded fractal

$\Sigma_\alpha$ is composed from the totality of usual fractals [7, 8]:

$$\Sigma_\alpha\left(\mathbf{a}_{\alpha j}^{qk}\right), \quad j,k=1,2,...,2^{p+q-2} \tag{1.1}$$

endowed with the functions (4).

For $q=\left[\frac{1}{2}\alpha\right]-p+2,\ \left[\frac{1}{2}\alpha\right]-p+3,...$, when composing the fractals (1.1) in order to construct the graded fractal $\Sigma_\alpha$, we have a problem with the points $z$, where the set $\left\{g_q^\alpha\left(a_{\alpha j}^{qk}\right)\right\}$ does not reduce to one point. We say that such a point is an *N-point* of our graded fractal. In order to make this notion precise, we formulate.

**Definition 1.** Consider the ordered pair of four sets contained in a neighbourhood $U$ of $z: U\subset Q_q^\alpha$:

$U\cap L^-$, where $L^-$ is the half-line $\zeta=z+(1-j)\tau,\ -\infty<\tau\le 0$;

$U\cap L^+$, where $L^+$ is the half-line $\zeta=z+(1-i)\tau,\ 0\le\tau<+\infty$;

$U\cap U^-\cup\{z\}$, where $U^-$ is the open half-plane with boundary $L^-\cup L^+$, containing $z_-=z-1-i$;

$U\cap U^+\cup\{z\}$, where $U^+$ is the open half-plane with boundary $L^-\cup L^+$, containing $z_+=z+1+i$ and of four constant functions:

$g_1(\xi)=s$ for $\zeta\in U\cap L^-$, $\quad g_3(\xi)=u$ for $\zeta\in U\cap U^-\cup\{z\}$,

$g_2(\xi)=t$ for $\zeta\in U\cap L^+$, $\quad g_4(\xi)=v$ for $\zeta\in U\cap U^+\cup\{z\}$.

If the set $\{s,t,u,v\}$ contains at least two distinct elements, the equivalence class of all such ordered pairs when $z$ ranges over $Q_q^\alpha$ will be called a *petal* (Figure 1), denoted by

$$\begin{pmatrix} s & & v \\ & \cdot & \\ u & & t \end{pmatrix};\ \text{in particular,}\ N(u\ v):=\begin{pmatrix} 0 & & v \\ & \cdot & \\ u & & 0 \end{pmatrix},$$

$$N\begin{pmatrix} s \\ t \end{pmatrix}:=\begin{pmatrix} s & & 0 \\ & \cdot & \\ 0 & & t \end{pmatrix},\ \begin{pmatrix} 0 & & 0 \\ & \cdot & \\ 0 & & 0 \end{pmatrix}=N(0\ 0)=N\begin{pmatrix} 0 \\ 0 \end{pmatrix}=0.$$

The definition can be formally generalized by replacing $Q_q^\alpha$ by an arbitrary connected and closed domain $D\subset\mathbb{C}$.

When letting $q\to\infty$, for $\alpha=1,2,...2p-1$, any $z$ on the diameter $L_\infty$:

$$\left[\frac{1}{2\sqrt{2}}(-1+i);\quad \frac{1}{2\sqrt{2}}(1-i)\right], \tag{1.2}$$

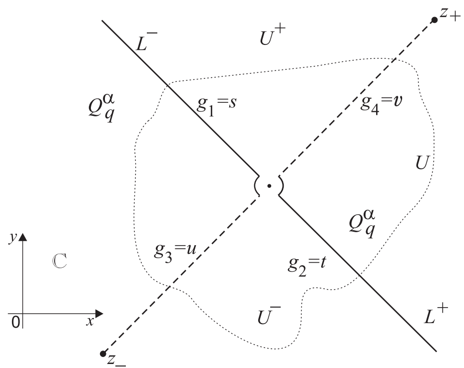


Figure 1: A petal

and any $\iota, \kappa \in \{0, 1, ..., 2^{p-1}\}$, we can choose subsequences $q_{\iota\ell}^{1\kappa}$, $q_{\iota\ell}^{2\kappa} \to \infty$, $\ell = 1, 2, ...$, such that

$$g_q^\alpha(z) = \begin{pmatrix} 0 & & 0 \\ & \cdot & \\ 0 & & 0 \end{pmatrix}, \quad \text{where } q = q_{\iota\ell}^{1\kappa},$$

$$\text{and/or } g_q^\alpha(z) = \begin{pmatrix} \eta a_{\alpha,\iota}^{\kappa} & & \eta a_{\alpha,\iota}^{\kappa+1} \\ & \cdot & \\ \eta a_{\alpha,\iota+1}^{\kappa} & & \eta a_{\alpha,\iota+1}^{\kappa+1} \end{pmatrix}, \quad \text{where } q = q_{\iota\ell}^{2k}, \qquad (1.3)$$

$\eta = 1$ or/and $-1$,

$$a_{\alpha\iota}^{\kappa} := a_{\alpha\iota}^{1\kappa},\ \left(a_{\alpha\iota}^{\kappa}; \iota, \kappa = 1, 2, ..., 2^{p-2}\right) = A_\alpha := A_\alpha^1,\ \alpha = 1, 2, ..., 2p-1.$$

and we have set $a_{\alpha\iota}^{\kappa} = 0$ for $\iota, \kappa = 0, 2^{p-1}, \iota \neq \kappa$. We are thus naturally lead to the following definition.

**Definition 2.** If, for each $q = 1, 2, ...$, a point $z,\ z \in L_\infty$, is not inside a square $Q_{qk}^{\alpha j}$ corresponding to $a_{\alpha j}^{qk} = 0$, that point is called a *Peano flower point*, their collection – a *Peano flower*; the whole object is called a fractal of Peano flower type [40, 9]; we denote it by $\Sigma_\alpha, \alpha = 1, 2, ..., 2p-1$.

Particularly interesting cases appear when $a_{\alpha\iota}^{\kappa} = 0$ for $\iota \neq \kappa$; $\iota$, $\kappa = 1, 2, ..., 2^{p+q-2}$, or $a_{\alpha\iota} = 0$ for $\iota = 1, 2, ..., 2^{p+q-2}$. Correspondingly, we introduce (cf. Figure 2 and Figure 3):

**Definition 3.** A fractal $\Sigma_\alpha$ *has Ola's flower at a point* $z \in L_\infty$ whenever (1.3) reduces to

$$g_q^\alpha(z) = N(0\ 0), \quad \text{where} \quad q = q_{\iota\ell}^{1\kappa},$$

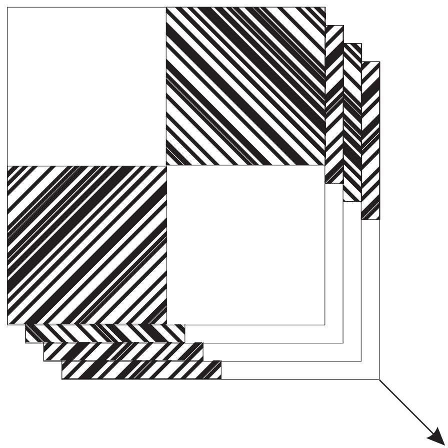

Figure 2: Ola's flower

$$\text{and/or } g_q^{\alpha}(z) = N\left(\eta a_{\alpha,\iota+1}^{\kappa}\ \eta a_{\alpha,\iota}^{\kappa+1}\right), \quad \text{where} \quad q = q_{\iota\ell}^{2\kappa}. \tag{1.4}$$

**Definition 4.** A fractal $\Sigma_\alpha$ *has Yaeko's flower at a point* $z \in L_\infty$ whenever (1.3) reduces to

$$g_q^{\alpha}(z) = N\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad q = q_{\iota\ell}^{1\kappa},$$
$$\text{and/or } g_q^{\alpha}(z) = N\begin{pmatrix} \eta a_{\alpha,\iota}^{\kappa} \\ \eta a_{\alpha,\iota+1}^{\kappa+1} \end{pmatrix}, \quad \text{where} \quad q = q_{\iota\ell}^{2\kappa}. \tag{1.5}$$

**Example 1.** If $(A_1^1, A_2^1, ..., A_{2p-1}^1) = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices), then $\Sigma_\alpha$, $\alpha = 1, 2$, have Ola's flowers at

$$z = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i), \ m = 0, \pm 1, ..., \pm\left(2^n - 1\right); \ n = 0, 1, ...; \tag{1.6}$$

the relations (1.4) reduce to

$$g_{q_\ell^1}^1(z) = N(0\ 0) \quad \text{and} \quad g_{q_\ell^2}^2(z) = N(\eta\ \eta),$$

and

$$g_{q_\ell^1}^2(z) = N(0\ 0) \quad \text{and} \quad g_{q_\ell^2}^2(z) = N(\eta i\ \ -\eta i),$$

respectively.

