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ON THE NUMERICAL RESOLUTION OF THE SYSTEMS OF NONLINEAR EQUATIONS USING A DESCENT LIKE-METHOD

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Abstract: We consider a system of nonlinear equations. We study the numerical resolution using a descent-like method. Based on an analysis of the way for choosing the step size, we present an algorithm which exploits better the results of the calculus, contributing to a decrease in the number of accomplished iterations necessary to approximate the exact solution of the above system within a given accuracy. Numerical examples solved using this algorithm are also presented.

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1. Introduction

In practice almost all occuring functional relations are nonlinear. There are still important phenomena such as saturation, solution branching, chaos, etc. and engineering problems like chosing the best one from a family of curves to fit data provided from some sample population, the optimal shape of a nuclear fusion reactor, etc. which can only be described by nonlinear models requiring the solution of nonlinear equations.

Solving nonlinear equations is a major task of numerical analysis. Even though it is difficult if not even impossible to exhibit the solution of such equa-

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tions in explicit form, numerical calculation provides means, where a solution may be found, or at least approximated as closely as desired.

We consider the following system of nonlinear equations written as canonical form

$$F(x) = 0\,, \tag{1}$$

where $F : D \subseteq \mathrm{R}^n \to \mathrm{R}^n$, D a convex and compact set, $x = (x_1, x_2, ..., x_n)^t$, $F = (f_1, f_2, ..., f_n)^t$, the component functions f_i, $i = 1, 2, ..., n$ mapping D into the real line R.

Observation 1. *(1) is equivalent with*

$$\begin{cases} f_1(x_1, x_2, ..., x_n) = 0\,, \\ f_2(x_1, x_2, ..., x_n) = 0\,, \\ \vdots \\ f_n(x_1, x_2, ..., x_n) = 0\,. \end{cases} \tag{2}$$

The analysed mathematical problem can be specified as follows: a vector $x^* \in D \subseteq \mathrm{R}^n$ is to be determined so that all component equations $f_i(x^*) = 0$, $i = 1, 2, ..., n$ are solved simultaneously.

In the sequel we consider the following hypotheses of work:

(H_1) $x^* \in D$ is an isolated solution for the problem (1);

(H_2) F is continuously differentiable on D;

(H_3) F is strictly convex on D.

Let the function $\Phi : D \to \mathbf{R}_+$ given by

$$\Phi(x) = \sum_{i=1}^{n} f_i^2(x_1, ..., x_n)\,. \tag{3}$$

Observation 2. *The solution of (1), or (2), corresponds to a global minimum of the function* Φ.

2. Problem Statement

An approximate descent-like method is usually iterative and has the following form Burden [2], Carasso [3], Ueberhuber [11]:

Beginning with a starting value $x^{(0)} \in D$, successive approximates $x^{(i)} \in D$, $i = 1, 2, ...$ to x^* are computed by using the following expression

$$x^{(i+1)} = x^{(i)} + \overline{\rho}\, d^{(i)}\,, \quad i = 0, 1, ...\,, \tag{4}$$

where

$$\begin{cases} d^{(i)} \text{ is a descent vector;} \\ \overline{\rho} \text{ satisfies the minimum problem :} \\ \min\limits_{\rho \geq 0} p(\rho) = p(\overline{\rho}), \;\; p(\rho) = \Phi\left(x^{(i)} + \rho\, d^{(i)}\right). \end{cases}$$

Observation 3. *With each iterration* $\Phi(x^{(i+1)}) < \Phi(x^{(i)})$.

The descent methods require the evaluation only of $\Phi(x)$ and $\nabla\Phi(x)$ and their total computing time is much less than with Newton's method. In addition these methods seek to obtain convergence for a large set of initial values $x^{(0)}$, Atkinson [1], Ortega [7].

Taking into consideration that the number of the accomplished iterations are proportional to the amount of work involved, to the computational cost, we are interested in using a descent-like iteration in order to determine approximately the solutions of a nonlinear system of equations, for a given accuracy requirement.

Definition 1. (Osborne [8]) One says that a set of vectors $d^{(i)}$ are downhill for a function Φ at the points $x^{(i)}$, if there exist a δ independent of i, such that

$$\frac{< \nabla\Phi(x^{(i)}),\, d^{(i)} >}{\| \nabla\Phi(x^{(i)}) \| \| d^{(i)} \|} \leq \delta \; < 0\,. \tag{5}$$

Concerning the descent-like methods, it is known that the direction and amount of greatest uphill slope is the gradient of the function, which is always perpendicular to the contour line of a contour map. The direction and amount of greatest downhill slope is then the negative of the gradient, Hostetter [4], Kelley [5].

