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ON THE MULTIPLE INTEGER KNAPSACK POLYHEDRA

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Abstract: In this paper we investigate the generation of valid inequalities for the multiple integer knapsack problem using the knowledge of the polyhedra structure of sets with two integer variables. We start by giving a complete characterization of the multiple integer knapsack polyhedra with two variables using the link between Hilbert bases and the coefficients of the facet-defining inequalities of such polyhedra. We prove that the number of faces of such polyhedra is polynomial and every such face can be obtained in polynomial time on the input data by presenting a polynomial algorithm that computes all the important information. The generation of valid inequalities for the multiple integer knapsack with $n > 2$ is discussed and a computational experience is reported.

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1. Introduction

The general multiple integer knapsack problem (MIKP) is known to be a strong NP-hard problem (see [5]). However, when the number of variables is fixed the problem becomes polynomial [10] and, when that number is small we may hope to find a complete polyhedral description of the convex hull of the corresponding feasible sets. Our approach consists in exploring the polyhedral structure of the MIKP with two integer variables in order to generate valid inequalities for the general case.

The multiple 2-integer knapsack problem (M2IKP) can be solved in polynomial time. The single constraint case was first considered by Hirschberg and Wong [8]. Kannan [9] developed the first polynomial time algorithm for the M2IKP with positive coefficients. Scarf [12] studied the case where no restriction on the sign of the coefficients is considered. Lenstra [10] proved that every integer programming problem can be solved in polynomial time once the number of variables is fixed. Since then several algorithms have been designed to improve the time complexity to solve the M2IKP, most of them trying to specialize the ideas proposed by Lenstra [10]. The most recent algorithm that we have knowledge is a linear-time algorithm from Eisenbrand and Laue [3]. In that paper a survey on the time complexity of the previous algorithms is presented. Our approach consists in the characterization of the convex hull of the feasible set of the M2IKP with non-negative integer coefficients. Doing so, we try to find properties on the coefficients of the faces of such polyhedra that allow us to construct valid inequalities for the general multiple integer knapsack polyhedra.

In Section 2 we summarize some properties of the coefficients of the facet-defining inequalities of such polyhedra. In particular, we summarize some results relating those coefficients with Hilbert bases and best approximations. In fact several polynomial algorithms developed to solve this problem use these coefficients. In Section 3 we present two algorithms that compute all the facets sequentially. These algorithms are based on a partition of a cone on the first quadrant onto several pointed cones and generate the integral Hilbert basis associated with each one. The second algorithm is a polynomial version of the first one. As an output we obtain a polyhedral characterization of the convex hull of the feasible set of the M2IKP. Finally, In Section 4 we discuss the lifting of the facet-defining inequalities for these polyhedra in order to obtain valid inequalities for the multiple integer knapsack set with $n > 2$ variables and report some computational results.

2. Best Approximations and Hilbert Bases

In this section we summarize results that will be essential for the description of the convex hull of the feasible set of the M2IKP. Hirschberg and Wong [8] established the link between the coefficients of the facet-defining inequalities of the single integer knapsack set with continued fractions. Weismantel [15] considered the single binary knapsack set with two different coefficients and proved that the coefficients of the facet-defining inequalities formed certain Hilbert bases. Given a real number α a fraction x/y in its reduced form is called a best upper (lower) approximation of α if $\frac{a}{b} - \alpha > \frac{x}{y} - \alpha$ (resp. $\alpha - \frac{a}{b} > \alpha - \frac{x}{y}$) for all $a/b > \alpha$ (resp. $a/b < \alpha$) with $1 \le b \le y$. In order to understand the sequences of best approximations as sequences of convergents of reduced regular continued fractions see [4]. Next we present an algorithm that computes the best approximations following [8].

Algorithm BA

Step 0. $k \leftarrow 1, (p_1^k, p_2^k) \leftarrow (\left\lfloor \frac{a_2}{a_1} \right\rfloor, 1), \ell \leftarrow 1, (q_1^\ell, q_2^\ell) \leftarrow (\left\lceil \frac{a_2}{a_1} \right\rceil, 1), r \leftarrow 1.$

Step 1. Repeat

Set $r_\le(p_1^k, p_2^k) \leftarrow -a_1 p_1^k + a_2 p_2^k$, $r_\ge(q_1^\ell, q_2^\ell) \leftarrow a_1 q_1^\ell - a_2 q_2^\ell$;

(i) if $r_\le(p_1^k, p_2^k) > r_\ge(q_1^\ell, q_2^\ell)$ set $k \leftarrow k+1, (p_1^k, p_2^k) \leftarrow (p_1^{k-1}, p_2^{k-1}) + r(q_1^\ell, q_2^\ell)$;