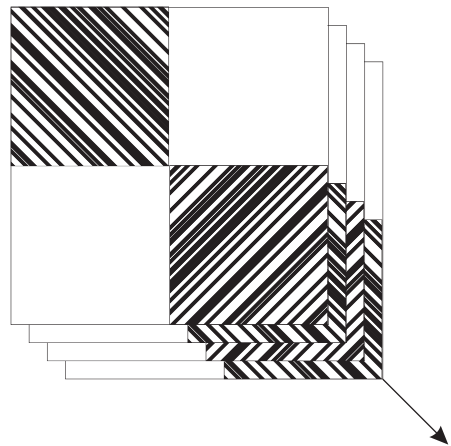

Figure 3: Yaeko's flower

**Example 2.** If $(A^1_1, A^1_2, ..., A^1_{2p-1}) = (\sigma_1, \sigma_2, \sigma_3)$, then $\Sigma_3$ has Yaeko's flowers at points $z$ of the form (1.6); the relations (1.5) reduce to

$$g^3_q(z) = N \begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \text{ where } q = q^{2\kappa}_{\iota\ell},$$

and

$$g^3_q(z) = N \begin{pmatrix} -\eta \\ -\eta \end{pmatrix}, \text{ where } q = q^{2,\kappa-1}_{\iota-1,\ell}.$$

At $z \in L_\infty$, but not of the form (1.6), $\Sigma_\alpha$ has Peano flower points $N \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ or $N \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; when restricting ourselves to $L_\infty$, we identify them with $-1$ and 1, respectively.

## 2. The Graded Clifford-Type Fractal Bundle

Passing to $A^q_{2r}$ with a fixed $r \in \{p, p+1, ...\}$, $q = r, r+1, ...$, we define the gradating function $g^{2r}_q$ with correspondingly attained values $1, -1, 0$, and the related functions

$$g^{2r,1}_q(z) = 1 \text{ if } g^{2r}_q(z) = 1;\; g^{2r,1}_q(z) = 0 \text{ if } g^{2r}_q(z) \neq 1;$$

$$g_q^{2r,2}(z) = -1 \text{ if } g_q^{2r}(z) = -1;\ g_q^{2r,2}(z) = 0 \text{ if } g_q^{2r}(z) \neq -1.$$

For the sake of convenience, let us shift $q$ according to (3) so that $q = 1, 2, \ldots$. When composing the both corresponding fractals, we have to take into account four kinds of N-points:

$$N\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ 1 \end{pmatrix};\ N\begin{pmatrix} -1 \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

In contrast to the previous cases, we have to consider two line segments, symmetric with respect to the diameter (1.2):

$$\left[\frac{1}{2\sqrt{2}}\left(-1+i-i\varepsilon_q^r\right);\ \frac{1}{2\sqrt{2}}\left(1-\varepsilon_q^r-i\right)\right] \tag{2.1}$$

and

$$\left[\frac{1}{2\sqrt{2}}\left(-1+\varepsilon_q^r+i\right);\ \frac{1}{2\sqrt{2}}\left(1-i+i\varepsilon_q^r\right)\right], \tag{2.2}$$

where $\varepsilon_q^r = 1/2^{q-1-r+p}$; more precisely: the union $L_h^1$, $h = p+q-1-r$, of subsegments of (2.1) with

$$\frac{1}{2\sqrt{2}}\left(-1+2\ell\varepsilon_q^r\right) \leq \text{re } z \leq \frac{1}{2\sqrt{2}}\left(-1+2\ell\varepsilon_q^r+\varepsilon_q^r\right),$$
$$\ell = 0, 1, \ldots, 2^{p+q-1-r}-1, \tag{2.3}$$

and the union $L_h^2$ of subsegments of (2.2) with

$$\frac{1}{2\sqrt{2}}\left(-1+2\ell\varepsilon_q^r+\varepsilon_q^r\right) \leq \text{re } z \leq \frac{1}{2\sqrt{2}}\left(-1+2\ell\varepsilon_q^r+2\varepsilon_q^r\right),$$
$$\ell = 0, 1, \ldots, 2^{p+q-1-r}-1. \tag{2.4}$$

When letting $q \to \infty$, for any $z \in L_h^1 \cup L_h^2$ we can choose $q_j^1, q_j^2 \to \infty$ so that

$$g_{q_j^1}^{2r} = N\begin{pmatrix} \eta \\ 0 \end{pmatrix},\ g_{q_j^2}^{2r}(z) = 0, \text{ or } g_{q_j^1}^{2r} = N\begin{pmatrix} 0 \\ -\eta \end{pmatrix},\ g_{q_j^2}^{2r}(z) = 0. \tag{2.5}$$

The situations (2.5) appear when

$$z = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) - \frac{1}{2\sqrt{2}}\varepsilon_q^r \text{ or } z = \frac{1}{2\sqrt{2}}\frac{m}{2^n}(1-i) + \frac{i}{2\sqrt{2}}\varepsilon_q^r,$$
$$m = 0, \pm 1, \ldots, \pm\left(2^n-1\right); n = 0, 1, \ldots \tag{2.6}$$

We are thus led to the following definition.

**Definition 2A.** The notions of Peano flower point and Peano flower as in Definition 2 are naturally extended to the case in question; the whole object is still called a fractal of Peano flower type and denoted by $\Sigma_\alpha$, $\alpha = 2r, r \in \{p, p+1, ...\}$.

Passing in turn to $A^q_{2r+1}$ with a fixed $r \in \{p, p+1, ...\}$, $q = r, r+1, ...,$ we define the gradating function $g^{2r+1}_q$ with correspondingly attained values $i, -i, 0$, and the related functions

$$g^{2r+1,1}_q(z) = i \text{ if } g^{2r+1}_q(z) = i, \quad g^{2r+1,1}_q(z) = 0 \text{ if } g^{2r+1}_q(z) \neq i;$$

$$g^{2r+1,2}_q(z) = -i \text{ if } g^{2r+1}_q(z) = -i, \quad g^{2r+1,2}_q(z) = 0 \text{ if } g^{2r+1}_q(z) \neq -i.$$

For the sake of convenience, let us shift $q$ according to (3) so that $q = 1, 2, \ldots$. When composing the both corresponding fractals, we have again to take into account four kinds of $N$-points:

$$N\begin{pmatrix} i \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ -i \end{pmatrix};\quad N\begin{pmatrix} -i \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ i \end{pmatrix}.$$

We have to consider the same collection of segments: $L^1_h$ and $L^2_h$ defined by (2.1), (2.3) and (2.2), (2.4), respectively. When letting $q \to \infty$, for any $z \in L^1_h \cup L^2_h$, we can choose subsequences $q^1_j, q^2_j \to \infty$ so that

$$g^{2r}_{q^1_j}(z) = N\begin{pmatrix} -\eta i \\ 0 \end{pmatrix},\ g^{2r}_{q^2_j}(z) = 0, \text{ or } g^{2r}_{q^2_j}(z) = N\begin{pmatrix} 0 \\ \eta i \end{pmatrix},\ g^{2r}_{q^2_j}(z) = 0. \quad (2.7)$$

Again, the situations (2.7) appear when $z$ is of the form (2.6). We are thus led to the following definition.

**Definition 2B.** The notions of Peano flower point and Peano flower as in Definition 2 are naturally extended to the case in question; the whole object is still called a fractal of Peano flower type and denoted by $\Sigma_\alpha$, $\alpha = 2r+1,\ r \in \{p, p+1, ..., \}$.

Since the graded fractals $\Sigma_\alpha$ rely upon the Clifford algebra generators $A^1_\alpha, \alpha = 1, 2, ..., 2p-1$, we admit the following definition.

