International Journal of Pure and Applied Mathematics

Volume 25 No. 3 2005, 289-306

STRUCTURE OF PERIODIC RINGS AND CERTAIN NEARRINGS

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Abstract: In the present paper, we first establish, in Section 2, some decomposition theorems for rings and its subsets (see Theorem 2.1 and Theorem 2.2, for details). Secondly, in Section 3, certain nearrings satisfying one of the defined properties (C) to (C_5), under appropriate conditions such nearrings are in fact shown to be commutative. Thirdly, in Section 4, it is proved that any nearring is decomposable into a direct sum of special sub nearrings and the Pierce-decomposition for rings is generalized. Finally, in Section 5, we provide some counterexamples which show that hypotheses of our theorems are not altogether superfluous.

AMS Subject Classification: 16Y30, 16U80
Key Words: decomposition theorem, J-ring, nil ring, periodic nearring, zero-commutative, distributively-generated $(d-g)$ nearing, D-nearring

1. Introduction

Throughout, in Section 2, R will represent an associative ring (may be without unity 1) and N the set of nilpotent elements of R. The symbol Z will denote the ring of integers, $Z[X]$ the ring of polynomials in one indeterminate with integer coefficients.

Recall: The concept of Boolean ring, as a ring in which every element is idempotent, was first introduced by Stone [27]. With this as motivation, McCoy and Montgomery [21] introduced the concept of p-ring (p prime) as a ring R

Received: August 22, 2005

in which $x^p = x$ and $px = 0$ for all $x \in R$. Hence, Boolean rings are simply 2-rings ($p = 2$). A concept that generalizes Boolean rings and p-rings is that of a periodic ring.

A ring R is called periodic if for each $x \in R$ there exist distinct positive integers $m = m(x), n = n(x)$ such that $x^m = x^n$, R is called zero-commutative if $xy = 0$ implies $yx = 0$, for all $x, y \in R$. An element $x \in R$ with the property $''x^{k(x)} = x''$ for some $k(x) > 1$, will be called potent. Let P be the set of all potent elements. We shall call R a J-ring when $P = R$. By a well-known theorem of Jacobson [14], J-rings are necessarily commutative. A sufficient condition for R to be periodic is Chacron's criterion [9]; for each $x \in R$, there exists an integer $p = p(x) > 1$ and a polynomial $f(X) \in Z[X]$ such that $x^p = x^{p+1}f(x)$.

It is well-known that Boolean condition leading to commutativity in a ring, for instance, $x^2 = x$ for all ring elements x has recently been weakened by Searcoid and MacHale [26], who established the commutativity of rings in which all products of two elements are potent. Latter it was pointed out by Ligh and Luh [20] that such rings are direct sum of J-rings, that is, rings satisfying Jacobson's "$x^{n(x)} = x$ theorem" (see [14] for references) and zero rings.

In 1989, Bell and Ligh [2] proved the direct sum decomposition of rings satisfying the property $xy = (xy)^2 f(x, y) \in XZ[X]$, where $f(X, Y) \in Z < X, Y >$, the ring of polynomials in two non-commuting indeterminates. Motivated by these observations, we define the properties in Section 2, and then establish the decomposition theorems for rings and its subsets.

In Section 3 to Section 5, An additively written group $(N, +)$ with a binary operation $(x, y) \to xy$, such that $(xy)z = x(yz)$ and $x(y + z) = xy + xz$ for all $x, y, z \in N$ is called a left nearring. A nearring N is said to be zero-symmetric if $0x = 0$ (left-distributive of N guarantees $x0 = 0$) for all $x \in N$. In what follows all nearrings are left nearrings with multiplicative center Z. We refer the reader to the books of Clay [11], Meldrum [22] and Piltz [23] for the basic results and definitions; such as, distributive nearrings, distributively generated $(d - g)$ nearrings and strongly distributively generated $(s - d - g)$ nearrings, of nearing theory and its applications.

A nearring N is periodic if for each $x \in N$, there exist distinct positive integers $m = m(x), n = n(x)$ such that $x^m = x^n$.

A nearring N is called D-nearring if every non-zero homomorphic image T of N satisfies the following properties:

(i) T has a non-zero right distributive element.

(ii) Additive group $(T, +)$ of T is Abelian implies that $(T, +, .)$ is a ring.

From the above definitions, it is clear that all distributive and $(d - g)$

nearrings are examples of D-nearring. However, there are D-nearring [19], Example 3 which are not necessarily $(d-g)$ nearrings. However, the Example 2.5 of Section 6 of [10] shows that the class of D-narrings is larger than that of the $(d-g)$ nearrings. Further, an ideal of N is defined to be a normal subgroup I of the additive group $(N,+)$ such that:

(i) $NI \subseteq I$.

(ii) $(x+a)y - xy \in I$ for all $x, y \in N$ and $a \in I$ (see [11], [23] for further information).