(ii) if $r_\le(p_1^k, p_2^k) < r_\ge(q_1^\ell, q_2^\ell)$ set $\ell \leftarrow \ell+1, (q_1^\ell, q_2^\ell) \leftarrow (q_1^{\ell-1}, q_2^{\ell-1}) + r(p_1^k, p_2^k)$;

(iii) if $r_\le(p_1^k, p_2^k) = r_\ge(q_1^\ell, q_2^\ell)$ set $k \leftarrow k+1, (p_1^k, p_2^k) \leftarrow (p_1^{k-1}, p_2^{k-1}) + r(q_1^\ell, q_2^\ell)$; $\ell \leftarrow \ell+1, (q_1^\ell, q_2^\ell) \leftarrow (q_1^{\ell-1}, q_2^{\ell-1}) + r(p_1^{k-1}, p_2^{k-1})$;

until $r_\le(p_1^k, p_2^k) = r_\ge(q_1^\ell, q_2^\ell) = 0$.

The coefficients computed by Algorithm BA satisfy, $q_1^1/q_2^1 > q_1^2/q_2^2 > \cdots > q_1^{n_2}/q_2^{n_2} = \mu = p_1^{n_1}/p_2^{n_1} > \cdots > p_1^2/p_2^2 > p_1^1/p_2^1$.

We denote by $k(\ell)$ the index of the pair (p_1, p_2) used in (ii) of Step 1 to obtain (q_1^ℓ, q_2^ℓ), that is $(q_1^\ell, q_2^\ell) = (q_1^{\ell-1}, q_2^{\ell-1}) + (p_1^{k(\ell)}, p_2^{k(\ell)})$. Similarly, we denote by $\ell(k)$ the index of the pair (q_1, q_2) used in (i) of Step 1 to obtain (p_1, p_2). Let n_1 and n_2 denote the number of distinct pairs (p_1, p_2) and (q_1, q_2) generated, respectively. Now we summarize some properties of these coefficients.

Proposition 1. *The coefficients computed by Algorithm BA satisfy (i) for all $j \in \{1, \ldots, n_2\}$, q_1^j/q_2^j is a best upper approximation of μ, (ii) for all $j \in \{1, \ldots, n_1\}$, p_1^j/p_2^j is a best lower approximation of μ.*

Notice that this algorithm is not polynomial since the situations $r_\le(p_1, p_2) \ge r_\ge(q_1, q_2)$ and $r_\le(p_1, p_2) < r_\ge(q_1, q_2)$ may occur in consecutive iterations.

To obtain a polynomial algorithm steps (i) and (ii) must be changed to: (i) If $r_{\leq}(p_1^k, p_2^k) > r_{\geq}(q_1^\ell, q_2^\ell)$, then $k \leftarrow k+1$, $p_1^k \leftarrow p_1^{k-1} + r \times q_1^\ell$, $p_2^k \leftarrow p_2^{k-1} + \rho \times q_2^\ell$, where $\rho = \lfloor r_{\leq}(p_1^k, p_2^k)/r_{\geq}(q_1^\ell, q_2^\ell) \rfloor$. (ii) If $r_{\leq}(p_1^k, p_2^k) < r_{\geq}(q_1^\ell, q_2^\ell)$, then $\ell \leftarrow \ell + 1$, $q_1^\ell \leftarrow q_1^{\ell-1} + \rho \times p_1^k$, $q_2^\ell \leftarrow q_2^\ell + \rho \times p_2^k$, where $\rho = \lfloor r_{\geq}(q_1^\ell, q_2^\ell)/r_{\leq}(p_1^k, p_2^k) \rfloor$.

This modified algorithm is in fact the Euclidean algorithm. It is well known that it runs in polynomial time (see [13]).

Another important notion is that of Hilbert bases. Weismantel [15] called the vectors (p_1, p_2) exchange vectors and proved they formed an integral Hilbert basis. A finite set of vectors $H = \{h_1, \ldots, h_t\}$, is called an *Hilbert basis* if each integer vector v in Cone$\{H\}$ can be obtained as a non-negative integer linear combination of the vectors $h_1, \ldots, h_t$. If the vectors of the basis are integer then H is called an *integer Hilbert basis*. It is known (see [13]) that each rational polyhedral cone C is generated by an integer Hilbert basis. Moreover, if C is pointed there is an unique minimal (with respect to inclusion) integer Hilbert basis generating C which will be denoted by $\mathcal{H}(C)$. In fact, the vectors computed by Algorithm BA also form integral Hilbert basis (see [15]).