**Definition 5.** The graded fractals $\Sigma_\alpha = \Sigma_\alpha(A_\alpha)$, $\alpha = 1, 2, ..., 2p-1$, will be called (*graded*) Clifford-type fractals; in particular, the fractals $\Sigma_\alpha = \Sigma_\alpha(\sigma_\alpha)$, $\alpha = 1, 2, 3$, will be called (graded) Pauli-type fractals. The corresponding sequence $(\Sigma_\alpha,\ \alpha = 1, 2, ...)$ will be called the (graded) Clifford-type fractal bundle, respectively: the (graded) *Pauli-type fractal bundle.*

Summing up, we have proved the following result.

**Lemma 1.** *The graded Clifford-type fractal bundle* $(\Sigma_\alpha)$ *is well defined.*

We have presented here a direct, constructive proof of Lemma 1. Alternatively, we can consider a Cuntz algebra $\mathcal{O}(4)$ [6] and two representations $\pi_1$ and $\pi_2$ of it on the fractal bundle in question, not necessarily the bundle $(\Sigma_\alpha)$; denote this bundle by $(\Sigma'_r)$. Given four isometries $S_1, ..., S_4$ on an infinite-dimensional Hilbert space, the Cuntz algebra $\mathcal{O}(4)$ is the $C^*$-algebra generated by $S_1, ..., S_4$ with the property that $S_1S_1^* + ... + S_4S_4^* = 1$. In our case we have to consider the situation

$$\pi_j(\sigma_1) = \pi_j\left(S_1S_2^* + S_2^*S_1\right),$$
$$\pi_j(\sigma_2) = i\pi_j\left(S_1S_2^* - S_2S_1^*\right), \quad \pi_j(\sigma_3) = i\pi_j\left(S_1S_1^* - S_2S_2^*\right).$$

Kakutani's Dichotomy Theorem [10] says that *these representations are equivalent if and only if their Hausdorff measures are equivalent.* If this is the case, our object $(\Sigma'_r)$ is well defined. The procedure, in a more general setting, is developed in [30].

Depending on the set $U \cap L^-$, $U \cap L^+$, $U \cap U^- \cup \{z\}$, or $U \cap U^+ \cup \{z\}$ (cf. Definition 1), when tending with $\zeta$ to $z$, the corresponding functions $g_\lambda$, $\lambda = 1, ..., 4$, will attain at $\zeta$ the values $s, t, u$, or $v$. The situation reminds the much developed case of *cluster sets* and *prime ends* [37, 1–5, 11, 12, 34, 35, 41, 42]. The notion of prime end is connected with the *accessibilty* of the point $z$ (e.g. [5], pp. 168–169). In our case, the point $z$ is accesible for $g_1, g_2, g_3, g_4$ along $L^-$ or $L^+$ and within $U^-$ or $U^+$, respectively (Figure 1 and [37]).

Graded fractal bundles seem also to play a crucial role in possible applications to value distribution theory, Picard's Theorems, universal coverings of compact Riemann surfaces, dynamical systems on infinite-dimensional Clifford algebras, and physical systems. As far as applications in physics are discussed, the generators (1) for $p = 2$ appear to be suitable to describe melting and roughness via a chosen particle considered in the heat bath of neighbouring particles, for $p = 4$ – to describe excitons and binary alloys, and for $p = 6$ – to describe ternary alloys [13–28, 33]. We continue these topics in separate papers [29, 31, 32, 36].

## 3. The Flower Structure

We are going to study more closely the convergence problems connected with the fractal bundle $(\Sigma_\alpha)$. Until now we were concerned with the limits when $q \to \infty$. Yet, our staff involves the limits $g_q^\alpha(\xi)$ when $\xi \to z \in L_\infty$. As we can now see the notation $L_\infty$ for (1.2) is motivated by the fact that $L_h^1 \cup L_h^2 \to \infty$ as $h \to \infty$. Denote the sets (1.6) and (2.6) by $L_\infty^0$ and $L_h^0$, respectively. It is clear that $L_h^0 \to L_\infty^0$ as $h \to \infty$. Hence our staff involves, in particular, the limits of $\left(g_q^\alpha(z)\right)_p$ when $q \to \infty$, i.e., of

$$\begin{aligned} \left(g^\alpha_{[\alpha/2]-p+1}(z),\ g^\alpha_{[\alpha/2]-p+2}(z), ...\right) \ &= \ \left(g^{2r}_{r-p+1}(z),\ g^{2r}_{r-p+2}(z), ...\right), \\ &\text{or} \ \left(g^{2r+1}_{r-p+1}(z),\ g^{2r+1}_{r-p+2}(z), ...\right). \end{aligned}$$

Consider a fractal $\Sigma_r' = \Sigma_r'(A_r')$ of Peano flower type related, as before, to a $2^{p+r-1}$-matrix $A_r' = \left(a'^k_{rj}\right)$:

$$A_r^1 = A_r',\ A_r^{q+1} = \begin{pmatrix} A_r^q & 0 \\ 0 & -A_r^q \end{pmatrix}, \tag{3.1}$$

not necessarily

$$A_r' = \begin{pmatrix} 0 & I_{p,r} \\ I_{p,r} & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & I_{p,r} \\ -iI_{p,r} & 0 \end{pmatrix}. \tag{3.2}$$

Clearly, the definitions of a petal, $N$-point, Peano flower point, Ola's flower, and Yaeko's flower remain unchanged.

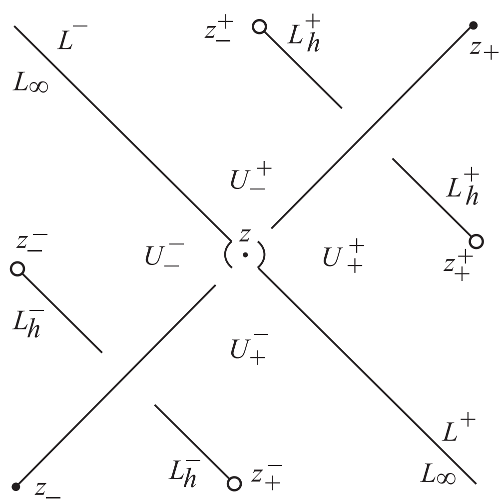


Figure 4: Convergence regions for constructing sepals

Suppose that all the Peano flower points are situated on $L_h^1 \cup L_h^2$. Decompose this set as $L_h^- \cup L_h^+$, where $L_h^- \subset U^-$, $L_h^+ \subset U^+$, $U^-$ and $U^+$ being related to $z \in L_\infty$ (Figure 4). Denote by:
$U_-^-$ – the part of $U^-$ lying between $L^-(z)$ and the segment $(z_-; z)$; $z \in L_\infty$;
$U_+^-$ – the part of $U^-$ lying between the segment $(z_-; z)$ and $L^+(z)$; $z \in L_\infty$;
$U_+^+$ – the part of $U^+$ lying between $L^+(z)$ and the segment $(z; z_+)$; $z \in L_\infty$;
$U_-^+$ – the part of $U^+$ lying between the segment $(z; z_+)$ and $L^-(z)$; $z \in L_\infty$.

Suppose that for a point $Z \in L_\infty^0$ the fractal $\Sigma_r'$ in question has Yaeko's flower at one of the points

$$z_-^- \in L_r^- \cap U_-^-, \quad z_+^- \in L_r^- \cap U_+^-, \quad z_+^+ \in L_r^+ \cap U_+^+, \quad z_-^+ \in L_r^+ \cap U_-^+. \tag{3.3}$$

Then, clearly, $\Sigma_r'$ has Yaeko's flower at the remaining points (3.3) and the condition for $g_q^\alpha(z)$, $q = q_{\iota\ell}^{2\kappa}$ in (1.5) can be either written in the form

$$N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa-1}_{r,\iota-1} \\ \eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa-1}_{r,\iota-1} \\ -\eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa}_{r,\iota} \\ -\eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},$$

respectively, or

$$N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa-1}_{r,\iota+1} \\ -\eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa-1}_{r,\iota-1} \\ -\eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix}, \tag{3.4}$$

or

$$N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa-1}_{r,\iota-1} \\ \eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa-1}_{r,\iota+1} \\ \eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},$$

or

$$N\begin{pmatrix} \eta a'^{\kappa}_{r,\iota} \\ \eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa-1}_{r,\iota-1} \\ -\eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} \eta a'^{\kappa-1}_{r,\iota-1} \\ \eta a'^{\kappa}_{r,\iota} \end{pmatrix},\ N\begin{pmatrix} -\eta a'^{\kappa}_{r,\iota} \\ -\eta a'^{\kappa+1}_{r,\iota+1} \end{pmatrix}. \tag{3.5}$$

Take into account the cases (3.4) and (3.5), specify the points $z_-^-$, $z_+^+$ to make the segment $(z_-^-;\ z_+^+)$ parallel to the $x$-axis, $z_+^-$, $z_-^+$ parallel to the $y$-axis, and take in (3.4), (3.5) the corresponding indices $\iota = \iota_-^-, \iota_+^-, \iota_+^+, \iota_-^+$ and $\kappa = \kappa_-^-, \kappa_+^-, \kappa_+^+, \kappa_-^+$. Then, by (3.1), in all cases in question $a'^{\kappa+1}_{r,\iota+1} = 0$ and $a'^{\kappa-1}_{r,\iota-1} = 0$. Moreover it appears that the case (3.4) corresponds to $A_r'$ symmetric, whereas (3.5) corresponds to $A_r'$ antisymmetric.