A well known theorem of Herstein states that a periodic ring is commutative if its nilpotent elements are central [13], and Ligh [20] has asked whether a similar result holds for distributively generated $(d-g)$ nearrings. In his paper [6], Bell gave an affirmative answer and proved that "if N is a $(d-g)$ nearing with its nilpotent elements lying in the center, then the set I of nilpotent elements forms an ideal; and if $N \setminus I$ is periodic, N must be commutative". Further, Quadri et al [25] established that a $(d-g)$ nearring satisfying the property $xy = xy^m x$ for $m = m(x,y) > 1$ is commutative.

In 2000, author [15] established that N is commutative if $(d-g)$ nearring N satisfying the property $xy = p(x,y)$, where $p(x,y)$ is a finite sum of terms of the form $xy = \alpha_i x^{p_i} y^{q_i} x^{r_i}$, where the number of summands and α_i, p_i, q_i, r_i are vary $x, y \in N$ and $\alpha_i, p_i, r_i \geq 1$ and $q_i > 1$.

In Section 3.1 and Section 3.2, we extend these for $(d-g)$ nearrings and D-nearrings respectively. Section 4 is devoted some decomposition theorems for certain nearrings. Finally, in Section 5, we provide some counterexamples for justification of our results.

2. Decomposition Theorems for Rings and Subsets

Here, we begin with the rings satisfying one of the following properties and establish some decomposition theorems.

(P) For each $x, y \in R$, there exist integers $p = p(x,y) \geq 0, q = q(x,y) \geq 0$ and a polynomial $f(X) \in XZ[X]$ such that $xy = x^p f(x^l y^m x) x^q$, where $l = 0$ and $m > 0$ are fixed integers.

(P_1) For each $x, y \in R$, there exist integers $p = p(x,y) \geq 0, q = q(x,y) \geq 0$ and a polynomial $f(X) \in XZ[X]$ such that $xy = x^p f(y^l x^m y) x^q$, where $l \geq 0$ and $m > 0$ are fixed integers.

Remark 2.1. In continuation of the properties, as a special case in the paper [16], author proved that if $l = m = 1$ in (P) or (P_1) and also in [17] it is shown that for $l = 1$ and $m = 2$ in the above conditions. In this context, we establish the following results for general values of l and m.

Now, we prove the following.

Theorem 2.1. *Let R be a ring satisfying one of the properties (P) and (P_1). Then $R = P \oplus N$, R is a direct sum of a J-ring and a nil ring.*

In order to develop the proof of the above theorem, we need the following lemmas.

Lemma 2.1. (see [8], Lemma 1) *Let R be a periodic ring. Then each element $x \in R$ can be represented as a sum of potent element and a nilpotent element.*

Lemma 2.2. (see [6], Theorem 3) *If in a periodic ring R every element $x \in R$ can be expressed uniquely in the form $x = b + c$, where $b \in P$ and $c \in N$, then P and N are both ideals and $R = P \oplus N$.*

In the sequel, we establish some results to be used in the proof of Theorem 2.1.

Lemma 2.3. *Let R be a ring satisfying (P). Then R is zero-commutative.*

Proof. Let $xy = 0$. Then there exist integers $p' = p(y, x) \geq 0, q' = q(y, x) \geq 0$ and a polynomial $h(X) \in XZ[X]$ such that $yx = y^{p'}h(y^l x^m y)y^{q'}$ $= 0$. This completes the proof. □

Lemma 2.4. *Let R be a ring satisfying (P). Then $RN = NR = 0$.*

Proof. Replacing y by x in (P), we get

$$x^2 = x^r g(x), \text{for } g(X) \in Z[X] \text{ and } r(x) \geq 2. \tag{2.1}$$

Let $c \in N$ and $x \in R$. Then choose $p_1 = p(c, x) \geq 0, q_1 = q(c, x) \geq 0$ and a polynomial $f_1(X) \in XZ[X]$ such that

$$cx = f_1(c^l x^m c)c^{q_1}. \tag{2.2}$$

In view of (2.1), one can trivially see that $c^2 = 0$, and hence $0 = xc^2 = (xc)c$. Lemma 2.3 gives that $c(xc) = 0$, that is, $(xc)^2 = 0$. This, together with (2.2), yields that $cx = 0$ and again Lemma 2.3 forces that $xc = 0$ for all $x \in R, c \in N$. This gives the required result. □

Proof of Theorem 2.1. Clearly, if y is replaced by x in any one of the properties (P) and (P_1), the condition $f(X) \in XZ[X]$ guarantees that R satisfies the Chacron's criterion for periodicity and hence the ring satisfying any one of the properties (P) and (P_1) is necessarily periodic. Since R is periodic, every element $x \in R$ can be written in the form $x = b + c$, where $b \in P$ and $c \in N$ by Lemma 2.1.

To complete the proof of the theorem it suffices to show that this representation is unique. Taking $a + c = b + d$ for some $a, b \in P$ and $c, d \in N$.

Then

$$a - b = d - c. \tag{2.3}$$

If $a, b \in P$, then there exist at least one odd of the positive integers $r = r(a)$ and $s = s(b)$ such that $a^r = a$ and $b^s = b$.

