International Journal of Pure and Applied Mathematics

Volume 25 No. 4 2005, 531-540

REGULARITY OF FEEDBACK AND ERIVATIVE FEEDBACK STRATIFICATION

Mª Isabel García-Planas[1 §], Sonia Tarragona[2]

[1,2]Departament de Matemàtica Aplicada, I
Universitat Politècnica de Catalunya
C. Minería 1, Esc C, 1º-3ª
Barcelona, 08038, SPAIN
[1]e-mail: maria.isabel.garcia@upc.edu

Abstract: In the space M of triples of matrices $(E, A, B) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C})$ representing singular time-invariant linear systems $E\dot{x} = Ax + Bu$ we consider the equivalence relation defined by the action of Lie group $\mathcal{G} = Gl(n; \mathbb{C}) \times \mathbb{C} \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ acting on M,

$$\begin{aligned} \alpha : \mathcal{G} \times M &\longrightarrow M\,, \\ ((P, Q, R, U, V), (E, A, B)) &\longrightarrow (QEP + QBU, P^{-1}AP + QBV, QBR)\,. \end{aligned}$$

We prove that the partition by discrete invariants of the space M is a finite constructible stratification. Other regularity properties are also proved.

AMS Subject Classification: 15A21, 93B52
Key Words: triples of matrices, singular systems, standardizable systems, feedback, derivative feedback, orbits, strata

1. Introduction

Let M be the differentiable manifold of triples of matrices (E, A, B), where $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, which represent singular time-invariant linear systems in the form

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

We will consider in M the equivalence relation that we call "feedback-

Received: Ocotber 17, 2005

§Correspondence author

equivalence". The triples (E, A, B) and (E', A', B') are said to be feedback-equivalent in the casew, here

$$(E', A', B') = (QEP + QBU, QAP + QBV, QBR) \qquad (2)$$

for some $P, Q \in Gl(n; \mathbb{C})$, $R \in \mathbb{C}$, $U, V \in M_{1\times n}(\mathbb{C})$. In a matrix form:

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \left(\begin{smallmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{smallmatrix}\right)$$

In the case where $n = 2$ and $m = 1$, a canonical reduced form describing the equivalence classes is given.

Proposition 1. *Let $(E, A, B) \in M$. Then, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $R \in \mathbb{C}$, $U, V \in M_{1\times n}(\mathbb{C})$ in such a way that this triple is equivalent to one of*

1. $\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$
2. $\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) \ \forall a \in \mathbb{C}$
3. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$
4. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$
5. $\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right) \ \forall \mu_1, \mu_2 \in \mathbb{C}$
6. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$
7. $\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu & 0 \\ 1 & \mu \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right) \ \forall \mu \in \mathbb{C}$
8. $\left(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right) \ \forall \lambda \in \mathbb{C}$
9. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$
10. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$
11. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mu \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right) \ \forall \mu \in \mathbb{C}$
12. $\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$
13. $\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$
14. $\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right)$

Now, for all n and m, restricting ourselves the set M to the open and dense set S of triples defining standardizable systems (triples such that by means a derivative feedback may be standardized, i.e. there exists $U \in M_{m\times n}(\mathbb{C})$ such that $E + BU$ is an invertible matrix), one dispose also, of canonical reduced form representing equivalence classes.

Proposition 2. *Let $(E, A, B) \in S$. Then, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $R \in \mathbb{C}$, $U, V \in M_{1\times n}(\mathbb{C})$ in such a way that this triple is equivalent to (I_n, A_1, B_1), where (A_1, B_1) is a pair of matrices in its Brunovsky-Kronecker reduced form.*

Remark 1. The standardizable character is invariant under equivalence relation considered (see García-Planas [3]).

2. Orbits

The orbits are equivalence classes of triples of matrices under the relation of feedback-equivalence:

$$\mathcal{O}(E, A, B) = \{(E_1, A_1, B_1) = (QEP + QBU, QAP + QBV, QBR)\},$$

$\forall P, Q \in Gl(n;\mathbb{C}), R \in Gl(m;\mathbb{C}), U, V \in M_{m\times n}(\mathbb{C})$.