Proposition 2. *$H_{\leq} = \{(p_1^1, p_2^1), \ldots, (p_1^{n_1}, p_2^{n_1})\}$ and $H_{\geq} = \{(q_1^1, q_2^1), \ldots, (q_1^{n_2}, q_2^{n_2})\}$ are integral Hilbert bases.*

For the two dimension some important properties relating Hilbert bases hold. Sebö [14] makes several conjectures stating that these properties also hold in higher dimensions and proves them for dimensions ≤ 3. Some of these conjectures have been disproved. Next we state some of these properties that will be used later.

Theorem 3. *Given a integral pointed cone $C \in \mathbb{Z}^2$ and an integral Hilbert basis $\mathcal{H}$ generating C there is an unimodular partition of C, $P^j = cone\{h_j, h_{j+1}\}, j \in J$ where $\mid det(h_j, h_{j+1}) \mid = 1$, formed by the elements of $\mathcal{H}$. That is (i) $C = \bigcup_{j \in J} P^j$; (ii) $P^i \cap P^j$ is a face of P^i and P^j, $\forall i, j \in J$.*

These properties have a geometric interpretation using klein polyhedra (see [4]). A fraction x/y is a best upper (lower) approximation of α if and only if (x, y) lies on the upper (lower) convex envelope of $\{(x, y) : y \geq \alpha x, (x, y) \neq (0, 0)\}$ $(\{(x, y) : y \leq \alpha x, (x, y) \neq (0, 0)\})$. For any rational pointed cone in $\mathbb{R}^2$ the integral points lying on the lower convex hull of Conv$(C \cap \mathbb{Z}^2 \setminus \{0\})$ forms an integral Hilbert basis (see [7]).

Another intuitive way to understand the properties of the coefficients generated by Algorithm BA is to notice that both the sequences of fractions $q_1/q_2 - \lfloor \frac{a_2}{a_1} \rfloor$ and $p_1/p_2 - \lfloor \frac{a_2}{a_1} \rfloor$ are subsequences of the Farey sequence for $n = p_2^{n_1} = q_2^{n_2}$ (see [11]). The fractions that occur in the Farey sequences are always in their reduced form. Every reduced fraction x/y such that $0 \leq x/y \leq 1$

with $y \leq n$ occurs in the sequence for n. If x/y and x'/y' are two consecutive fractions in the sequence for n then (i) $x'y - xy' = 1$; (ii) $y + y' > n$; (iii) considering all the fractions with value between x/y and x'/y', $(x+x')/(y+y')$ is the unique fraction with lower denominator. This last property implies that $\{(x, y), (x', y')\}$ is an integral Hilbert basis of $\text{Cone}\{(x, y), (x', y')\}$.

3. The Multiple 2-Integer Knapsack Problem

In this section we consider the M2IKP: $\min\{\pi_1 y_1 + \pi_2 y_2 : (y_1, y_2) \in Y_M\}$ where $Y_M = \{(y_1, y_2) \in \mathbb{N}_0^2 : a_1^i y_1 + a_2^i y_2 \leq D^i,\ i = 1, \ldots, m\}$. We assume that all the coefficients are positive integers and denote by Q_M the convex hull of Y_M. We construct an algorithm that sequentially generates all the extreme points. This information is enough to describe the facets of Q_M. Notice that the fact that the extreme points are obtained sequentially is essential to describe Q_M. We suppose that every constraint is non-redundant for the linear relaxation of Y_M and suppose that its coefficients satisfy $\frac{a_2^1}{a_1^1} < \frac{a_2^2}{a_1^2} < \cdots < \frac{a_2^m}{a_1^m}$. Thus, there exist $m+1$ points, $t_1 > t_2 > \cdots > t_m > t_{m+1} = 0$ such that, if (y_1, y_2) belongs to $\{(y_1, y_2) \in \mathbb{N}_0^2 : t_{j+1} \leq y_1 \leq t_j\}$ and satisfies $a_1^j y_1 + a_2^j y_2 \leq D^j$ then (y_1, y_2) satisfies $a_1^i y_1 + a_2^i y_2 \leq D^i$, for all $i = 1, \ldots, m$. That is, constraint j dominates all other constraints on $[t_{j+1}, t_j]$. Kannan [9] presents an algorithm with complexity $\mathcal{O}(m^2)$, that sorts the constraints in this form.