Let $t = (r-1)s-(r-2) = (s-1)r-(s-2)$ be an odd positive integer. Thus it is clear that $a^t = a$ and $b^t = b$. Also $e_1 = a^{t-1}, e_2 = b^{t-1}$ are idempotents in R with $e_1 a = a$ and $e_2 b = b$. Multiplying (2.3) by a and b from both sides and using the result of Lemma 2.3, we get $a^2 = ab = ba$ and $b^2 = ab = ba$. This gives $a^2 = b^2$ and hence $e_1 = e_2$.

If t is even and $a^t = a$, then $a^{2(t-1)+1} = a$, where $2(t-1)+1$ is odd, so this gives the required result. Left -multiplying (2.3) by e_1 now yields $a = b$, and gives the required result. □

Similar arguments can be used if R satisfies the property (P_1).

Remark 2.2. Note that the nilpotent elements of R annihilate R on both sides and hence central, and thus, J-rings are commutative (see Lemma 2.3 for details). So, Theorem 2.1 at once gives the following corollary which extends the main results of [7], [5], [9], [17], [16], [20], [26].

Corollary 2.1. *Let R be a ring satisfying any one of the properties (P) and (P_1). Then R is commutative.*

In view of Theorem 2.1, it is natural to ask a question: What can we say about the direct sum decomposition of the ring R if any one of the properties (P) and (P_1) is assumed to be satisfied by an appropriate proper subset S of R?

The following Theorem 2.2 and Theorem 2.3 are attempts to answer this question.

Theorem 2.2. *Let R be a ring with $N \neq 0$, and S the additive subgroup of R with $S \subseteq N$. If for each $x, y \in R \setminus S$, there exists $p(X) \in XZ[X]$ such that (P) or (P_1) is satisfied, then R is a direct sum of a J-ring and a nil ring.*

In Theorem 2.2 and Theorem 2.3, we shall prove the result for (P) and the proof for (P_1) follows similarly.

Proof of Theorem 2.2. Take $x, y \in R \setminus S$. Replacing y by x in (P), we get

$$x^2 = x^r g(X) \in Z[X] \text{ and } r(x) \geq 2. \tag{2.4}$$

Since elements of S are nilpotent, for each $x \in S, x^{r'} = 0$ for some $r' > 1$ and we have $x^{r'} = x^{r'+k} = 0$ for any integer k. But R is periodic by Chacron criterion. Now, in view of (2.4), we have

$$x \in N \setminus S, \text{ implies } x^2 = 0\,. \tag{2.5}$$

Next, it suffices to show that N is an ideal which annihilates R on both sides.

First, we claim that

$$x \in N \setminus S, y \in R \text{ and } \ xy = 0 \text{ imply } \ yx = 0\,. \tag{2.6}$$

Let $y \in R \setminus S$, Clearly, we suppose $xs = 0$, where $x \in N \setminus S$ and $s \in S$.

Next $(x + s) \notin S$, and $x(x + s) = x^2 + xs = 0$; and hence $(x + s)x = 0 = x^2 + xs = sx$.

Secondly, we prove that N is an ideal: Suppose that $x, y \in S$. Then $x - y \in S \subseteq N$. On the other hand, if $x \in N \setminus S$, then we see from (2.5) and (2.6) that $x^2 z = 0$ for all $z \in R$. Thus, we get

$$xRx = 0. \tag{2.7}$$

Let $y \in N$. Then $y^t = 0$ for some integer $t > 1$ and we have $(x - y)^{2t} = 0$. This implies that $x - y \in N$. Further, let $x \in N \setminus S$. Then in view of (2.7), one gets $(xz)^2 = 0 = (zx)^2$ for all $z \in R$. Suppose $s \in S$ and $x \in N \setminus S$. Then we write $sz = (x + s)z - xz$. This implies that $sz \in N$; hence N is an ideal.

On the lines of the proof given in case of Theorem 2.1, it is sufficient to show that N annihilates R on both sides, that is, $RN = 0 = NR$.

Let $x \in N \setminus S$ and $y \in R \setminus S$. Then (P) and the fact that N is an ideal imply that $xy = 0$.

Using this that every element of S is a difference of two elements of $R \setminus S$, it is easy to show that $RN = 0 = NR$. □

From the above result, we can conclude that nilpotent elements of the ring R annihilate R on both sides and hence are central. Since J-rings are commutative, the consequences of Theorem 2.2 is the following corollary which generalizes the main result of Bell [1] and the author [17].

Corollary 2.2. *Let R be a ring with $N \neq 0$ and S be an additive subgroup of R with $S \subseteq N$. If for each $x, y \in R \setminus S$ either of the conditions (P) or (P_1) holds, then R is commutative.*

The following result is a direct consequence of Theorem 2.2.

Theorem 2.3. *Let R be a 2-torsion -free ring and J its Jacobson radical and S an additive subgroup of R with $S \subseteq J$. If for each $x, y \in R \setminus S$, there exists $p(X) \in XZ[X]$ such that (P) or (P_1) is satisfied, then R is a direct sum of a J-ring and a zero-ring.*

3. Some Commutativity Results for Nearrings

In this context: we define the following conditions.