The equivalence relation may be seen as induced by Lie group action. Let us consider the following Lie group $\mathcal{G} = Gl(n;\mathbb{C}) \times Gl(n;\mathbb{C}) \times Gl(m;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ acting on M. The product $\star$ in $\mathcal{G}$ is given by

$$\begin{aligned}(P_1, Q_1, R_1, U_1, V_1) \star (P_2, Q_2, R_2, U_2, V_2)\\ = (P_1P_2, Q_2Q_1, R_1R_2, U_1P_2 + R_1U_2, V_1P_2 + R_1V_2)\end{aligned} \quad (3)$$

being $e_2 = (I_n, I_n, I_m, 0, 0)$ its unit element.

The action is defined as follows:

$$\begin{aligned}\alpha : \mathcal{G} \times M &\longrightarrow M\,,\\ ((P, Q, R, U, V), (E, A, B)) &\longrightarrow\\ &(QEP + QBU, QAP + QBV, QBR)\,.\end{aligned} \quad (4)$$

Proposition 3. *Any orbit is a constructible subset of M.*

Proof. Given a triple $(E, A, B) \in M$ we take $\alpha_{(E,A,B)} : \mathcal{G} \longrightarrow M$ to be the mapping $(P, Q, R, U, V) \longrightarrow (QEP + QBU, QAP + QBV, QBR)$. The orbit through $(E, A, B) \in M$ is the image of the constructible set $\mathcal{G} \subset M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_m(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$. □

Corollary 1. *The orbits are embedded submanifolds of M.*

Proof. Any constructible set has at least one non-singular point. However the orbits verify the following homogeneity property: given any two points on one orbit there is a diffeomorphism of M mapping the one point to the other and preserving orbits. Then every point is non-singular. Consequently an orbit is a non-singular constructible set, hence every orbit is a manifold. □

For a triple $(E, A, B) \in M$, we denote by

$$T_{(E,A,B)} = \{(EP + QE + BU, AP + QA + BV, BR + QB)\}$$

for all $P, Q \in M_n(\mathbb{C}), U, V \in M_{m\times n}(\mathbb{C}), R \in M_m(\mathbb{C})$, the tangent space at (E, A, B) to the orbit through (E, A, B).

Now we will use the description of the orthogonal complementary subspace to the tangent space to orbit for explicitly obtaining miniversal deformations.

First, we recall the definition of versal deformations. Let M be a smooth manifold.

Definition 1. Let $\mathcal{U}_0$ be a neighborhood of the origin of $\mathbb{C}^\ell$. A *deformation* $\varphi(\lambda)$ of x_0 is a smooth mapping

$$\varphi : \mathcal{U}_0 \longrightarrow M$$

such that $\varphi(0) = x_0$. The vector $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{U}_0$ is called the parameter vector.

The deformation $\varphi(\lambda)$ is also called *differentiable family* of elements of M.

Let $\mathcal{G}$ be a Lie group acting smoothly on M. We denote the action of $g \in \mathcal{G}$ on $x \in M$ by $g \circ x$.

Definition 2. The deformation $\varphi(\lambda)$ of x_0 is called *versal* if any deformation $\varphi'(\xi)$ of x_0, where $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{U}'_0 \subset \mathbb{C}^k$ is the parameter vector, can be represented in some neighborhood of the origin as

$$\varphi'(\xi) = g(\xi) \circ \varphi(\phi(\xi)), \quad \xi \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \tag{5}$$

where $\phi : \mathcal{U}''_0 \longrightarrow \mathbb{C}^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0)$ is the identity element of $\mathcal{G}$. Expression (5) means that any deformation $\varphi'(\xi)$ of x_0 can be obtained from the versal deformation $\varphi(\lambda)$ of x_0 by an appropriate smooth change of parameters $\lambda = \phi(\xi)$ and an equivalence transformation $g(\xi)$ smoothly depending on parameters.

A versal deformation having minimal number of parameters is called *miniversal.*

The following result was proved by Arnold [1], in the case where $\mathrm{Gl}(n;\mathbb{C})$ acts on $M_{n\times n}(\mathbb{C})$, and was generalized by Tannenbaum [5] in the case where a Lie group acts on a complex manifold. It provides the relationship between a versal deformation of x_0 and the local structure of the orbit.

Theorem 1. (Tannenbaum) *1. A deformation $\varphi(\lambda)$ of x_0 is versal if and only if it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.*

2. Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in $\mathcal{M}$, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.

Let $\{v_1, \dots, v_\ell\}$ be a basis of any arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ (for example, $(T_{x_0}\mathcal{O}(x_0))^\perp$).