The key point of our algorithm is to compute an integral Hilbert basis generating the integral vectors in $C = \text{Cone}\{(\lfloor \frac{a_2^1}{a_1^1} \rfloor, 1), (1, 0)\}$. To do that we start by covering C with pointed cones: two cones associated with the same constraint i, $\text{Cone}\{(\lfloor \frac{a_2^i}{a_1^i} \rfloor, 1), (\frac{a_2^i}{a_1^i}, 1)\}$, $\text{Cone}\{(\frac{a_2^i}{a_1^i}, 1), (\lceil \frac{a_2^i}{a_1^i} \rceil, 1)\}$, a cone considering two consecutive constraints (possible the empty set) $\text{Cone}\{(\lceil \frac{a_2^i}{a_1^i} \rceil, 1), (\lfloor \frac{a_2^{i+1}}{a_1^{i+1}} \rfloor, 1)\}$ and a final cone, $\text{Cone}\{(\lceil \frac{a_2^m}{a_1^m} \rceil, 1), (1, 0)\}$. Then generate the Hilbert basis associated with each one of the previous cones. In the final step of the algorithm the extreme points of Q_M are generated in sequence by testing the vectors from the Hilbert basis.

For each constraint i we consider the ordered set C^i of the vectors generated by Algorithm BA: $C^i = \{(p_1^{i,1}, p_2^{i,1}), \ldots, (p_1^{i,n_1(i)}, p_2^{i,n_1(i)}), (q_1^{i,n_2(i)},\ q_2^{i,n_2(i)}), \ldots, (q_1^{i,1}, q_2^{i,1})\}$, where $n_1(i)$ (respectively, $n_2(i)$) are the number of vectors (p_1, p_2) (respectively, (q_1, q_2)) generated by Algorithm BA for the coefficients of constraint i. Notice that these vectors satisfy: $\frac{p_1^{i,1}}{p_2^{i,1}} < \cdots < \frac{p_1^{i,n_1(i)}}{p_2^{i,n_1(i)}} = \frac{a_2^i}{a_1^i} =$

$\frac{q_1^{i,n_2(i)}}{q_2^{i,n_2(i)}} < \cdots < \frac{q_1^{i,1}}{q_2^{i,1}}$. As $\frac{a_2^1}{a_1^1} < \frac{a_2^2}{a_1^2} < \cdots < \frac{a_2^m}{a_1^m}$ then $\frac{p_1^{1,1}}{p_2^{1,1}} = \left\lfloor \frac{a_2^1}{a_1^1} \right\rfloor \leq \frac{p_1^{2,1}}{p_2^{2,1}} = \left\lfloor \frac{a_2^2}{a_1^2} \right\rfloor \leq \cdots \leq \frac{p_1^{m,1}}{p_2^{m,1}} = \left\lfloor \frac{a_2^m}{a_1^m} \right\rfloor$. Let C be the ordered set whose elements belong to $C^1 \cup C^{1,2} \cup C^2 \cup \cdots \cup C^m \cup \{(0,1)\}$, where set $C^{i,i+1}$ is obtained as $C^{i,i+1} = \{(p_1^{i,1}+1,1),(p_1^{i,1}+2,1),\ldots,(p_1^{i+1,1}-1,1)\}$, if $p_1^{i+1} > p_1^i + 1$, and it is the empty set otherwise. We denote by (v_1^s, v_2^s) the s^{th} element of this ordered set such that $\frac{v_1^s}{v_2^s} \leq \frac{v_1^r}{v_2^r}$ for all s, r with $r > s$. From the discussion in Section 2 we have the following result.

Lemma 4. *Let $J = \{1,\ldots,\mid C \mid -1\}$ and let $P^j = \text{Cone}\{(v_1^s, v_2^s), (v_1^{s+1}, v_2^{s+1})\}$, $s \in J$. Then $P^j, j \in J$ is an unimodular partition of C.*

Next we develop an algorithm (denoted by MK1) that generates all extreme points of Q_M. This algorithm starts with a trivial extreme point (a point with one of its coordinates equal to zero) and, in sequence, it generates all other extreme points until it reaches the other trivial extreme point, the point with the other coordinate equal to zero.

Algorithm MK1

Initialization: $k \leftarrow 1$; $s \leftarrow 1$; $(y_1^k, y_2^k) \leftarrow (\lfloor D^1/a_1^1 \rfloor, 0)$.
Compute $C = \{(v_1^i, v_2^i), i = 1,\ldots,c\}$ with $c = \mid C \mid$.
While $y_1^k > 0$ and $s < c$ do
(a) Set $\gamma^i(y_1^k, y_2^k) = D^i - a_1^i y_1^k - a_2^i y_2^k$, $r_{\leq}^i(v_1^s, v_2^s) = -a_1^i v_1^s + a_2^i v_2^s$, for $i = 1,\ldots,m$.
(b) $I = \{i \in \{1,\ldots,m\} : \frac{a_2^i}{a_1^i} > \frac{v_1^s}{v_2^s}\}$.
(c) Set $n \leftarrow \min\{\lfloor y_1^k / v_1^s \rfloor, \min_{i \in I}\{\lfloor \gamma^i(y_1^k, y_2^k)/r_{\leq}^i(v_1^s, v_2^s) \rfloor\}\}$.
(d) If $n > 0$ then set $(y_1^{k+1}, y_2^{k+1}) \leftarrow (y_1^k, y_2^k) + n(-v_1^s, v_2^s)$, $k \leftarrow k+1$.
(e) $s \leftarrow s + 1$.