(C) For each pair of elements $x, y \in N$, there exist integers $p = p(x,y) \geq 0, q = q(x,y) \geq 0$ and $r = r(x,y) > 0$ such that

$$xy = x^p(x^l y^m x)^r x^q, \text{ where } l \geq 0 \text{ and } m > 1 \text{ are integers.}$$

(C_1) For each pair of elements $x, y \in N$, there exist integers $p = p(x,y) \geq 0, q = q(x,y) \geq 0$ and $r = r(x,y) > 0$ such that

$$xy = x^p(y^l x^m y)^r x^q, \text{ where } l \geq 0 \text{ and } m > 1 \text{ are integers.}$$

(C_2) For each pair of elements $x, y \in R$, there exist positive integers $p = p(x,y) \geq 1$, $q = q(x,y) \geq 1$ and $f(xyx)$ denotes an element of a nearring N which is finite sum of powers $(xyx)^t$, $t \geq 2$ with additive inverses of such powers such that

$$xy = x^p f(xyx) x^q \, .$$

(C_3) For each pair of elements $x, y \in R$, there exist positive integers $p = p(x,y) \geq 1$, $q = q(x,y) \geq 1$ and $f(xyx)$ denotes an element of a nearring N which is finite sum of powers $(yxy)^t$, $t \geq 2$ with additive inverses of such powers such that

$$xy = x^p f(yxy) x^q.$$

(C_4) For each pair of elements $x, y \in R$, there exist positive integers $p = p(x,y) \geq 1$ and $f(xyx)$ denotes an element of a nearring N which is finite sum of powers $(xyx)^t$, $t \geq 2$ with additive inverses of such powers such that

$$xy = (xyx)^p f(xyx) \, .$$

(C_5) For each pair of elements $x, y \in R$, there exist positive integers $p = p(x,y) \geq 1$ and $f(yxy)$ denotes an element of a nearring N which is finite sum of powers $(yxy)^t$, $t \geq 2$ with additive inverses of such powers such that

$$xy = (xyx)^p f(yxy) \, .$$

Theorem 3.1. *Let N be a $(d-g)$ nearing satisfying (C) or (C_1). Then N is commutative.*

Before proving our theorems, we state the following lemmas proved in [12] and [4] respectively.

Lemma 3.1. *A $(d-g)$ nearring N is distributive if and only if N^2 is additively commutative.*

Lemma 3.2. *A $(d-g)$ nearring N with unity 1 is a ring if N is distributive or $(N,+)$ is commutative.*

Lemma 3.3. *Let N be a $(d-g)$ nearring with its nilpotent elements lying in the center. Then the set S of nilpotent elements of N forms an ideal. Moreover, if $N \setminus S$ is periodic then N must be commutative.*

Now we establish the following lemma.

Lemma 3.4. *Let N be a $(d-g)$ nearring satisfying (C), (C_1), (C_2), (C_3), (C_4) or (C_5). Then $S \subseteq Z$.*

Proof. Let N be a $(d-g)$ nearing satisfying (C). By hypothesis, we have

$$xy = x^p(x^l y^m x)^r x^q = x^p \underbrace{(x^l y^m x)(x^l y^m x)...(x^l y^m x)}_{r-\text{times}} x^q.$$

Taking $a \in S$ and $x \in N$. Then there exist integers $p_1 = p(x,a) \geq 1, l_1 = l(x,a) \geq 1, m_1 = m(x,a) > 1, q_1 = q(x,a) \geq 1$ and $r_1 = r(x,a) \geq 1$ such that

$$xa = x^{p_1}(x^{l_1} a^{m_1} x)^{r_1} x^{q_1}.$$

Choose $p_2 = p(x^{p_1}, a^{m_1}) \geq 1, l_2 = l(x^{p_1}, a^{m_1}) \geq 1, m_2 = m(x^{p_1}, a^{m_1}) > 1, q_2 = q(x^{p_1}, a^{m_1}) \geq 1$ and $r_2 = r(x^{p_1}, a^{m_1}) > 1$ such that

$$x^{p_1} a^{m_1} = x^{p_1 p_2}(x^{l_1 l_2} a^{m_1 m_2} x)^{r_1 r_2} x^{q_1 q_2}.$$

Continuing this process for an arbitrary integer t, we get $p' = p_1 p_2 ... p_t \geq 1, l' = l_1 l_2 ... l_t \geq 1, q' = q_1 q_2 ... q_t \geq 1, r' = r_1 r_2 ... r_t > 1$ and $m' = m_1 m_2 ... m_t > 1$, such that

$$xa = x^{p'}(x^{l'} a^{m'} x)^{r'} x^{q'}.$$

Since $a \in S$, we can see that $a' = a^{m_1 m_2 ... m_t} = 0$. This implies that $xa = 0$ for all $a \in S$ and $x \in N$. Clearly N is a zero-commutative, the nilpotent elements of N annihilate N on both sides and then central.

Similar arguments can be used to show that $S \subseteq Z$ if N satisfy (C_1). □

Theorem 3.2. *Let N be a $(d-g)$ nearing satisfying (C) or (C_1). If $N^2 = N$, then N is a commutative ring.*

Proof. Let N satisfy (C). In view of [1], Lemma 1, and Lemma 3.2 we obtain S is a two sided ideal. Also, by Lemma 3.3, N becomes a commutative ring.