Observation 4. *The downhill vector* $d^{(i)}$ *can be chosen as the negative of the gradient* $\nabla\Phi(x^{(i)})$, *or as a vector whose angle with the negative of the gradient* $\nabla\Phi(x^{(i)})$ *remains essentially under* $\pi/2$.

Concerning the descent-like methods the main question is the choice of the coefficient $\overline{\rho}$, knowing that he determines the magnitude of the descent step at the current iteration, being important for the convergence of the method, Hostetter [4], Ostrowski [9], Nougier [6], Reinboldt [10]. If $\overline{\rho}$ is too large, a minimum will be repeatedly overshot, with the search making large excursions back and forth past the minimum. If $\overline{\rho}$ is too small, convergence to a relative minimum will be exceedingly slow and in consequence the amount of work involved to approach the solution within a given accuracy is significant.

3. The Step Choice

At each iterration i, one has to solve the following problem:

$$\min_{\rho \geq 0} p(\rho) = p(\overline{\rho}) \,,$$

where

$$\begin{cases} p(\rho) = \Phi(x^{(i)} + \rho\, d^{(i)}); \\ \text{the descent vector } d^{(i)} \text{ satisfies (5).} \end{cases}$$

Proposition 1. *The step $\overline{\rho}$ is characterized by:*

$$(-\delta)\frac{\parallel \nabla\Phi(x^{(i)}) \parallel}{\parallel H_\Phi(x^{(i)}) \parallel\parallel d^{(i)} \parallel} \leq \overline{\rho} \leq \Big(-\frac{1}{\delta}\Big)\frac{\Phi(x^{(i)})}{\parallel \nabla\Phi(x^{(i)}) \parallel\parallel d^{(i)} \parallel}\,, \tag{6}$$

where $\big(H_\Phi(x)\big)_{ij} = \dfrac{\partial^2\Phi(x)}{\partial x_i \partial x_j}$, $1 \leq i, j \leq n$, *represents the Hessien matrix of* Φ.

Proof. $\overline{\rho}$ represents the minimum for the function $p(\rho)$. Thus

$$\frac{dp}{d\overline{\rho}} = 0 \Leftrightarrow < \nabla\Phi(x^{(i)} + \overline{\rho} d^{(i)}),\, d^{(i)} >= 0 \Leftrightarrow$$

$$< \nabla\Phi(x^{(i)}),\, d^{(i)} > + \overline{\rho} < H_\Phi(x^{(i)}) d^{(i)},\, d^{(i)} >= 0 \Rightarrow$$

$$\overline{\rho} = -\frac{< \nabla\Phi(x^{(i)}),\, d^{(i)} >}{< H_\Phi(x^{(i)}) d^{(i)},\, d^{(i)} >} \geq (-\delta)\frac{\parallel \nabla\Phi(x^{(i)}) \parallel}{\parallel H_\Phi(x^{(i)}) \parallel\parallel d^{(i)} \parallel}\,.$$

On the other hand, imposing that $\Phi(x^{(i)} + \overline{\rho} d^{(i)}) = 0$ and using a development limited at first order, it follows:

$$\overline{\rho} = -\frac{\Phi(x^{(i)})}{< \nabla\Phi(x^{(i)}),\, d^{(i)} >} \leq \Big(-\frac{1}{\delta}\Big)\frac{\Phi(x^{(i)})}{\parallel \nabla\Phi(x^{(i)}) \parallel\parallel d^{(i)} \parallel}\,. \qquad \square$$

Proposition 2. *Suppose the function p being continuous differentiable, bounded below and strictly convex. Let $0 < m_d < m_s < 1$. Then, the step $\overline{\rho}$ lies in the following non empty interval:*

$$I = \{\rho \geq 0 \mid p(0) + m_s\, p'(0)\rho \leq p(\rho) \leq p(0) + m_d\, p'(0)\rho\}\,.$$