Example 5. Consider the integer set of those points $(y_1, y_2) \in \mathbb{N}_0^2$, satisfying two knapsack constraints $R1 : 8y_1 + 15y_2 \leq 202$, $R2 : 4y_1 + 27y_2 \leq 302$. Using Algorithm BA we obtain $C^1 = \{(1,1),(3,2),(5,3),(7,4),(9,5),(11,6),(13,7),(15,8),(2,1)\}$; $C^2 = \{(6,1),(13,2),(20,3),(27,4),(7,1)\}$; $C^{1,2} = \{(3,1),(4,1),(5,1)\}$. Consider the ordered set obtained from the union of these sets: $C = \{(1,1),(3,2),(5,3),(7,4),(9,5),(11,6),(13,7),(15,8),(2,1),(3,1),(4,1),(5,1),(6,1),(13,2),(20,3),(27,4),(7,1),(1,0)\}$. Algorithm MK1 starts at $(25,0)$ and, testing the vectors in C, generates the extreme points $(14,6)$, $(6,10)$, $(1,11)$, $(0,11)$ (see the following table).

s	1	2	3	4	5	6	7	8	9
(y_1, y_2)	(25,0)	(25,0)	(25,0)	(25,0)	(25,0)	(14,6)	(14,6)	(14,6)	(6,10)
(v_1, v_2)	(1,1)	(3,2)	(5,3)	(7,4)	(9,5)	(11,6)	(13,7)	(15,8)	(2,1)
(γ^1, γ^2)	(2,202)	(2,202)	(2,202)	(2,202)	(2,202)	(0,84)	(0,84)	(0,84)	(4,8)
$(r^1_{\leq}, r^2_{\leq})$	(7,23)	(6,42)	(5,61)	(4,80)	(3,99)	(2,118)	(1,137)	(0,156)	(-1,19)
I	{1,2}	{1,2}	{1,2}	{1,2}	{1,2}	{1,2}	{1,2}	{2}	{2}
n	0	0	0	0	0	1	0	0	1

s	10	11	12	13	14	15	16	17	18
(y_1, y_2)	(6,10)	(6,10)	(1,11)	(1,11)	(1,11)	(1,11)	(1,11)	(1,11)	(0,11)
(v_1, v_2)	(3,1)	(4,1)	(5,1)	(6,1)	(13,2)	(20,3)	(27,4)	(7,1)	(1,0)
(γ^1, γ^2)	(4,8)	(4,8)	(29,1)	(29,1)	(29,1)	(29,1)	(29,1)	(29,1)	(37,5)
$(r^1_{\leq}, r^2_{\leq})$	(-9,15)	(-17,11)	(-25,7)	(-33,3)	(-74,2)	(-115,1)	(-156,0)	(-41,-1)	(-8,-4)
I	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}
n	0	0	1	0	0	0	0	0	1

In this case $\mathrm{Conv}(Y) = \{(y_1, y_2) \in \mathbb{R}^2_0 : 6y_1 + 11y_2 \leq 150, y_1 + 2y_2 \leq 26, y_1 + 5y_2 \leq 56, y_2 \leq 11, y_1 \geq 0, y_2 \geq 0\}$.

Lemma 6. *Consider the s-th iteration of Step 3, and let (y_1^k, y_2^k) be the point generated in the previous iteration with $n > 0$ (if $s = 1$ consider $(\lfloor D^1/a_1^1 \rfloor, 0)$ obtained for $k = 1$). If there exists $(y_1, y_2) \in Y_M$ with $y_1 < y_1^k$ and $y_2 > y_2^k$ then the vector $(c, d) = (y_1^k - y_1, y_2 - y_2^k)$ belongs to $R = \mathrm{Cone}\{(1,0), (v_1^s, v_2^s)\}$, where (v_1^s, v_2^s) is the vector in C tested in that iteration.*