Corollary 2. *The deformation*

$$x : \mathcal{U}_0 \subset \mathbb{C}^\ell \longrightarrow \mathcal{M}, \quad x(\lambda) = x_0 + \sum_{i=1}^{\ell} \lambda_i v_i \tag{6}$$

is a miniversal deformation.

In the case where $n = 2$ and $m = 1$, an explicit miniversal deformation for any triple is given. Recall that, the homogeneity property of the orbits permits us to take the triple in its canonical reduced form.

Proposition 4. *Let $(E, A, B) \in M$ in its canonical reduced form. A miniversal deformation$\Gamma = (E, A, B) + \{(X, Y, Z)\}$, is given by: if (E, A, B) is as $i)$ in Proposition 1.1.*

1. $i = 1$, *the orbit is structurally stable*

2. $i = 2$, *by the family of triples:* $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y_1 & a+y_3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$.

3. $i = 3$, *by the family of triples:* $\left(\begin{pmatrix} 0 & 0 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

4. $i = 4$, *by the family of triples:* $\left(\begin{pmatrix} 0 & 0 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y_2 & y_4 \end{pmatrix}, \begin{pmatrix} 1 \\ z_2 \end{pmatrix}\right)$.

5. $i = 5$, *by the family of triples:* $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu_1+y_1 & 0 \\ 0 & \mu_2+y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

6. $i = 6$, *by the family of triples:* $\left(\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

7. $i = 7$, *by the family of triples:* $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu+y_4 & y_3 \\ 1 & \mu \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

8. $i = 8$, *by the family of triples:* $\begin{cases} a) \text{ if } \lambda = 0: \left(\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right). \\ b) \text{ if } \lambda \neq 0: \left(\begin{pmatrix} \lambda+x_1 & 0 \\ 0 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right). \end{cases}$

9. $i = 9$, *by the family of triples:* $\left(\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

10. $i = 10$, *by the family of triples:* $\left(\begin{pmatrix} x_1 & x_3 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

11. $i = 11$, *by the family of triples:* $\left(\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y_1 & y_3 \\ y_2 & \mu+y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

12. $i = 12$, *by the family of triples:* $\left(\begin{pmatrix} x_1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

13. $i = 13$, *by the family of triples:* $\left(\begin{pmatrix} 0 & 1 \\ x_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y_2 & y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

14. $i = 14$, *by the family of triples:* $\left(\begin{pmatrix} 0 & 1 \\ x_2 & 0 \end{pmatrix}, \begin{pmatrix} y_1 & 0 \\ y_2 & 1 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

Finally, for all n and m, and for (E, A, B) in S, we have the following proposition.

Proposition 5. *Let $(E, A, B) \in S$ in its canonical reduced form. Then, a miniversal deformation is given by $(E, A, B) + \{(0, Y, Z)\}$, where $(A, B) + \{(Y, Z)\}$ is a miniversal deformation of the pair of matrices (A, B) under block-similarity.*

Explicit miniversal deformations for pairs of matrices are presented in Garcí-Planas [2].

3. The Strata

The space M of all triples of matrices is formed by the disjoint union of all orbits of the triples and the frontier of each orbit is formed by orbits of strictly lower dimension. Given two triples (E_i, A_i, B_i) in M, we can ask when the closure of $\mathcal{O}(E_1, A_1, B_1)$ includes the closure of $\mathcal{O}(E_2, A_2, B_2)$. A necessary condition is $\dim T_{(E_1,A_1,B_1)} > \dim T_{(E_2,A_2,B_2)}$.

We remark that this partition is not finite (as example, for $n = 2, m = 1$ in a neighborhood of any triple in the orbits with canonical reduced form 2,5,7,8-b or 11, there are infinite orbits the same type, i.e. varying only in the continuous invariant).

In order to obtain a finite partition preserving the orbit structure, we group the orbits with the same type, we call this set stratum in M. There are only finitely many strata, each an uncountable union of orbits or a unique orbit partitioning M.

Now we are going to describe in M with $n = 2, m = 1$, the collection of all strata:

1. $\mathcal{E}_1 = \mathcal{O}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$
2. $\mathcal{E}_2 = \bigcup_a \mathcal{O}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$
3. $\mathcal{E}_3 = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$
4. $\mathcal{E}_4 = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$
5. $\mathcal{E}_5 = \bigcup_{\mu_i} \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
6. $\mathcal{E}_6 = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
7. $\mathcal{E}_7 = \bigcup_{\mu} \mathcal{O}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 1 & \mu \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
8. $\mathcal{E}_{8-a} = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
9. $\mathcal{E}_{8-b} = \bigcup_{\lambda \neq 0} \mathcal{O}\left(\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
10. $\mathcal{E}_9 = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
11. $\mathcal{E}_{10} = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
12. $\mathcal{E}_{11} = \bigcup_{\mu} \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
13. $\mathcal{E}_{12} = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
14. $\mathcal{E}_{13} = \mathcal{O}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$
15. $\mathcal{E}_{14} = \mathcal{O}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$.