**Definition 6.** A fractal $\Sigma_r' = \Sigma_r'(A_r')$ *has Maria's flower at a point* $z \in L_\infty^0$ whenever it has Yaeko's flower at one of the points (3.3) and the matrix $A_r'$ is symmetric (Figure 5).

**Definition 7.** A fractal $\Sigma'_r = \Sigma'_r(A'_r)$ *has Sanae's flower at a point* $z \in L^0_\infty$ whenever it has Yaeko's flower at one of the points (3.3) and the matrix $A'_r$ is antisymmetric (Figure 6).

Coming back to the fractals $\Sigma_\alpha$ for $\alpha = 2r,\ 2r+1;\ r \in \{p, p+1, ...\}$, we can see that any of them has Yaeko's flower at each point (3.3). Since, by (1), $A_{2r}^{r-p+1}$ is symmetric whereas $A_{2r+1}^{r-p+1}$ is antisymmetric, we arrive at

**Lemma 2.** *Each fractal* $\Sigma_\alpha,\ \alpha = 2r,\ r \in \{p, p+1, ...\}$, *i.e., the fractal*

$$\Sigma'_r\left(A'_r\right) = \Sigma_{2r}\left(A_{2r}^{r-p+1},\ A_{2r}^{r-p+2}, ...\right), \quad 2r = \alpha, \tag{3.6}$$

*has Maria's flower at any point* $z \in L^0_\infty$. *Analogously, each fractal* $\Sigma_\alpha,\ \alpha = 2r+1,\ r \in \{p, p+1, ...\}$, *i.e., the fractal*

$$\Sigma'_r\left(A'_r\right) = \Sigma_{2r+1}\left(A_{2r+1}^{r-p+1},\ A_{2r+1}^{r-p+2}, ...\right), \quad 2r+1 = \alpha, \tag{3.7}$$

*has Sanae's flower at any point* $z \in L^0_\infty$.

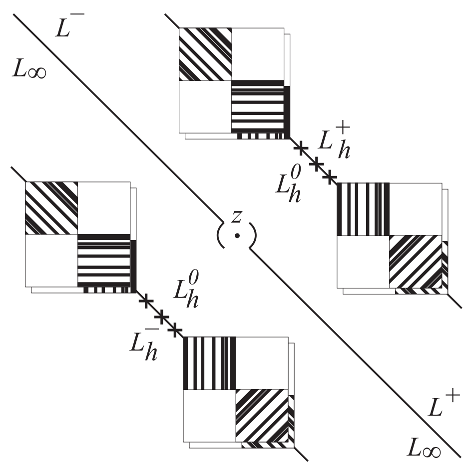


Figure 5: Maria's flower

The limit processes $q \to \infty$ and $r \to \infty$ ($\alpha = 2r$, $\left[\frac{1}{2}\alpha\right] = r \to \infty$; $\alpha = 2r+1$, $\left[\frac{1}{2}\alpha\right] = r \to \infty$), as far as applied to petals, lead to new kinds of them which, in general, differ from those already appearing in the flowers in question. Continuing our flower terminology, we come to the concept of sepals of $\Sigma'_r$.

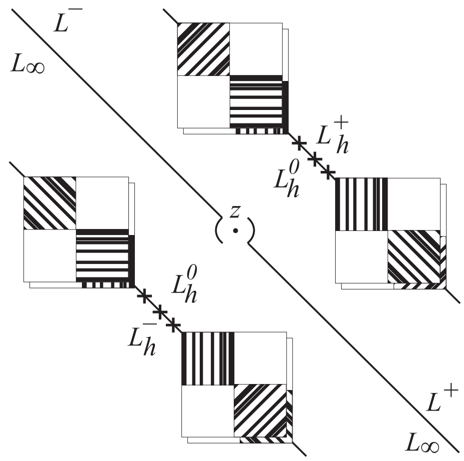


Figure 6: Sanae's flower

**Definition 8.** A fractal $\Sigma_r' = \Sigma_r'(A_r')$ has a *sepal at a point* $z \in L_\infty^0$, whenever for a sequence of indices $h = h_j \to \infty$ and distinct points

$$z_h^1,\ z_h^2 \in \{z_-^-(h),\ z_+^-(h),\ z_+^+(h),\ z_-^+(h)\}, \quad h = p + q - r, \tag{3.8}$$

it has there Yaeko flowers with the conditions (1.5) reducing to:

$$g_q^\alpha\left(z_h^1\right) = N\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad q = q_{\iota\ell}^{1\kappa},$$
$$\text{and/or } g_q^\alpha\left(z_h^1\right) = N\begin{pmatrix} \eta a_{\alpha,\iota}'^{\kappa} \\ 0 \end{pmatrix}, \quad \text{where} \quad q = q_{\iota\ell}^{2\kappa},$$

and

$$g_q^\alpha\left(z_h^2\right) = N\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where} \quad q = q_{i\ell}^{1\kappa},$$
$$\text{and/or } g_q^\alpha\left(z_h^2\right) = N\begin{pmatrix} 0 \\ \eta' a_{\alpha,i+1}'^{\kappa+1} \end{pmatrix}, \quad \text{where} \quad q = q_{i\ell}^{2\kappa},$$

$\eta' = 1$ or/and $-1$, respectively. We say that it is *related to the direction* $\begin{pmatrix} - & - \\ - & + \end{pmatrix}$, $\begin{pmatrix} - & + \\ + & + \end{pmatrix}$, $\begin{pmatrix} + & + \\ + & - \end{pmatrix}$, or $\begin{pmatrix} + & - \\ - & - \end{pmatrix}$ according to (3.8) and denote it by $N\begin{pmatrix} s \\ t \end{pmatrix}$, where $s = \eta a_{\alpha,\iota}'^{\kappa}$ and $t = \eta' a_{\alpha i}'^{\kappa}$.

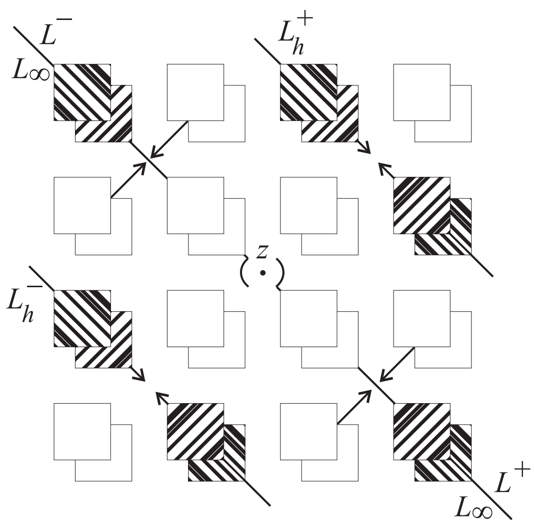


Figure 7: Maria's sepal

**Definition 9.** *Maria's* (respectively: *Sanae's*) *sepal at a point* is a sepal corresponding to Maria's (respectively: Sanae's) flower at that point (Figure 7 and Figure 8).

It is straightforward to verify the following proposition.