Proof. As $c > 0$ and $d > 0$ the vector (c, d) belongs to the first quadrant. Thus we must show that it satisfies $\frac{c}{d} \geq \frac{v_1^s}{v_2^s}$. If $s = 1$ then $(v_1^s, v_2^s) = (p_1^{1,1}, p_2^{1,1})$ and $k = 1$. There cannot exist (y_1, y_2) such that $(c, d) = (y_1^k - y_1, y_2 - y_2^k) \notin R$ because, otherwise, (c, d) would satisfy $0 < \frac{c}{d} < \lfloor \frac{a_2^1}{a_1^1} \rfloor \Rightarrow \frac{c+1}{d} \leq \lfloor \frac{a_2^1}{a_1^1} \rfloor \leq \frac{a_2^1}{a_1^1} \Rightarrow -ca_1^1 + da_2^1 \geq a_1^1 \Rightarrow r^1_{\leq}(c, d) \geq 1 > \gamma^1(y_1^1, y_2^1)$. Thus (y_1^1, y_2^1) would violate constraint with index 1. Suppose that Lemma 6 does not hold. Let k be the index of the first point and let s be the index of the first vector such that $(c, d) \notin \mathrm{Cone}\{(1,0), (v_1^s, v_2^s)\}$, where $(c, d) = (y_1^k - y_1, y_2 - y_2^k)$. To prove that it cannot occur $\frac{c}{d} \leq \frac{v_1^{s-1}}{v_2^{s-1}}$ we consider two cases: (i) $(y_1^k, y_2^k) = (y_1^{k-1}, y_2^{k-1}) + t(-v_1^{s-1}, v_2^{s-1}), t \geq 1$ and integer, (ii) either $k = 1$ or $(y_1^k, y_2^k) = (y_1^{k-1}, y_2^{k-1}) + t(-v_1^{s-p}, v_2^{s-1}), t \geq 1, k > 1, p > 1$. Case (i) occurs when (y_1^k, y_2^k) is obtained using (v_1^{s-1}, v_2^{s-1}) and Case (ii) occurs otherwise. In this last case the contradiction to Lemma 6 would have occurred with indexes k and $s - 1$. Therefore we consider Case (i).

As $(y_1, y_2) = (y_1^k, y_2^k) + (-c, d)$ then $(y_1, y_2) = (y_1^{k-1}, y_2^{k-1}) + (-c', d')$ where $(-c', d') = (y_1, y_2) - (y_1^{k-1}, y_2^{k-1}) = (y_1^k, y_2^k) + (-c, d) - (y_1^{k-1}, y_2^{k-1}) = (-c - t \times v_1^{s-1}, d + t \times v_2^{s-1})$. Noticing that (y_1, y_2) belongs to Y_M and $\frac{c'}{d'} < \lfloor \frac{v_1^{s-1}}{v_2^{s-1}} \rfloor$, we conclude that the contradiction would have occurred with (y_1^{k-1}, y_2^{k-1}) and (c', d') that satisfies $\frac{c'}{d'} < \frac{v_1^{s-1}}{v_2^{s-1}}$. Thus $\frac{v_1^{s-1}}{v_2^{s-1}} < \frac{c}{d} < \frac{v_1^s}{v_2^s}$.

Now consider the case $\frac{v_1^{s-1}}{v_2^{s-1}} < \frac{c}{d} < \frac{v_1^s}{v_2^s}$. Let $j = \max\{k \in \{1,\ldots,m\} : \frac{v_1^{s-1}}{v_2^{s-1}} > \frac{a_2^k}{a_1^k}\}$. We know that $(y_1^k, y_2^k) + (-v_1^{s-1}, v_2^{s-1})$ is not feasible. The case in which the unique inequality that is not satisfied is $y_1 \geq 0$ leads to an absurd since, from Lemma 4, $(c,d) = \beta_1(v_1^{s-1}, v_2^{s-1}) + \beta_2(v_1^s, v_2^s)$ with $\beta_1 \geq 1, \beta_2 \geq 1$ and $\beta_1, \beta_2 \in \mathbb{Z}$, thus, $c > v_1^s$. Suppose that there exists a constraint $i \in \{1,\ldots,m\}$ that is not satisfied. The case $i \leq j$ cannot occur because $\frac{v_1^{s-1}}{v_2^{s-1}} > \frac{a_2^i}{a_1^i}$ and, as (y_1^k, y_2^k) satisfies constraint i, then it is also satisfied by $(y_1^k, y_2^k) + (-v_1^{s-1}, v_2^{s-1})$ because $a_1^i(y_1^k - v_1^{s-1}) + a_2^i(y_2^k + v_2^{s-1}) = a_1^i y_1^k + a_2^i y_2^k - a_1^i v_1^{s-1} + a_2^i v_2^{s-1} < a_1^i y_1^k + a_2^i y_2^k < D^i$. Therefore, $i > j$. In this case, using Lemma 4 that ensures $(c,d) = \beta_1(v_1^{s-1}, v_2^{s-1}) + \beta_2(v_1^s, v_2^s)$ with $\beta_1 \geq 1, \beta_2 \geq 1$ and β_1, β_2 integers, and noticing that $\frac{a_2^i}{a_1^i} \geq \frac{v_1^s}{v_2^s} \Rightarrow a_2^i v_2^s \geq a_1^i v_1^s$ and $\frac{a_2^i}{a_1^i} > \frac{v_1^{s-1}}{v_2^{s-1}} \Rightarrow a_2^i v_2^{s-1} > a_1^i v_1^{s-1}$, we obtain $a_1^i(y_1^k - v_1^{s-1}) + a_2^i(y_2^k + v_2^{s-1}) \leq a_1^i(y_1^k - v_1^{s-1}) + a_2^i(y_2^k + v_2^{s-1}) + (\beta_1 - 1)(-a_1^i v_1^{s-1} + a_2^i v_2^{s-1}) + \beta_2(-a_1^i v_1^s + a_2^i v_2^s) = a_1^i(y_1^k - \beta_1 v_1^{s-1} - \beta_2 v_1^s) + a_2^i(y_2^k + \beta_1 v_2^{s-1} + \beta_2 v_2^s) = a_1^i(y_1^k - c) + a_2^i(y_2^k + d) = a_1^i y_1 + a_2^i y_2 \leq D^i$. This contradicts the hypotheses that $(y_1^k - v_1^{s-1}, y_2^k + v_2^{s-1})$ violates constraint i. □