Proposition 6. *Any stratum is a constructible and connected subset of M.*

Proof. The strata 1, 3, 4, 6, 8-a, 9, 10, 12, 13, 14 are orbits. For strata 2, let (E, A, B) in canonical reduced form with parameter $a \in \mathbb{C}$, and we consider the map $\rho_2 : \mathcal{G} \times \mathbb{C} \longrightarrow M$ such that $\rho_2((P, Q, R, U, V), a) = \alpha((P, Q, R, U, V), (E, A, B))$. The domain $\mathcal{G} \times \mathbb{C}$ is constructible in $\mathbb{C}^{2n^2+m^2+2nm} \times \mathbb{C}$, and the mapping is regular rational, so by Chevalley's theorem its image (stratum) is constructible. □

Note also that this stratum is connected since the set $\mathcal{G} \times \mathbb{C}$ is connected and ρ_2 is continuous.

Analogously, we prove that the strata 5, 7, 8-b and 11 are also constructible and connected.

Lemma 1. *Let $\varphi_1 : \Lambda \longrightarrow M$ be a deformation of (E,A,B) minitransversal to the orbit $\mathcal{O}(E,A,B)$ given in Section 2. Let $V \subset \mathcal{G}$ a subvariety minitransversal to* $\mathrm{St}(E,A,B) = \{(P,Q,R,U,V) \in \mathcal{G} \mid \alpha((P,Q,R,U,V),(E,A,B)) = (E,A,B)\}$. *Then:*

$$\begin{aligned}&\beta : \Lambda \times V \longrightarrow M\,,\\ &(\lambda,(P,Q,R,U,V)) \longrightarrow Q\begin{pmatrix}E(\lambda) & A(\lambda) & B(\lambda)\end{pmatrix}\left(\begin{smallmatrix}P&0&0\\0&P&0\\U&V&R\end{smallmatrix}\right)\,,\end{aligned}$$

with $\lambda = (\lambda_1,\dots,\lambda_s)$ and $(E(\lambda),A(\lambda),B(\lambda)) = \varphi_1(\lambda)$, is a diffeomorphism at $(0,I)$.

Proof. The inverse function theorem ensures that β is a local diffeomorphism at $(0,I)$, if and only if $d\beta_{(0,I)}$ is a diffeomorphism. □

Taking into account that $\dim(V\times\Lambda) = 2n^2 + mn = \dim M$, it suffices to observe that $d\beta$ is surjective.

Lemma 2. *Let (E,A,B) be a triple in M, $\mathcal{O}(E,A,B)$ its orbit, $\mathcal{E}_i$ its stratum, and Γ the variety transversal to the orbit considered in proposition 2.2. Then, in a neighborhood of (E,A,B), $\mathcal{E}_i$ is a subvariety regular en (E,A,B) if and only if $\mathcal{E}_i\cap\Gamma$ is.*

Proof. Suppose $\mathcal{E}_i$ regular at (E,A,B). Taking into account that Γ is transversal to $\mathcal{O}_1(E,A,B)$, it also is transversal to $\mathcal{E}_i$. Then, $\mathcal{E}_i\cap\Gamma$ is regular at (E,A,B).

Conversely, suppose $\mathcal{E}_i\cap\Gamma$ regular at (E,A,B). The local triviality given in Lemma 1, we have

$$\mathcal{E}_i = \beta((\mathcal{E}_i\cap\Gamma)\times V)$$

locally in (E,A,B). Then $\mathcal{E}_i$ is regular at (E,A,B). □

Now we analyze $\mathcal{E}_i\cap\Gamma$.

Proposition 7. *Let $(E,A,B)\in\mathcal{E}_i$ for some i.*

i) If $(X,Y,Z)\neq(0,0,0)$, then $(E,A,B)+(X,Y,Z)\notin\mathcal{O}(E,A,B)$.