**Proposition 1.** *Each fractal (3.6) has, any point $z \in L^0_\infty$, Maria's sepals*

$$N\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } N\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ related to } \begin{pmatrix} - & - \\ - & + \end{pmatrix}, \begin{pmatrix} - & + \\ + & + \end{pmatrix},$$
$$\begin{pmatrix} + & + \\ + & - \end{pmatrix}, \text{ and } \begin{pmatrix} + & - \\ - & - \end{pmatrix}.$$

*Each fractal* (3.7) *has, at any point $z \in L^0_\infty$, Sanae's sepals*

$$N\begin{pmatrix} -i \\ i \end{pmatrix} \text{ and } N\begin{pmatrix} i \\ -i \end{pmatrix} \text{ related to } \begin{pmatrix} - & - \\ - & + \end{pmatrix}, \text{ and } \begin{pmatrix} + & + \\ + & - \end{pmatrix}.$$

It is clear that, in the case of (1), we cannot forget about the generators $A^1_\alpha = A_\alpha$ that started the whole construction and form its natural nucleus. Therefore we are led to

**Definition 10.** We shall call the sets $L^0_{p+q-1-r}$, $q = 1, 2, ...$ – the *stamens* of $\Sigma'_r$, $r = p, p+1, ...$, and their limit set $L^0_\infty$ – the *pistil* of $\Sigma'_r$. In the case of (1), i.e., in the cases of (3.6) and (3.7), it is natural to call the initial Clifford algebra $Cl_{2p-1}(\mathbb{C})$ – the *ovary* of $(\Sigma_\alpha,\ \alpha = 1, 2, ...)$, and its generators $A^1_\alpha = A_\alpha$,

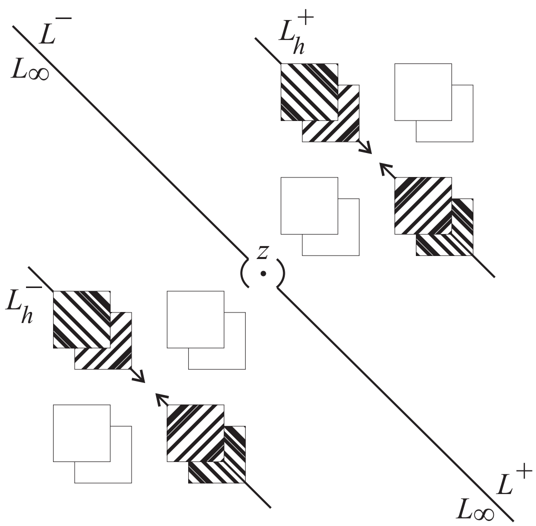


Figure 8: Sanae's sepal

$\alpha = 1, 2, ...2p-1$, – the *ovules* of $(\Sigma_\alpha,\ \alpha = 1, 2, ...)$.

### 4. Perodicity on the Pistil

We turn our attention to the problem whether the sequences of the gradating functions

$$g_1^\alpha(z), g_2^\alpha(z), ... \text{ for } z \in L_\infty^0 \text{ and a fixed } \alpha < 2p \tag{4.1}$$

are periodic. We have

**Theorem 1.** *If $a_{\alpha\lambda}^\lambda \neq 0$ for $\lambda = 2^{p+q-2}$, the sequences (4.1) are periodic of period 2. The periods are*

$$N\begin{pmatrix} \eta a_{\alpha\lambda}^\lambda \\ -\eta a_{\alpha 1}^1 \end{pmatrix},\ N\begin{pmatrix} -\eta a_{\alpha\lambda}^\lambda \\ -\eta a_{\alpha 1}^1 \end{pmatrix},\ \textit{where } \eta = 1 \textit{ or } -1.$$

*If $m$ in (1.6) is odd, the periodicity of the sequence (4.1) starts from its $(n+1)$-th term or earlier. If $m$ is even, of the form $m = \mu \cdot 2^\nu$, where $\mu$ is odd, the periodicity of (4.1) starts from the $(n-\nu+1)$-th term or earlier. Geometrically, in each case we get Yaeko's flower.*

*Proof.* The assertion is a direct consequence of Lemma 1 and Lemmas 3–5 below which can be proved by a direct verification: induction with respect to

$\nu = 0, 1, ..., n-1$, for a fixed $n \in \{2, 3, ...\}$, and then induction with respect to $n$. □

**Lemma 3.** *For any $z \in L^0_\infty$, i.e., of the form (1.6), where $m = \mu \cdot 2^\nu, \nu = 0, 1, ..., n-1$; $n = 1, 2, ...,$ we have*

$$\begin{aligned} g^\alpha_{n-\nu-1}(z) &= \begin{pmatrix} a^{\lambda'}_{\alpha,\lambda'} & a^{\lambda'+1}_{\alpha,\lambda'} \\ & . \\ a^{\lambda'}_{\alpha,\lambda'+1} & a^{\lambda'+1}_{\alpha,\lambda'+1} \end{pmatrix}, \\ g^\alpha_{n-\nu}(z) &= \begin{pmatrix} a^{\lambda''}_{\alpha,\lambda''} & a^{\lambda''+1}_{\alpha,\lambda''} \\ & . \\ a^{\lambda''}_{\alpha,\lambda''+1} & a^{\lambda''+1}_{\alpha,\lambda''+1} \end{pmatrix}, \end{aligned} \tag{4.2}$$

*or*

$$\begin{aligned} g^\alpha_{n-\nu-1}(z) &= \begin{pmatrix} a^{\lambda'''}_{\alpha,\lambda'''} & a^{\lambda'''+1}_{\alpha,\lambda'''} \\ & . \\ a^{\lambda'''}_{\alpha,\lambda'''+1} & a^{\lambda'''+1}_{\alpha,\lambda'''+1} \end{pmatrix}, \\ g^\alpha_{n-\nu}(z) &= \begin{pmatrix} -a^{\lambda''}_{\alpha,\lambda''} & -a^{\lambda''+1}_{\alpha,\lambda''} \\ & . \\ -a^{\lambda''}_{\alpha,\lambda''+1} & -a^{\lambda''+1}_{\alpha,\lambda''+1} \end{pmatrix}, \end{aligned} \tag{4.3}$$

*where*

$$\lambda' = 2^{p+q-4},\ \lambda'' = 2 \cdot 2^{p+q-4}\ \lambda''' = 3 \cdot 2^{p+q-4}.$$

**Lemma 4.** *Let $z$ be as in Lemma 3. If, for some $\kappa \in \{0, 1, ...\}$, we have*

$$g^\alpha_{n-\nu+1}(z) = \begin{pmatrix} \eta a^\lambda_{\alpha\lambda} \\ -\eta a^1_{\alpha 1} \end{pmatrix},\ g_{n-\nu+2}(z) = N \begin{pmatrix} -\eta a^\lambda_{\alpha\lambda} \\ -\eta a^1_{\alpha 1} \end{pmatrix}, \tag{4.4}$$

*where $\lambda = 2^{p+q-2}$ and $\eta = 1$ or $-1$.*

**Lemma 5.** *Let $z$ be as in Lemma 3. If, for some $\kappa \in \{1, 2, ...\}$, we have*

$$g^\alpha_{n-\nu+2\kappa}(z) = g^\alpha_{n-\nu+1}(z),\ g^\alpha_{n-\nu+2\kappa}(z) = g^\alpha_{n-\nu+2}(z),$$

*then the relations remain true with the same $\eta$ in (4.2) when replacing $\kappa$ by $\kappa + 1$.*

Theorem 1 and Lemma 3 imply

**Corollary 1.** *If $m$ in* (1.6) *is odd and*

$$\begin{pmatrix} a^{\lambda''}_{\alpha,\lambda''} & & a^{\lambda''+1}_{\alpha,\lambda''} \\ & \cdot & \\ a^{\lambda''}_{\alpha,\lambda''+1} & & a^{\lambda''+1}_{\alpha,\lambda''+1} \end{pmatrix} = N \begin{pmatrix} -a^{\lambda}_{\alpha\lambda} \\ -a^{1}_{\alpha 1} \end{pmatrix},$$

$$\text{respectively: } \begin{pmatrix} -a^{\lambda''}_{\alpha,\lambda''} & & -a^{\lambda''+1}_{\alpha,\lambda''} \\ & \cdot & \\ -a^{\lambda''}_{\alpha,\lambda''+1} & & -a^{\lambda''+1}_{\alpha,\lambda''+1} \end{pmatrix} = N \begin{pmatrix} a^{\lambda}_{\alpha\lambda} \\ a^{1}_{\alpha 1} \end{pmatrix}, \tag{4.5}$$