Using the similar ways to establish N is commutative if N satisfy (C_1). □

Remark 3.1. Given condition $N^2 = N$ of Theorem 3.2 replaced by N has unity then the result follows trivially by Lemma 2 and Theorem 3.2. In other words, if R is $(s-d-g)$-nearring satisfying the same conditions, then by Theorem 3.2 and Lemma 1, N^2 is additively commutative. Hence the additive group

$(N,+)$ of $(s-d-g)$-nearing is also commutative. Thus N is a commutative ring.

Theorem 3.3. *Let N be a D-nearring satisfying (C). If idempotent elements of N are central, then N is commutative.*

Before proving this result we need the following lemmas which are proved in [9] and [1].

Lemma 3.5. *Let N be a zero symmetric D-nearrings in which for each $x \in N$ there exists a positive integer $m = m(x) > 1$ such that $x^m = x$. Then N is a commutative ring.*

Lemma 3.6. *Let N be a zero commutative nearrings. Then:*

(i) $ab = 0$ implies that $anb = 0$ for all $n \in N$.

(ii) The annihilator of any non-empty subset of N is an ideal.

Proof of Theorem 3.3. Let N satisfy (C). Clearly, N is zero commutative, and hence left and right annihilators of N coincide. If annihilator is denoted by ann(N), then by Lemma 3.2, ann(N) is an ideal. Take $a \in N$ with $a^2 = 0$. Then, for any $x \in N$ there exist integers $p = p(x,a) \geq 0, n = n(x,a) \geq 0, q = q(x,a) \geq 0$ and $r = r(x,a) > 1$ such that $xa = 0$. So, $a \in$ann(N). Being a homomorphic image of $N, N\backslash$ann(N) is a D-nearring. Next, we set positive integers p', q' and r' such that $x^2 = x^{p'+q'+r'(l+m+1)}$ this implies that $x(x - x^{p'+q'+r'(l+m+1)-1}) = 0$. Zero-commutativity in N implies that $(x - x^{p'+q'+r'(l+m+1)-1})x = 0$. Further, we found $(x - x^{p'+q'+r'(l+m+1)-1})^2 = 0$.

In view of Lemma 3.1, we observe that $N\backslash$ann(N) is a commutative ring. So, $x(xy - yx) = 0$, i.e., $xyx = x^2y,\ \ x \in N$. Now, $x^2 = x^{p'+q'+r'(l+m+1)}$ implies that $x^{p'+q'+r'(l+m+1)-2}$ is idempotent and central.

$$\begin{aligned}
yx^2 = yx^{p'+q'+r'(l+m+1)} &= yx^{p'+q'+r'(l+m+1)-2}x^2 = x^{p'+q'+r'(l+m+1)-2}yx^2 \\
&= x^{p'+q'+r'(l+m+1)-3}x^2yx = x^{p'+q'+r'(l+m+1)-2}xyx \\
&= x^{p'+q'+r'(l+m+1)-2}x^2y = x^{p'+q'+r'(l+m+1)y} = x^2y.
\end{aligned}$$

Hence we have

$$x^2y = xyx = yx^2.$$

This implies that

$$x^ty = yx^t \text{ for all } x, y \in R \text{ and all } t \geq 2. \tag{3.1}$$

This yields that

$$x^ty^{t'} = y^{t'}x^t \text{ "} x, y \in N, t \geq 2 \text{ and } t' \geq 1. \tag{3.2}$$

By hypothesis, we have $xy = x^p(x^l y^m x)^r x^q$ and $yx = y^{p_1}(y^{l_1}x^{m_1}y)^r y^{q_1}$ for some positive integers p_1, l_1, m_1, and q_1. In view of (3.1) and (3.2), we have

$$\begin{aligned} xy = x^{p+lr}y^{mr-1}(yx)x^{r+q-1} &= x^{p+lr}y^{mr-1}[y^{p_1}(y^{l_1}x^{m_1}y)^{r_1}y^{q_1}]x^{r+q-1} \\ &= x^{p+lr}y^{mr-1}(y^{p_1+l_1r_1}x^{m_1r_1}y^{r_1+q_1})x^{r+q-1} \\ &= y^{p_1+l_1r_1-1}(x^{p+lr}y^{mr}x^{r+q}x^{m_1r_1-1})y^{r_1+q_1} \\ = y^{p_1+l_1r_1-1}(x^p(x^1y^mx)^rx^q)x^{m_1r_1-1}y^{r_1+q_1} &= y^{p_1+l_1r_1-1}(xy)x^{m_1r_1-1}y^{r_1+q_1} \\ = y^{p_1+l_1r_1-1}(x^{m_1r_1}y)y^{r_1+q_1} = y^{p_1+l_1r_1}x^{m_1r_1}y^{r_1+q_1} &= y^{p_1}(y^{l_1}x^{m_1}y)^{r_1}y^{q_1} = yx. \end{aligned}$$

Hence, N is commutative. □

Remarks 3.1. Since the zero commutativity is not implied by the condition (C_1), then by using the same arguments as we have used to prove Theorem 3.3 with necessary variations, we get the following result.