Proof. Following the property of convexity of the function p, the set

$$D = \{\rho \geq 0 \mid p(0) + p'(0)\rho \leq p(\rho) \leq p(0)\}$$

is non empty. Thus

$$\overline{\rho} \in D \Leftrightarrow \overline{\rho} \in \{\rho \geq 0 \mid p(\rho) = p(0) + \alpha\, p'(0)\rho, \quad \alpha \in [0,1]\}\,.$$

In order to choose an approximation of the step $\overline{\rho}$ not too large, such that the function Φ decreases sufficiently from $x^{(i)}$ to $x^{(i+1)}$, one set $m_d \in]0,1[$ and one imposes:

$$\overline{\rho} \in D_1 = \{\rho \in D \mid p(\rho) \leq p(0) + m_d\, p'(0)\rho\} \Leftrightarrow$$
$$\overline{\rho} \in \{\rho \geq 0 \mid p(\rho) = p(0) + (\beta m_d + 1 - \beta)\, p'(0)\rho,\ \beta \in]0,\ 1]\}\,. \tag{7}$$

Taking also into consideration the fact that the approximation ρ should not be too small such that $x^{(i+1)}$ should be sufficiently different from $x^{(i)}$, one has to impose that $p'(0) \neq p'(\rho)$.

In consequence taking $m_s \in]0,\ 1[$ one gets:

$$\overline{\rho} \in D_2 = \{\rho \in D_1 \mid p(\rho) \geq p(0) + m_s\, p'(0)\rho\}\,. \tag{8}$$

(7) and (8) yield

$$\overline{\rho} \in \{\rho \geq 0 \mid p(0) + m_s\, p'(0)\rho \leq p(\rho) \leq p(0) + m_d\, p'(0)\rho\}$$

with $0 < m_d < m_s < 1$.

The property of the function p to be unbounded below insures the unicity of the researching minimum. □

Corollary 1. *Let $\rho \geq 0$. If $\rho \notin D_1$ there is an interval non empty, D_L, at the left side of ρ such that $D_l \subset D_1$. If $\rho \in D_1$ and $\rho \notin D_2$, there is an interval non empty, D_r, at the right side of ρ such that $D_r \subset D_2$.*

Proof. Let

$$(d):\ \ y(x) = p(0) + m_d\, p'(0)x$$

be the equation of the line passing to point $(0,\, p(0))$ and having the slope $m_d\, p'(0)$. Following this line the interval is divided in two regions:

$$D_1 = \{\rho \geq 0 \mid p(\rho) \leq y(\rho)\} \Leftrightarrow \{\rho \geq 0 \mid p(\rho) \leq p(0) + m_d\, p'(0)\rho\}\,,$$
$$D \setminus D_1 = \{\rho \geq 0 \mid p(\rho) > y(\rho)\} \Leftrightarrow \{\rho \geq 0 \mid p(\rho) > p(0) + m_d\, p'(0)\rho\}\,.$$

If an arbitrary $\rho \geq 0$ lies in the second set, i.e.

$$p(\rho) \geq p(0) + m_d\, p'(0)\rho\,.$$

It follows immediately that at the left side of ρ there are elements from the first set D_1.

The proof is analogue for D_2. □

4. Practical Implementation

Based on the above analysis we can construct the following descent-like algorithm for the numerical resolution of the systems of nonlinear equations:

1. Let $x^{(i)}$ and $d^{(i)}$ be known values; set $m_d \in]0;\, 1[$, $m_s \in]m_d;\, 1[$, ρ, $\rho_l = 0$, $\rho_r = 0$;

2. Set $x = x^{(i)} + \rho\, d^{(i)}$;

3. If $p(\rho) \leq p(0) + m_d\, p'(0)\, \rho$ then

3.1. if $p(\rho) \geq p(0) + m_s\, p'(0)\, \rho$ then

3.1.1 $x^{(i+1)} = x$;

3.1.2 stop.

3.2 $\rho_l = \rho$; go to Step 2 with a ρ obtained by interpolation or by extrapolation of ρ_l and ρ_r;

4. $\rho_r = \rho$; go to Step 2 with a ρ obtained by interpolation or by extrapolation of ρ_l and ρ_r.

Observation 5. *The above algorithm is applied until the stopping criterion has reached, i.e.*

$$\| x^{(i+1)} - x^{(i)} \| \leq \varepsilon\,, \quad \textit{or} \quad \frac{\| x^{(i+1)} - x^{(i)} \|}{\| x^{(i+1)} \|} \leq \varepsilon\,.$$

$\varepsilon > 0$ being the prescribed accuracy. For this purpose, any convenient norm can be used.