*then the periodicity of the sequence (4.1) starts from its n-th term or earlier. Such a situation appears in the case of Pauli's matrix $\sigma_3$, where the periodicity starts exactly from the n-th term. If $m$ is even, of the form $m = \mu \cdot 2^{\nu}$, where $\mu$ is odd, and* (4.5) *holds, the periodicity of* (4.1) *starts from the $(n-\nu)$-th term or earlier. In particular, such a situation appears in the case of $\sigma_3$, where the periodicity starts exactly from the $(n-\nu)$-th term.*

*Proof.* It the case of $\sigma_3$ we have $g^3_{n-1}(z) = -g^3_{n+1}(z)$, respectively: $g^3_{n-\nu-1}(z) = -g^3_{n-\nu+1}(z)$, which suffices to conclude the proof. □

**Corollary 2.** *If $m$ in* (1.6) *is odd and, in addition to* (4.5)*, we have*

$$\begin{pmatrix} a^{\lambda'}_{\alpha,\lambda'} & & a^{\lambda'+1}_{\alpha,\lambda'} \\ & \cdot & \\ a^{\lambda'}_{\alpha,\lambda'+1} & & a^{\lambda'+1}_{\alpha,\lambda'+1} \end{pmatrix} = N \begin{pmatrix} a^{\lambda}_{\alpha\lambda} \\ -a^{1}_{\alpha 1} \end{pmatrix},$$

$$\text{respectively: } \begin{pmatrix} a^{\lambda'''}_{\alpha,\lambda'''} & & a^{\lambda'''+1}_{\alpha,\lambda'''} \\ & \cdot & \\ a^{\lambda'''}_{\alpha,\lambda'''+1} & & a^{\lambda'''+1}_{\alpha,\lambda'''+1} \end{pmatrix} = N \begin{pmatrix} -a^{\lambda}_{\alpha\lambda} \\ a^{1}_{\alpha 1} \end{pmatrix}, \tag{4.6}$$

*then the periodicity of the sequence* (4.1) *starts from its $(n-1)$-th term or earlier. Such a situation appears in the cases of Pauli's matrices $\sigma_1$ and $\sigma_2$, where the periodicity starts exactly from the $(n-1)$-th term. If $m$ is even, of the form $m = \mu \cdot 2^{\nu}$, where $\mu$ is odd, and both* (4.5), (4.6) *hold, the periodicity of* (4.1) *starts from the $(n-\nu-1)$-th term or earlier.*

*Proof.* analogous to that of Corollary 1. □

If $a^{\lambda}_{\alpha\lambda} = 0$ for $\lambda = 2^{p+q-2}$, the sequences (4.1) are not only constant-valued, starting from some term, but even trivial: $N\binom{0}{0}$. Therefore it is reasonable to replace them by

$$\hat{g}_1^\alpha(z_-^1), \hat{g}_2^\alpha(z_-^2), \ldots \quad \text{for} \quad z \in L_\infty^0 \text{ and a fixed } \alpha < 2p \tag{4.7}$$

and

$$\hat{g}_1^\alpha(z_+^1), \hat{g}_2^\alpha(z_+^2), \ldots \quad \text{for} \quad z \in L_\infty^0 \text{ and a fixed } \alpha < 2p. \tag{4.8}$$

Here

$$\hat{g}_q^\alpha(z_-^q) = \left(g_q^\alpha\left(z_-^-(q)\right), g_q^\alpha\left(z_-^+(q)\right)\right), \quad \hat{g}_q^\alpha(z_+^q) = \left(g_q^\alpha\left(z_+^-(q)\right), g_q^\alpha\left(z_+^+(q)\right)\right),$$

$$z_-^-(q), z_+^-(q), z_+^+(q), z_-^+(q) \in L_{p+q-3}^0, q = 1, 2, \ldots,$$

and the stamens $L_{p+q-3}^0$ are related to $z$. A reasoning analogous to that applied for proving Theorem 1 gives the following result.

**Theorem 2.** *If $a_{\alpha\lambda}^\lambda = 0$ for $\lambda = 2^{p+q-2}$, the sequences* (4.7) *are periodic of period* 2*. If $a_{\alpha,\lambda-1}^{\lambda-1} = a_{\alpha 2}^2 = a_{\alpha 1}^1 = 0$ (the condition will be removed in the forthcoming paper with a modification of the assertion (4.9)), the periods are:*

$$N\left(\eta a_{\alpha\lambda}^{\lambda-1}, \eta a_{\alpha,\lambda-1}^\lambda\right), \quad N\left(-\eta a_{\alpha\lambda}^{\lambda-1}, -\eta a_{\alpha,\lambda-1}^\lambda\right), \quad \text{where} \quad \eta = 1 \text{ or } -1. \tag{4.9}$$

*The periodicity of the sequences* (4.7) *always starts from their n-th term or earlier whenever $m$ in* (1.6) *is odd. If $m$ is even, of the form $m = \mu \cdot 2^\nu$, where $\mu$ is odd, the periodicity of* (4.7) *starts from the $(n-\nu)$-th term or earlier. In particular, such a situation appears in the cases of Pauli matrices $\sigma_1$ and $\sigma_2$, where the periodicity, for $m$ odd, starts exactly from the n-th term. For $m$ even the periodicity starts exactly from the $(n-\nu)$-th term. Geometrically, in each case we get Ola's flower (cf. Figure 2).*

It is convenient to introduce (cf. Figure 9).

**Definition 11.** If a fractal $\Sigma_\alpha$ determines two adjacent Ola's flowers with the only common point $z \in L_\infty$ and the union of these flowers determines some Yaeko's (composed) flower, the resulting (composed) flower is called *Ania's flower*.

Again, a reasoning similar to that applied for proving Theorem 1 yields the next result.

**Theorem 3.** *If $a_{\alpha\lambda}^\lambda = 0$ for $\lambda = 2^{p+q-2}$, the sequences* (4.8) *are constant-valued, starting from some term; if $a_{\alpha,\lambda-1}^{\lambda-1} = a_{\alpha 2}^2 = a_{\alpha 1}^1 = 0$ (the condition will be removed in the forthcoming paper with a modification of the assertion (4.10)), it amounts at:*

$$N\left(-\eta a_{\alpha 2}^1, -\eta a_{\alpha 1}^2\right), \quad \text{where} \quad \eta = 1 \text{ or } -1. \tag{4.10}$$

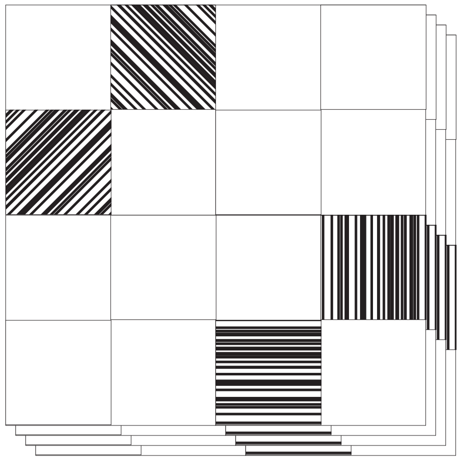

Figure 9: Ania's flower

*The constancy of the sequences* (4.8) *always starts from their $n$-th term or earlier whenever $m$ in* (1.6) *is odd. If $m$ is even, of the form $m = \mu \cdot 2^{\nu}$, where $\mu$ is odd, the constancy of* (4.8) *starts from the $(n - \nu)$-th term or earlier. In particular, such a situation appears in the cases of Pauli matrices $\sigma_1$ and $\sigma_2$, where the constancy, for $m$ odd, starts exactly from the $n$-th term. For $m$ even the constancy starts exactly from the $(n - \nu)$-th term. Geometrically, in each case we get Ola's flower. Moreover, given $z \in L_\infty$, the choices* (4.9) *and* (4.10) *are mutually correlated according to the same $\eta$: they appear together for $(z_q^-)$ and $(z_q^+)$, giving rise to Ania's flower at $z$.*