Theorem 3.4. *Let N be a zero-symmetric D-nearring satisfying (C_1). If idempotent elements of N are central, then N is commutative.*

Theorem 3.5. *Let N be a D-nearring with unity satisfying (C). Then N is a commutative ring.*

Proof. Let N satisfies (C). Then by using the same argument as we have used to prove Lemma 3.4, we get $S \subseteq Z$. Since R is periodic, hence by Lemma 3.2., $(N, +)$ Abelian. Hence by definition of D-nearring N turn out to be a ring which is periodic with central nilpotent elements. Hence N is a commutative ring. □

Similarly, we can prove the following.

Theorem 3.6. *Let N be a zero-symmetric D-nearring satisfying (C_1). Then N is a commutative ring.*

4. Some Decomposition Theorems for Nearrings

In a decomposition theorem, it is known that the nearing N is the direct sum of, $S \oplus B$, the subnearrings S and B if every element of N can be expressed as a unique sum of elements $s + b, s \in S$ and $b \in B$. It is clear that, given any two nearrings S, B; a nearing N can be formed which is the direct sum of S and B by defining coordinate wise addition and multiplication on $S \times B$.

The nearring N has similar properties to the corresponding direct sum of two rings. An idem potent is an element $e \in N$ such that $e^2 = e$. The result, which asserts that: Every element $n \in N$, and e an idempotent in N, has two unique decompositions $n = en + (-en + n) = (n - en) + en$. Thus

$N = S \oplus B = B \oplus S$, where $S = \{en|n \in N\}$ and $B = \{n \in N|en = 0\}$. This result generalizes the Peirce decomposition for rings. Motivated by this observation, we say that ; a nearring N is an orthogonal sum of sub-nearrings A and B denoted by $N = A + B$ if $AB = BA = \{0\}$, and each element of R has unique representation of the form $a + b, a \in A, b \in B$ (see [5] for details).

In this context: we establish some decomposition theorems which satisfies one of the properties (C), $(C_1), (C_2), (C_3)$, (C_4) and (C_5) (see Section 3 for details of properties).

Theorem 4.1. *Let R be a nearring satisfying (C). If the idempotents of R are multiplicatively central, then A is a subnearring with $(A, +)$ Abelian and S is a subnearring with trivial multiplication and $N = A + S$.*

Theorem 4.2. *Let N be a zero-symmetric nearring satisfying (C_1). If the idempotent elements of N are multiplicatively central, then S is a subnearring with trivial multiplication, A is a subnearring with $(A, +)$ Abelian and $N = S + A$.*

Theorem 4.3. *Suppose that N is a $(d - g)$ nearring satisfying one of the conditions $(C_2), (C_3)$, (C_4) and (C_5). Then N is periodic and commutative. In otherwords, $N = M + S$, where M is a subring, S is a subnearring with trivial multiplication.*

We state the following known results.

Lemma 4.1. (see [1], Lemma (C)) *Let N be a zero-commutative nearring. Then the set S of nilpotent elements is an ideal if and only if S is a subgroup of the additive group $(N, +)$.*

Lemma 4.2. (see [7], Theorem 1) *Let N be a periodic nearring with multiplicative identity. If $S \subseteq Z$, center of N, then $(N, +)$ is Abelian.*

Lemma 4.3. (see [5], Lemma 1) *Let N be a nearring in which the idempotents are multiplicatively central. If e_1 and e_2 are any idempotents, then there exists an idempotent e_3 such that $e_3e_1 = e_1$ and $e_3e_2 = e_2$.*

The following lemmas are proved in [1], [6], [5] and [9].

Lemma 4.4. *Let N be a zero symmetric nearring satisfying the following properties:*

(i) For each $x \in N$, there exists an integer $n(x) > 1$ xuch that $x^{n(x)} = x$.

(ii) Every non trivial homomorphic image of N contains a non-zero central idempotent.

Then $(N, +)$ is commutative.

Lemma 4.5. *If N is zero-commutative periodic nearring, then $N = A+S$.*

Lemma 4.6. *Let N be a $(d - g)$ nearring such that for each $x \in N$, there exist a positive integer $n = n(x)$ and an element s in the subnearring generated by x, for which $x^n = x^n s$. If $N \subset Z$, then N is periodic and commutative.*

Lemma 4.7. *Let N is a zero commutative nearring. Then S is an ideal of N.*

Now we establish the following lemma.

Lemma 4.8. *Let N be a nearring satisfying one of the conditions (C_2), (C_3), (C_4) and (C_5). Then $NS = SN = \{0\}$.*

Proof. Let N satisfy (C_2) with $xy = 0$. Then there exist integers $m' = m(y,x) \geq 1$ and $n' = n(y,x) \geq 1$ such that $yx = y^{m'} f(yxy) y^{n'} = 0$, i.e., N is zero-commutative.

Taking y in place of x in the hypothesis (C_2), we get

$$x^2 = x^p f(x^3) x^{p'}, \text{where } p = p(x) \geq 1,\ p' = p'(x) \geq 1\,. \tag{4.1}$$

Let $a \in S$. Then making repeated use of (4.1), we have $a^2 = 0$. Moreover, for any $x \in N$ by property (C_2), we find $xa = a^r f(axa) a^{r'} = 0$. Zero-commutativity implies that $xa = 0$ for $a \in S$ and $x \in N$. This completes the proof. □

Using the similar arguments we can show that the following lemma if N satisfies (C_3), (C_4) or (C_5).