One concludes that $x^* = x^{(i+1)}$, within the accuracy ε.

5. Numerical Examples

Example 4.1. We consider the following system of nonlinear equations:

$$\begin{cases} 15x_1 + x_2^2 - 4x_3 - 13 = 0\,, \\ x_1^2 + 10x_2 - x_3 - 11 = 0\,, \\ x_2^3 - 25x_3 + 22 = 0\,, \end{cases}$$

$$D = \{(x_1, x_2, x_3) \big|\, x_1, x_2, x_3 \in [-3; 3]\}\,.$$

For the accuracy $\varepsilon = 5e-4$, using the presented algorithm, one obtains:

starting values $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$	approximated solution (x_1, x_2, x_3)	number of iterations
(2; 1; 1)	(1.0364064; 1.0856838; 0.9311912)	21
(-2.; -1; 0)	(1.0363817; 1.0857046; 0.9311873)	17
(-3; -2; -1)	(1.0363983; 1.0857100; 0.9311861)	18
(3; -2; 2)	(1.0363989; 1.0857102; 0.9311864)	23
(-3; 3; 3)	(1.0364166; 1.0856638; 0.9311688)	29

Using the steepest descent algorithm, within the same accuracy, one obtains:

starting values $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$	approximated solution (x_1, x_2, x_3)	number of iterations
(2; 1; 1)	(1.0364162; 1.0856574; 0.9311817)	25
(-2; -1; 0)	(1.0364162; 1.0856574; 0.9311817)	27
(-3; -2; -1)	(1.0364145; 1.0856544; 0.9311904)	31
(3; -2; 2)	(1.0364118; 1.0856780; 0.9311741)	33
(-3; 3; 3)	(1.0364091; 1.0856661; 0.9312008)	39

Observation 6. *The exact solution $x^* \in D$ is given by* $x^* = (1.0364006;\ 1.0857065;\ 0.9311914)^t$.

Example 4.2. We consider the following system of nonlinear equations:

$$\begin{cases} x_1 + x_2^2 - 2x_2x_3 - 0.1 = 0\,, \\ x_2^2 - x_2^2 - 3x_1x_3 + 0.2 = 0\,, \\ x_3 + x_3^2 + 2x_1x_2 - 0.3 = 0\,, \end{cases}$$

$$D = \{(x_1, x_2, x_3) \big|\, x_1 \in [-1; 2];\ x_2 \in [-1; 1],\ x_3 \in [0; 2]\}\,.$$

For the accuracy $\varepsilon = 2e-3$, using the presented algorithm, one obtains:

starting values $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$	approximated solution (x_1, x_2, x_3)	number of iterations
(2; 0.5; 0.5)	(0.0114509; -0.1768680; 0.2445945)	10
(-0.5; 0; 0)	(0.0128757; -0.1780509; 0.2445531)	9
(-0.5; 1; 2)	(0.0112747; -0.1779036; 0.2451864)	10
(1; 0; 1)	(0.0135881; -0.1791872; 0.2454590)	6

Using the steepest descent algorithm, within the same accuracy, one obtains:

starting values $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$	approximated solution (x_1, x_2, x_3)	number of iterations
$(2; 0.5; 0.5)$	(0.0125847; -0.1785199; 0.2447789)	15
(-0.5; 0; 0)	(0.0122886; -0.1774749; 0.2446872)	14
(-0.5; 1; 2)	(0.0118828; -0.1771471; 0.2446047)	19
(1; 1; 0)	(0.0140142; -0.1785640; 0.2447554)	10

Observation 7. *The exact solution $x^* \in D$ is given by* $x^* = (0.0128242; -0.1778007; 0.2446881)^t$.

6. Conclusions

In this paper we give a descent-like algorithm for the numerical resolution of the systems of nonlinear equations, based on an analysis of the step size $\overline{\rho}$, which determines the magnitude of the descent step at each iteration.

We characterize this step as function of the iterated function Φ, the norm of his gradient and his Hessian matrix, and the associated downhill vector d. We present the set of admissible steps $\overline{\rho}$.

The numerical experiments performed by practical implementation of the above algorithm lead to the conclusion that this one offers a fast tool, from the point of view of the number of accomplished iterations necessary to approximate the exact solution x^*, within a prescribed accuracy ε.

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