## 5. Periodicity on the Stamens

Hereafter we turn our attention to the problem whether the sequences of the gradating functions

$$\hat{g}_1^{2r}(z_1^-),\ \hat{g}_2^{2r}(z_2^-), ...;\quad \hat{g}_1^{2r}(z_1^+),\ \hat{g}_2^{2r}(z_2^+), ...;\quad 2r = \alpha \geq 2p \tag{5.1}$$

and

$$\hat{g}_1^{2r+1}(z_1^-),\ \hat{g}_2^{2r+1}(z_2^-), ...;\quad \hat{g}_1^{2r+1}(z_1^+),\ \hat{g}_2^{2r+1}(z_2^+), ...;\quad 2r+1 = \alpha > 2p \tag{5.2}$$

are periodic for $z$ of the pistil $L_\infty^0$ and a fixed $r$. Here

$$\hat{g}_q^{\alpha}(z_q^-) = \left(g_q^{\alpha}\left(z_-^-(h)\right),\ g_q^{\alpha}\left(z_+^-(h)\right)\right),\ \hat{g}_q^{\alpha}\left(z_q^+\right) = \left(g_q^{\alpha}\left(z_-^+(h)\right),\ g_q^{\alpha}\left(z_+^+(h)\right)\right),$$

$$z_-^-(h), z_+^-(h), z_+^+(h), z_-^+(h) \in L_h^0, \quad h = p + q - 1 - r,\ q = 1, 2, ...,$$

and the stamens $L_h^0$ are related to $z$. We have the following theorem.

**Theorem 4.** (i) *The sequences* (5.1) *and* (5.2) *are periodic of period* 2. *The periods are:*

$$\left(N\begin{pmatrix}1\\0\end{pmatrix}, N\begin{pmatrix}0\\-1\end{pmatrix}\right), \quad \left(N\begin{pmatrix}-1\\0\end{pmatrix}, N\begin{pmatrix}0\\-1\end{pmatrix}\right), \tag{5.3}$$

*or*

$$\left(N\begin{pmatrix}-1\\0\end{pmatrix}, N\begin{pmatrix}0\\1\end{pmatrix}\right), \quad \left(N\begin{pmatrix}1\\0\end{pmatrix}, N\begin{pmatrix}0\\1\end{pmatrix}\right), \tag{5.4}$$

*in the cases of* (5.1) *with* $\left(\hat{g}_q^{2r}(z_q^-)\right)$ *and* $\left(\hat{g}_q^{2r}(z_q^+)\right)$,

$$\left(N\begin{pmatrix}-i\\0\end{pmatrix}, N\begin{pmatrix}0\\i\end{pmatrix}\right), \quad \left(N\begin{pmatrix}i\\0\end{pmatrix}, N\begin{pmatrix}0\\i\end{pmatrix}\right), \tag{5.5}$$

*or*

$$\left(N\begin{pmatrix}i\\0\end{pmatrix}, N\begin{pmatrix}0\\-i\end{pmatrix}\right), \quad \left(N\begin{pmatrix}-i\\0\end{pmatrix}, N\begin{pmatrix}0\\-i\end{pmatrix}\right), \tag{5.6}$$

*in the case of* (5.2) *with* $\left(\hat{g}_q^{2r+1}(z_q^-)\right)$, *and* (5.6) *or* (5.5) *in the case of* (5.2) *with* $\left(\hat{g}_q^{2r+1}(z_q^+)\right)$.

(ii) *Given a point* $z \in L_\infty^0$, *either choice:* (5.3) *or* (5.4), *being identical for* $(z_q^-)$ *and* $(z_q^+)$, *gives rise to two petals of the same Maria's flower and two Maria's sepals at* $z$. *The other choice leads to further two petals of Maria's flower and two Maria's sepals at* $z$. *Given* $z \in L_\infty^0$, *the choices* (5.5) *and* (5.6) *are mutually correlated: they appear together for* $(z_q^-)$ *and* $(z_q^+)$, *giving rise to two petals of the same Sanae's flower and two Sanae's sepals at* $z$.

(iii) *If* $m$ *in* (1.6) *is odd, the periodicity of the sequences* (5.1) *and* (5.2) *starts from their* $(n+1)$*-th terms. If* $m$ *is even, of the form* $m = \mu \cdot 2^\nu$, *where* $\mu$ *is odd, the periodicity of* (5.1) *and* (5.2) *starts from their* $(n - \nu + 1)$*-th terms.*

*Proof.* Analogous to that of Theorem 1. Let us set $\rho = n - \nu$. For the sequences (5.1) the formulae (4.2) and (4.3) have to be replaced by

$$\hat{g}_{\rho-1}^{2r}\left(z_{\rho-1}^-\right) = \hat{g}_\rho^{2r}\left(z_\rho^-\right) = \hat{g}_{\rho-1}^{2r}\left(z_{\rho-1}^+\right)$$

$$= \hat{g}_\rho^{2r}\left(z_\rho^+\right) = \begin{cases} \left(N\begin{pmatrix}1\\0\end{pmatrix}, N\begin{pmatrix}0\\1\end{pmatrix}\right) & \text{in the case of (5.3)}, \\ \left(N\begin{pmatrix}-1\\0\end{pmatrix}, N\begin{pmatrix}0\\-1\end{pmatrix}\right) & \text{in the case of (5.4)}. \end{cases}$$

For the sequances (5.2) the formulae (4.2) and (4.3) have to be replaced by

$$\hat{g}_{\rho-1}^{2r+1}\left(z_{\rho-1}^{-}\right)=\hat{g}_{\rho}^{2r+1}\left(z_{\rho}^{-}\right) \text{ and } \hat{g}_{\rho-1}^{2r+1}\left(z_{\rho-1}^{+}\right)=\hat{g}_{\rho}^{2r+1}\left(z_{\rho}^{+}\right)$$

$$\text{equal}\begin{cases}\left(N\begin{pmatrix}-i\\0\end{pmatrix}, N\begin{pmatrix}0\\-i\end{pmatrix}\right) & \text{in the case of (5.5),}\\ \left(N\begin{pmatrix}i\\0\end{pmatrix}, N\begin{pmatrix}0\\i\end{pmatrix}\right) & \text{in the case of (5.6).}\end{cases}$$

In the above relation *equal* the conjunction *and* has to remind that the both cases never occur simultaneously: if one of them corresponds to (5.5), the other corresponds to (5.6).

Finally we turn our attention to the problem whether the sequences of gradating functions

$$\hat{g}_1^{2r}(z_-^1),\ \hat{g}_2^{2r}(z_-^2),...;\quad 2r=\alpha\geq 2p, \tag{5.7}$$

$$\hat{g}_1^{2r}(z_+^1),\ \hat{g}_2^{2r}(z_+^2),...;\quad 2r=\alpha\geq 2p, \tag{5.8}$$

and

$$\hat{g}_1^{2r+1}(z_-^1),\ \hat{g}_2^{2r+1}(z_-^2),...;\quad 2r+1=\alpha> 2p, \tag{5.9}$$

$$\hat{g}_1^{2r+1}(z_+^1),\ \hat{g}_2^{2r+1}(z_+^2),...;\quad 2r+1=\alpha> 2p, \tag{5.10}$$

are periodic for $z$ of the pistil $L_\infty^0$ and a fixed $r$. Here

$$\hat{g}_q^{\alpha}\left(z_-^q\right)=\left(g_q^{\alpha}\left(z_-^{-}(h)\right), g_q^{\alpha}\left(z_-^{+}(h)\right)\right),\ \hat{g}_q^{\alpha}\left(z_+^q\right)=\left(g_q^{\alpha}\left(z_+^{-}(h)\right), g_q^{\alpha}\left(z_+^{+}(h)\right)\right),$$

$$z_-^{-}(h),\ z_+^{-}(h),\ z_+^{+}(h),\ z_-^{+}(h)\in L_h^0,\ h=p+q-1-r,\ q=1,2,...,$$

and the stamens $L_h^0$ are related to $z$. □

We have the following theorem.