Lemma 4.5. *Let N be a zero-symmetric nearring satisfies one of the conditions (C_3), (C_4) and (C_5). Then $NS = SN = \{0\}$.*

Proof of Theorem 4.1. Without loss of generality:

We first establish that: (I) If N is a nearring satisfying the property (C), then the set S of nilpotent elements of N is an ideal.

Clearly, we observe that a nearring satisfying (C) is necessarily zero-symmetric as well as zero-commutative. Let $s \in S$ and n be an arbitrary element of N. Then there exist integers $p = p(n,s) \geq 0, q = q(n,s) \geq 0, r = r(n,s) > 1$ such that

$$ns = n^p (n^l s^m n)^r n^q\,, \tag{4.1}$$

where $l \geq 0$ and $m > 1$ are integers.

Next, choose integers $p' = p(n) \geq 0, q' = q(n) \geq 0, r' = r(n) > 1$ such that

$$n^2 = n^{p'+q'+r'(l+m+1)}\,, \tag{4.2}$$

for all $n \in N$ and $p' + q' + r'(l + m + 1) \geq 5$.

Since (4.2) gives that $s^2 = 0$ for any $s \in S$, we obtain that $s(sn) = s^2 n = 0$ and the zero-commutativity in N gives $(sn)s = 0$. Thus, by using (4.1), we find that $ns = 0$ for all $n \in N$ and also zero-commutativity of N implies $sn = 0$, for all $n \in N$, that is,

$$NS = SN = \{0\}. \tag{4.3}$$

Equation (4.3) indicates that the nilpotent elements of N annihilate N on both sides and hence, in particular, $S^2 = \{0\}$ and $S \subseteq Z$. If $a, b \in S$, then $(a-b)^2 = 0$, yields that $a - b \in S$ and S is a subgroup of the additive group $(N, +)$. By Lemma 4.1, this gives the required result.

Secondly we claim that: (II) Suppose that N is a nearring satisfying the property (C_1). If the idempotents of N are multiplicatively central, then A is a subnearring with $(A, +)$ Abelian.

Taking $a, b \in A$. Then there exist integers $u' = u(a) > 1, v' = v(b) > 1$ such that $au' = a$ and $bv' = b$. Let $w = (u'-1)v' - (u'-2) = (v'-1)u' - (v'-2) > 1$.

Then it is clear that $a^w = a$ and $b^w = b$. Note also that $e_1 = a^{w-1}$ and $e_2 = b^{w-1}$ are central idempotents in N with $e_1 a = a$ and $e_2 b = b$. Also, in view of (C_1), we find that

$$ab = (e_1 a)(e_2 b) = (e_1 e_2)(ab) = (e_1 e_2)^p ((ab)^m (e_1 e_2))^r (e_1 e_2)^q$$

for some integers $p = p(e_1 e_2, ab) \geq 0, q = q(e_1 e_2, ab) \geq 0$ and $r = r(e_1 e_2, ab) > 1$. This gives that $ab = e_1 e_2 (ab)^{mr} e_1 e_2$, so $ab \in A$. Moreover, since $N \setminus A$ has $x^n = x$ property, we have integer $i > 1$ such that

$$(a-b)i = a - b + s, \text{ where } a, b \in A \text{ and } s \in S. \tag{4.4}$$

Since e_1 and e_2 are central idempotents in N and, by Lemma 4.3, there exists an idempotent $e \in N$ such that $ee_1 = e_1$ and $ee_2 = e_2$. This implies that $ea = a$ and $eb = b$. Since (4.3) is still valid in the present situation, multiply (4.4) by e to get $(a-b)i = a-b$, and hence $a - b \in A$. Also, eN is a periodic nearring with multiplicative identity element in which nilpotent elements are multiplicative central. Thus by Lemma 4.2, $(eN, +)$ is Abelian. Therefore, $ea + eb = eb + ea$, that is $a + b = b + a$, and hence $(A, +)$ is Abelian.

Thirdly, we show that: (III) $N = S + A$.

Let $n \in N$. Then in view of (4.2), if $n^2 = ni$, where $I = p' + q' + r'(l+m+1) \geq 5$, then, clearly $n^j = n^{j+w(i-2)}$ for all $j \geq 2$ and $w \geq 1$ it follows at once that $(n^{i-1})^{i-1} = n^{i-1}$; so $n^{i-1} \in A$: it also follows that $(n - n^{i-1})^2 = 0$ and $n - n^{i-1} \in S$. Hence we can write $n = n - n^{i-1} + n^{i-1}$ and observe that $N = S + A$.

Finally, in view of (I) and (II), it remains only to show that each element of N has the unique representation in the form $s + b$ where $s \in S, a \in A$.