So, for 1, 3, 4, 6, 8-a, 9, 10, 12, 13, 14, $(X,Y,Z)=(0,0,0)$.

ii) $(E,A,B)+(X,Y,Z)\in\mathcal{E}_i(E,A,B)$ if and only if:

a) Case $i=2$: for $y_1=0$ and for all y_3.

b) Case $i=5$: for $z_1=z_2=0$ and for all y_1,y_4.

c) Case $i=7$: for $z_1=z_2=0$ and $y_4^2+4y_3=0$.

d) Case $i=8-b$: for $x_4=z_1=z_2=0$ and for all x_1.

e) Case $i=11$: for $x_1=y_1=y_2=y_3=z_1=z_2=0$.

Theorem 2. *The strata are submanifolds of M.*

Proof. Let $(E,A,B)\in M$.

It is obvious for strata $\mathcal{E}_i$ with 1, 3, 4, 6, 8-a, 9, 10, 12, 13, 14.

We are going to proof for strata $i = 2, 5, 7, 8-b, 11$. The homogeneity of the orbits we can suppose the triples (E, A, B) in its canonical reduced form.

By Lemma 3.2 it suffices to prove that $\mathcal{E}(E, A, B) \cap \Gamma$, is regular at (E, A, B).

For each $\mathcal{E}_i$-stratum we define the map $\phi_{\mathcal{E}_i} : \mathbb{C}^{n_i^2} \longrightarrow M$ in the following manner:

for $i = 2$

$$\begin{array}{rl} \phi_{\mathcal{E}_2} : \mathbb{C}^1 & \longrightarrow M\,, \\ y_3 & \longrightarrow \left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & a+y_3 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \right). \end{array}$$

for $i = 5$

$$\begin{array}{rl} \phi_{\mathcal{E}_5} : \mathbb{C}^2 & \longrightarrow M\,, \\ (y_1, y_4) & \longrightarrow \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu_1+y_1 & 0 \\ 0 & \mu_2+y_4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \right). \end{array}$$

for $i = 7$

$$\begin{array}{rl} \phi_{\mathcal{E}_7} : \mathbb{C} & \longrightarrow M\,, \\ y_4 & \longrightarrow \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mu+y_4 & -\frac{y_4^2}{4} \\ 1 & \mu \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \right). \end{array}$$

for $i = 8-b$

$$\begin{array}{rl} \phi_{\mathcal{E}_{8-b}} : \mathbb{C} & \longrightarrow M\,, \\ x_1 & \longrightarrow \left(\left(\begin{smallmatrix} \lambda+x_1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \right). \end{array}$$

for $i = 11$

$$\begin{array}{rl} \phi_{\mathcal{E}_{11}} : \mathbb{C} & \longrightarrow M\,, \\ y_4 & \longrightarrow \left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mu_4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \right). \end{array}$$

For each i the map ϕ_i is differentiable and in a neighborhood $\mathcal{U}_i \subset \mathbb{C}^{n_i}$ of origin is, in fact, a regular parametrization of $\mathcal{E}_i \cap \Gamma$, then each $\mathcal{E}_i$ is a differentiable manifold. □

4. Regularity Properties

We recall the definition of Whitney regularity. Take first the case when M is a vector space. Let X, Y submanifolds of M with dimension p and q respectively, and let $x \in X$. We say that Y is Whitney regular over X at x when the following condition holds: Let (x_i), (y_i) be sequences in X, Y respectively, both converging to x with $x_i \neq y_i$ for all i. We consider the line (L_i) spanned by $x_i - y_i$, and T_i the tangent space $T_{y_i}Y$. If (L_i) converges to L (in the Grassmannian of 1-dimensional subspaces of M) and (T_i) converges to T (in the Grassmannian of q-dimensional subspaces of M) then $L \subseteq T$. It is easy to prove that the Whitney regularity is invariant under diffeomorphisms, so the notion can be defined when M is a manifold. We say that Y is Whitney regular over X when it is so at every point in X.

Finally, let $\mathcal{E}$ be a stratification of a subset V of a manifold, and let $\mathcal{E}_i$ be a stratum. We say that the stratification $\mathcal{E}$ is Whitney regular over $\mathcal{E}_i$ when every stratum $\mathcal{E}_j$ is Whitney regular over $\mathcal{E}_i$.