**Theorem 5.** (i) *The sequences* (5.7) *and* (5.9) *are periodic of period* 2. *The periods are:*

$$\left(N\begin{pmatrix}1\\0\end{pmatrix}, N\begin{pmatrix}1\\0\end{pmatrix}\right),\quad \left(N\begin{pmatrix}-1\\0\end{pmatrix}, N\begin{pmatrix}-1\\0\end{pmatrix}\right), \tag{5.11}$$

*or*

$$\left(N\begin{pmatrix}-1\\0\end{pmatrix}, N\begin{pmatrix}-1\\0\end{pmatrix}\right),\quad \left(N\begin{pmatrix}1\\0\end{pmatrix}, N\begin{pmatrix}1\\0\end{pmatrix}\right), \tag{5.12}$$

*in the case of* (5.7), *and*

$$\left(N\begin{pmatrix}-i\\0\end{pmatrix},\ N\begin{pmatrix}i\\0\end{pmatrix}\right),\quad \left(N\begin{pmatrix}i\\0\end{pmatrix},\ N\begin{pmatrix}-i\\0\end{pmatrix}\right),\tag{5.13}$$

*or*

$$\left(N\begin{pmatrix}i\\0\end{pmatrix},\ N\begin{pmatrix}-i\\0\end{pmatrix}\right),\quad \left(N\begin{pmatrix}-i\\0\end{pmatrix},\ N\begin{pmatrix}i\\0\end{pmatrix}\right),\tag{5.14}$$

*in the case of* (5.9).

(ii) *Given a point* $z \in L^0_\infty$, *either choice:* (5.11) *or* (5.12) *in the case of* (5.7), *respectively:* (5.13) *or* (5.14) *in the case of* (5.9), *gives rise to two petals of the same Maria's flower* (*also of the same flower as in Theorem* 4) *and two Maria's sepals at* $z$, *respectively: to two petals of the same Sanae's flower* (*also of the same flower as in Theorem* 4) *and no Sanae's sepals at* $z$.

(iii) *If* $m$ *in* (1.6) *is odd, the periodicity of the sequences* (5.7) *and* (5.9) *starts from their* $(n+1)$*-th terms. If* $m$ *is even, of the form* $m = \mu \cdot 2^\nu$, *where* $\mu$ *is odd, the periodicity of* (5.7) *and* (5.9) *starts from their* $(n-\nu+1)$*-th terms.*

*Proof.* analogous to that of Theorem 1. Let us set $\rho = n - \nu$. For the sequence (5.7) the formulae (4.2) and (4.3) have to be repalced by

$$\hat{g}^{2r}_{\rho-1}\left(z_-^{\rho-1}\right) = \left(N\begin{pmatrix}0\\0\end{pmatrix},\ N\begin{pmatrix}0\\0\end{pmatrix}\right),$$

$$\hat{g}^{2r}_{\rho}\left(z_-^{\rho}\right) = \left(N\begin{pmatrix}0\\1\end{pmatrix},\ N\begin{pmatrix}0\\1\end{pmatrix}\right) \quad \text{in the case of (5.11),}$$

and by

$$\hat{g}^{2r}_{\rho-1}\left(z_-^{\rho-1}\right) = \left(N\begin{pmatrix}1\\1\end{pmatrix},\ N\begin{pmatrix}1\\1\end{pmatrix}\right),$$

$$\hat{g}^{2r}_{\rho}\left(z_-^{\rho}\right) = \left(N\begin{pmatrix}0\\-1\end{pmatrix},\ N\begin{pmatrix}0\\-1\end{pmatrix}\right) \quad \text{in the case of (5.12).}$$

For the sequence (5.9) the formulae (4.2) and (4.3) have to be replaced by

$$\hat{g}^{2r+1}_{\rho-1}\left(z_-^{\rho-1}\right) = \left(N\begin{pmatrix}0\\0\end{pmatrix},\ N\begin{pmatrix}0\\0\end{pmatrix}\right),$$

$$\hat{g}^{2r}_{\rho}\left(z_-^{\rho}\right) = \left(N\begin{pmatrix}0\\-i\end{pmatrix},\ N\begin{pmatrix}0\\-i\end{pmatrix}\right) \quad \text{in the case of (5.13),}$$

and by

$$\hat{g}^{2r+1}_{\rho-1}\left(z_-^{\rho-1}\right) = \left(N\begin{pmatrix}-i\\-i\end{pmatrix},\ N\begin{pmatrix}i\\i\end{pmatrix}\right),$$

$$\hat{g}_{\rho}^{2r+1}\left(z_{-}^{\rho}\right)=\left(N\begin{pmatrix}0\\ i\end{pmatrix},\ N\begin{pmatrix}0\\ -i\end{pmatrix}\right)\quad \text{in the case of (5.14).}$$

We also have the following result.

**Theorem 6** (i). *The sequences* (5.8) *and* (5.10) *are constant-valued, starting from some term; it amounts at:*

$$\left(N\begin{pmatrix}0\\ -1\end{pmatrix},\ N\begin{pmatrix}0\\ -1\end{pmatrix}\right), \tag{5.15}$$

*or*

$$\left(N\begin{pmatrix}0\\ 1\end{pmatrix},\ N\begin{pmatrix}0\\ 1\end{pmatrix}\right), \tag{5.16}$$

*in the case of* (5.8), *and*

$$\left(N\begin{pmatrix}0\\ i\end{pmatrix},\ N\begin{pmatrix}0\\ -i\end{pmatrix}\right), \tag{5.17}$$

*or*

$$\left(N\begin{pmatrix}0\\ -i\end{pmatrix},\ N\begin{pmatrix}0\\ i\end{pmatrix}\right), \tag{5.18}$$

*in the case of* (5.10).

(ii) *Given a point* $z\in L_{\infty}^{0}$, *either choice:* (5.15) *or* (5.16) *in the case of* (5.8), *respectively:* (5.17) *or* (5.18) *in the case of* (5.10), *gives rise to two petals of the same of the same Maria's flower* (*also of the same flower as in Theorem* 4 *and Theorem* 5) *and two Maria's sepals at* $z$, *respectively: to two petals of the same Sanae's flower* (*also of the same flower as in Theorem* 4 *and Theorem* 5) *and no Sanae's sepals at* $z$.

(iii) *If* $m$ *in* (1.6) *is odd, the constancy of the sequences* (5.8) *and* (5.10) *starts from their* $(n+1)$*-th terms. If* $m$ *is even, of the form* $m=\mu\cdot 2^{\nu}$, *where* $\mu$ *is odd, the constancy of* (5.8) *and* (5.10) *starts from their* $(n-\nu+1)$*-th terms.*

*Proof.* analogous to that of Theorem 1. Let us set $\rho=n-\nu$. For the sequence (5.8) the formulae (4.2) and (4.3) have to be replaced by

$$\hat{g}_{\rho-1}^{2r}\left(z_{+}^{\rho-1}\right)=\left(N\begin{pmatrix}1\\ 1\end{pmatrix},\ N\begin{pmatrix}1\\ 1\end{pmatrix}\right),$$

$$\hat{g}_{\rho}^{2r}\left(z_{+}^{\rho}\right)=\left(N\begin{pmatrix}1\\ 0\end{pmatrix},\ N\begin{pmatrix}1\\ 0\end{pmatrix}\right)\quad \text{in the case of (5.15),}$$

and by

$$\hat{g}^{2r}_{\rho-1}\left(z_+^{\rho-1}\right) = \left(N\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right),$$
$$\hat{g}^{2r}_{\rho}\left(z_+^{\rho}\right) = \left(N\begin{pmatrix} -1 \\ 0 \end{pmatrix},\ N\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) \quad \text{in the case of (5.16).}$$

For the sequence (5.10) the formulae (4.2) and (4.3) have to be replaced by

$$\hat{g}^{2r+1}_{\rho-1}\left(z_+^{\rho-1}\right) = \left(N\begin{pmatrix} -i \\ -i \end{pmatrix},\ N\begin{pmatrix} i \\ i \end{pmatrix}\right),$$
$$\hat{g}^{2r+1}_{\rho}\left(z_+^{\rho}\right) = \left(N\begin{pmatrix} -i \\ 0 \end{pmatrix},\ N\begin{pmatrix} i \\ 0 \end{pmatrix}\right) \quad \text{in the case of (5.17),}$$

and by

$$\hat{g}^{2r+1}_{\rho-1}\left(z_+^{\rho-1}\right) = \left(N\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ N\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$
$$\hat{g}^{2r+1}_{\rho}\left(z_+^{\rho}\right) = \left(N\begin{pmatrix} i \\ 0 \end{pmatrix},\ N\begin{pmatrix} -i \\ 0 \end{pmatrix}\right) \quad \text{in the case of (5.18).}$$

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