Proposition 7. *If (y_1, y_2) is generated by Algorithm MK1 then it is an extreme point of Q_M.*

Proof. We prove by induction. First, notice that (y_1^1, y_2^1) is an extreme point of Q_M, because is the unique point that maximizes functions $\pi(y_1, y_2) = \pi_1 y_1 + \pi_2 y_2$ with $\pi_1 > 0$ and $\pi_2 < 0$. Now we prove that if (y_1^k, y_2^k) is extreme point of Q_M then (y_1^{k+1}, y_2^{k+1}) is also extreme point of Q_M. Suppose that (y_1^k, y_2^k) is an extreme point of Q_M. Let (v_1^s, v_2^s) be the vector in C used to obtain (y_1^{k+1}, y_2^{k+1}). Now, consider $\pi_1 > 0, \pi_2 > 0$ such that $\frac{\pi_2}{\pi_1} > \frac{v_1^s}{v_2^s}$ and, if there exists (v_1^{s+1}, v_2^{s+1}), i.e. $s <\mid C \mid$, then consider $\frac{\pi_2}{\pi_1} < \frac{v_1^{s+1}}{v_2^{s+1}}$ (setting, for instance, $\pi_1 = v_1^s + v_1^{s+1}, \pi_2 = v_2^s + v_2^{s+1}$). Suppose that $\pi_1 y_1 + \pi_2 y_2$ is maximized by $(y_1^*, y_2^*) \neq (y_1^{k+1}, y_2^{k+1})$. Three cases may occur:

a) $y_1^* < y_1^{k+1}, y_2^* > y_2^{k+1}$. This case implies the existence of (v_1^{s+1}, v_2^{s+1}). Let $(c,d) = (y_1^{k+1} - y_1^*, y_2^* - y_2^{k+1})$. Then, from Lemma 6, $(c,d) \in \text{Cone}\{(1,0), (v_1^{s+1}, v_2^{s+1})\}$. Hence, $\frac{c}{d} \geq \frac{v_1^{s+1}}{v_2^{s+1}} > \frac{\pi_1}{\pi_2}$ which implies $\pi(y_1^*, y_2^*) < \pi(y_1^{k+1}, y_2^{k+1})$, contradicting the hypotheses that (y_1^*, y_2^*) maximizes $\pi_1 y_1 + \pi_2 y_2$.

b) $y_1^k > y_1^* > y_1^{k+1}, y_2^k < y_2^* < y_2^{k+1}$. Let $(c,d) = (y_1^k - y_1^*, y_2^* - y_2^k)$. Then, from Lemma 6, $(c,d) \in \text{Cone}\{(1,0), (v_1^s, v_2^s)\}$. Therefore, $\frac{\pi_1}{\pi_2} > \frac{v_1^s}{v_2^s} \geq \frac{c}{d}$, which implies $\pi(y_1^*, y_2^*) < \pi(y_1^k, y_2^k)$.

c) $y_1^* > y_1^k, y_2^* < y_2^k$. In this case we will conclude that (y_1^k, y_2^k) cannot be an extreme point because the value of any function at this point is always lower than the value of the function either at (y_1^{k+1}, y_2^{k+1}) or (y_1^*, y_2^*).