We show first that the stratification $\mathcal{E}$ defined in Section 3, is Whitney regular over any stratum $\mathcal{E}_i$ with 1, 3, 4, 6, 8-a, 9, 10, 12, 13, 14. Recall that this strata are orbits and posses the following homogeneity property: given two points in $\mathcal{E}_i$ there exists a diffeomorphism of M which maps the one point to the other preserving strata. It follows that if the stratification is Whitney regular over one point in $\mathcal{E}_i$, then it is Whitney regular over every point in $\mathcal{E}_i$. Now it suffices to apply the Whitney Theorem.

Theorem 3. (Whitney) *Let X, Y be constructible submanifolds of M such that $X \cap \overline{Y} \neq \emptyset$. Then, there exists a point $x \in X$, such that Y is Whitney regular over X at x.*

In order to analyze the strata $\mathcal{E}_i$ with $i = 2, 5, 7$, 8-b, 11 we have the following lemma

Lemma 3. *Let (E, A, B) be a triple and $\mathcal{E}_i$ its stratum. The stratification $\mathcal{E}$ is Whitney regular over $\mathcal{E}_i$ at (E, A, B), if and only if $\mathcal{E} \cap \Gamma$ is Whitney regular over $\mathcal{E}_i \cap \Gamma$ at (E, A, B).*

Proof. Necessity. the inclusion map $\Gamma \longrightarrow M$ is transverse to the stratification on M.

Sufficiency. If $\bigcup(\mathcal{E}_i \cap \Gamma)$ is Whitney regular over $\mathcal{E}_i \cap \Gamma$ in (E, A, B), the stratification $\bigcup(\mathcal{E}_i \cap \Gamma) \times V$ also is Whitney regular over $(\mathcal{E}_i \cap \Gamma) \times V$ en (E, A, B). Now it suffices to consider the diffeomorphism β defined in Lemma 3.1. $\square$

Now, we are going to proof the Whitney regularity of $\mathcal{E}$ over $\mathcal{E}_2$ at $(E, A, B) \in \mathcal{E}_2$.

We can suppose (E, A, B) in its canonical reduced form. The variety Γ at this point is, in a small neighborhood U, is $\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}y_1&a+y_3\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)$ so

$$\mathcal{E} \cap \Gamma = \begin{cases} \mathcal{E}_1 \cap \Gamma = \left\{\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}y_1&a+y_3\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right),\ y_1 \neq 0\right\}, \\ \mathcal{E}_2 \cap \Gamma = \left\{\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&a+y_3\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)\right\}. \end{cases}$$

Let

$$(E_1, A_1, B_1) = \left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&a+y_1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right), \quad (E_2, A_2, B_2) = \left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&a+y_2\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right)$$

be two elements in $\mathcal{E}_2 \cap \Gamma$, and we define the map

$$\begin{aligned} h : U &\longrightarrow U, \\ (E, A, B) &\longrightarrow (E, A + Y, B), \end{aligned}$$

with $Y = \left(\begin{smallmatrix} 0 & y_2 - y_1 \\ 0 & 0 \end{smallmatrix}\right)$. Obviously, h is a diffeomorphism such that $h(E_1, A_1, B_1) = (E_2, A_2, B_2)$ preserving strata.

Then, we have the following proposition.

Proposition 8. *The stratification $\mathcal{E}$ is Whitney regular over $\mathcal{E}_2$ at (E, A, B).*

Analogously we can prove the regularity of the de la stratification over the strata $i = 5, 7, 8 - b, 11$.

Finally, for all n and $m = 1$, the induced stratification in $\mathcal{E} \cap S$, is Whitney regular because of the stratification of the space of pairs of matrices under block-similarity is Whitney regular (see García-Planas [2]).

References

[1] V.I. Arnold, On matrices depending on parameters, *Russian Math. Surveys*, **26**, No. 2 (1971), 29-43.

[2] Mª Isabel García-Planas, *Estudio Geométrico de Familias Diferenciables de Parejas de Matrices*, Tesis Doctoral, Barcelona (1994).

[3] Mª I. García-Planas, Standardization of a descriptor system under discrete derivative feedback, In: *The Book Mathematics and Simulation with Biological, Economical and Musicoacoustical Applications*, World Scientific and Engineering Society Press (2001), 45-50.

[4] C.G. Gibson, Regularity of the Segre stratification, *Math. Proc. Camb. Phil. Soc.*, **80** (1976), 91-97.

[5] A. Tannenbaum, *Invariance and System Theory: Algabraic and Geometric Aspects*, Lecture Notes in Math., **845**, Springer-Verlag (1981).