Taking $s + a = t + b$, where $s, t \in S$ and $a, b \in A$. Then $-t + a = b - s \in S \bigcap A = \{0\}$. This gives that $a = b$ and $s = t$. $\square$

Remark 4.1. If a nearring N satisfies (C_1), then it can be easily verified that N need not be zero-commutative. However, a zero-symmetric nearring satisfying (C_1) is necessarily zero-commutative. Hence, for a zero-symmetric nearring satisfying (C_1); conditions (I) and (II) of the proof of Theorem 4.1 may be proved easily in the same fashion.

By using similar argument used to prove Theorem 4.1, with necessary variations we can establish Theorem 4.2 and omit the details of the proof.

Proof of Theorem 4.3. Let N satisfy (C_2). Then, by using Lemma 4.7, we have $S \subseteq Z$ and $S^2 = \{0\}$. Putting y by x in (C_2) we obtain an element b in the subnearring generated by x such that $x^3 = x^3 b$. Thus, by Lemma 4.4, N is periodic and commutative. In view of Lemma 4.3, every element $x \in N$ can be expressed in the form $x = m + a$, where $m \in M$ and $a \in S$.

We first claim that M is a subring: Take $x, y \in M$ and slect positive integers $r = r(x) > 1$ and $t = t(y) > 1$ such that $x^r = x$ and $y^t = y$. Let $k = (r-1)t - (r-2) = (t-1)r - (t-2)$. Then it is obvious that $x^k = x$ and $y^k = y$. Notice that $e_1 = x^{k-1}$ and $e_2 = y^{k-1}$ are idempotents with $e_1 x = x$ and $e_2 y = y$. Using same techniqe as used in the proof of Theorem 4.1, we can write $(xy)^k = x^k y^k = xy$, thus $xy \in M$. Further, since $N \setminus S$ has $x^n = x$ property we have an integer $i > 1$ such that

$$(x-y)^i = x - y + a,\ a \in S\,. \tag{4.2}$$

In view of Lemma 4.5, we can set an idempotent e_3 for which $e_3 e_1 = e_1$ and $e_3 e_2 = e_2$. So, $e_3 x = x$ and $e_3 y = y$.

Multiplying (4.2) by e_3 we get $(x-y)^i = x - y$, that is-$x - y \in M$. Also, by Lemma 4.2, $(M, +)$ is Abelian. Thus M is a subring.

Obviously, $M \bigcap S = \{0\}$. Take $x + a = y + b$, where $x, y \in M$ and $a, b \in S$. Then $x - y = b - a \in M \bigcap S = \{0\}$. This gives $x = y$ and $b = a$ Therefore, $N = M + S$.

If N satisfies either of the properties (C_3), (C_4) or (C_5), the proof runs on the same lines as in proof of Theorem 4.3. □

Remark 4.2. Example 5.3, due to Clay [10], Section 29, (2.5), demonstrates that one cannot get a direct sum decomposition under the hypotheses of the Theorem 4.3 even in case of distributive nearrings.

5. Counterexamples

The following example shows that the centrality of idempotents in the hypotheses of Theorem 4.1 is not superfluous.

Example 5.1. Take $N = \{0, u, v, w\}$ with addition and multiplication tables defined as follows:

$+$	0	u	v	w
.	.	.	.	.
0	0	u	v	w
u	u	0	w	v
v	v	w	0	u
w	w	v	u	0

.	0	u	v	w
0	0	0	0	0
u	0	u	0	u
v	0	0	0	0
w	0	w	0	w

It is obvious to see that $(N, +, .)$ is a nearring satisfying the condition $xy = x(yxy)xx = x(yx)^2x$ for any $x, y \in N$. However, the set $A = \{0, u, w\}$ is not a sub-nearring of R.

Example 5.2. Take $N = \{0, a, b, u, v, w\}$ with addition and multiplication tables defined as follows:

$+$	0	a	b	u	v	w
0	0	a	b	u	v	w
a	a	0	w	v	u	b
b	b	v	0	w	a	u
u	u	w	u	0	b	a
v	v	b	v	a	w	0
w	w	u	a	b	0	v

.	0	a	b	u	v	w
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	b	b	b	0	0
u	0	u	u	u	0	0
v	0	0	0	0	0	0
w	0	0	0	0	0	0

Then N has only one non-trivial normal subgroup, namely $S = \{0, v, w\}$ for which $NS = \{0\} \subseteq S$. It can be easily seen that $(x + i)y - xy = S$ for all $x, y \in N$ and $i \in S$; and hence S is the only proper ideal of N. Thus $T = N \setminus S$ is the only non-trivial homomorphic image of N. Also $T \cong Z^2$, the field of order 2. Hence, N is a non-commutative D-nearring. However, N satisfies (C_1) for all $x, y \in N$ in which idempotents a, b, u are central.

Example 5.3. Take the non-Abelian group $(N, +)$, isomorphic to the symmetric group S_3 and define the multiplication in N as follows:

.	0	a	b	c	u	v
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
u	0	0	0	0	0	0
v	0	0	0	0	0	0

Then $(N, +, .)$ is a noncommutative nearring satisfying $(xyx)^2 = x^2y^2x^2 = xy$ for all $x, y \in N$. Hence, $M = \{0, a\}$ is not an ideal.

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