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PROFINITE CROSSED n-FOLD EXTENSIONS

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Abstract: In this paper following F. Korkes and T. Porter [3] we will define the category of profinite crossed n-fold extensions.

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1. Introduction

The definition of a crossed complex was first introduced under the name "group system" by Blaker and systematically used by Whitehead [4]. More recently Brown ang Higgins have studied over a groupoid [1]. Crossed $n-$fold extensions were defined by J. Huesbschmann in [2].

It is obvious that one should adapt theory of crossed $n-$fold extensions for other algebraic structures such as profinite groups. Pro-algebraic structures occur in problems relating to number theory, commutative algebras, algebraic geometry and algebraic topology. Altough the category of profinite commutative algebras forms a natural extension of the category of finite commutative

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algebras. It carries a richer structures in that it has categorical objects and constructions which do not exist in finite case; e.g projective limits and free products.

2. Profinite Crossed Modules

Profinite crossed modules of groups were first introduced by E.J. Korkes and T. Porter in [3] as below.

Let R be a profinite group. *A pre-crossed module of profinite group* is an profinite R-module C, together with a continuous R-module morphism

$$\partial : C \longrightarrow R,$$

such that for all $c \in C$, $r \in R$:

$CM1)\qquad \partial(r \cdot c) = r(\partial c)r^{-1}.$

This is *a crossed module* if in addition, for all $c, c' \in C$,

$CM2)\qquad \partial c \cdot c' = c(c')c^{-1}.$

This second condition is called the *Peiffer identity.* We denote such a profinite crossed module by (C, R, ∂). Clearly any profinite crossed module is a profinite pre-crossed module.

A standard example of a profinite crossed module is any closed normal subgroup N in R giving an inclusion map the image $N = \partial C$ of C is an closed normal subgroup in R.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of profinite group morphisms,

$$\theta : C \longrightarrow C', \quad \psi : R \longrightarrow R',$$

such that

$$\theta(r \cdot c) = \psi(r) \cdot \theta(c) \ \text{ and } \ \partial'\theta(c) = \psi\partial(c).$$

In this case, we will say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity.

3. Profinite Crossed $n-$Fold Extensions

Let Q be a profinite group and A a $Q-$module. A profinite crossed $n-$fold extension of A by Q $(n \geq 1)$ is an exact sequence

$$0 \to A \stackrel{\chi}{\to} C_{n-1} \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} G \to Q \to 1$$

of profinite groups (all maps are continuous group homomorphisms) with the following properties:

(i) (C_1, G, ∂_1) is a profinite crossed module.

(ii) C_k is a profinite $Q-$ module, $1 < k < n$, and ∂_k and χ are $Q-$linear.

Note that, since the kernel of ∂_1 is a profinite $Q-$module, ∂_2 is $Q-$linear.

Let e, e' be two profinite crossed $n-$fold extensions. A morphism (α, β, ψ) : $e \to e'$ of profinite crossed $n-$fold extensions consists of continuous group homomorphisms $\psi : Q \to Q'$, $\beta_0 : G \to G'$, $\beta_k : C_k \to C'_k$, $1 < k < n$, and $\alpha : A \to A'$ such that

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & A & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta_{n-1}} & & & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & A' & \longrightarrow & C'_{n-1} & \longrightarrow & \cdots & \longrightarrow & C'_1 & \longrightarrow & G' & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

is a commutative diagram in ${}^{PG}\mathbf{Grp}$; here ${}^{PG}\mathbf{Grp}$ denotes the category of profinite groups on which PG acts continuously on the left, and the continuous action of G on all profinite groups occuring in the diagram is defined in the obvious way. So we have category $PE^n(Q, A)$ of profinite crossed $n-$fold extensions of A by Q.

References

[1] R. Brown, P.J. Higgins, Homotopies and tensor products for ω-groupoids and crossed complexes, *J. Pure and Appl. Algebra*, **47** (1987) 1-33.

[2] J. Huesbschmann, Crossed $n-$fold extensions of groups and cohomology, *Comment. Math. Helvetici*, **55** (1980), 302-314.

[3] K.J. Korkes, T. Porter, *Pro-C Crossed Modules*, University of Wales, U.V.N.W Pure Maths Preprints.

[4] J.H.C. Whitehead, Combinatorial homotopy II, *Bull. Amer. Math. Soc.*, **55** (1949), 213-245.