As $\pi_1 y_1^* + \pi_2 y_2^* \geq \pi_1 y_1^k + \pi_2 y_2^k$ then $\frac{y_1^* - y_1^k}{y_2^k - y_2^*} \geq \frac{\pi_2}{\pi_1}$. Consider $\overline{\pi}_1 > 0, \overline{\pi}_2 > 0$ such that (y_1^k, y_2^k) is the unique solution that maximizes $\overline{\pi}_1 y_1 + \overline{\pi}_2 y_2$ over Q_M. If $\frac{\overline{\pi}_2}{\overline{\pi}_1} \leq \frac{\pi_2}{\pi_1}$ then $\frac{y_1^* - y_1^k}{y_2^k - y_2^*} \geq \frac{\pi_2}{\pi_1} \geq \frac{\overline{\pi}_2}{\overline{\pi}_1} \Rightarrow \overline{\pi}_1(y_1^* - y_1^k) \geq \overline{\pi}_2(y_2^k - y_2^*) \Rightarrow \overline{\pi}(y_1^*, y_2^*) \geq \overline{\pi}(y_1^k, y_2^k)$. So, suppose $\frac{\overline{\pi}_2}{\overline{\pi}_1} > \frac{\pi_2}{\pi_1}$. As $\frac{\pi_2}{\pi_1} > \frac{v_1^s}{v_2^s}$, we have $\frac{\overline{\pi}_2}{\overline{\pi}_1} > \frac{v_1^s}{v_2^s} = \frac{y_1^k - y_1^{k+1}}{y_2^{k+1} - b^k}$, which implies $\overline{\pi}_2(y_2^{k+1} - y_2^k) > \overline{\pi}_1(y_1^k - y_1^{k+1}) \Rightarrow \overline{\pi}(y_1^{k+1}, y_2^{k+1}) > \overline{\pi}(y_1^k, y_2^k)$ contradicting the hypotheses. □

Proposition 8. *If $(y_1, y_2) \neq (0,0)$ is an extreme point of Q_M then it is generated by Algorithm MK1.*

Proof. Suppose there exist an extreme point (y_1^*, y_2^*) that is not generated by Algorithm MK1 and that satisfies $y_1^{k+1} < y_1^* < y_1^k$ and $y_2^{k+1} > y_2^* > y_2^k$ for some pair of points (y_1^k, y_2^k), (y_1^{k+1}, y_2^{k+1}) generated consecutively by Algorithm MK1. Let $\pi_1 y_1 + \pi_2 y_2$ with $\pi_1 > 0$, $\pi_2 > 0$, be a function that attains its unique maximum at (y_1^*, y_2^*). Let $(c,d) = (y_1^k - y_1^*, y_2^* - y_2^k)$, $(c', d') = (y_1^* - y_1^{k+1}, y_2^{k+1} - y_2^*)$, and let (v_1^s, v_2^s) be the vector used to obtain (y_1^{k+1}, y_2^{k+1}), i.e., $(y_1^{k+1}, y_2^{k+1}) = (y_1^k, y_2^k) + t(-v_1^s, v_2^s)$ for some non-negative integer t.

Lemma 6 implies that $\frac{c}{d} \geq \frac{v_1^s}{v_2^s}$. As $\frac{c+c'}{d+d'} = \frac{v_1^s}{v_2^s}$ then $\frac{c'}{d'} \leq \frac{v_1^s}{v_2^s}$. Therefore, $\frac{c}{d} \geq \frac{v_1^s}{v_2^s} \geq \frac{c'}{d'}$. If $\frac{\pi_2}{\pi_1} \geq \frac{v_1^s}{v_2^s}$, as $\frac{v_1^s}{v_2^s} \geq \frac{c'}{d'}$, we have $\frac{\pi_2}{\pi_1} \geq \frac{c'}{d'}$, which implies that (y_1^{k+1}, y_2^{k+1}) is a better solution than (y_1^*, y_2^*). If $\frac{\pi_2}{\pi_1} < \frac{v_1^s}{v_2^s}$ then $\frac{\pi_2}{\pi_1} < \frac{c}{d}$ and, hence (y_1^k, y_2^k) is a better solution than (y_1^*, y_2^*). In both cases (y_1^*, y_2^*) is not the unique solution that maximizes $\pi_1 y_1 + \pi_2 y_2$, contradicting the initial assumption. Cases $y_1^* > y_1^1 = \lfloor \frac{D^1}{a_1^1} \rfloor$ and $y_1^* < y_1^n = 0$ cannot occur. Using Lemma 6 we can also prove that the cases $y_1^* = y_1^k, y_2^* > y_2^k$ and $y_1^* > y_1^k, y_2^* = y_2^k$ cannot occur with the exception of the initial case, if $(v_1^1, v_2^1) = (0,1)$, and the final case, if $(v_1^{|C|}, v_2^{|C|}) = (1,0)$. But in these two situations (y_1^*, y_2^*) cannot be an extreme point. □

Algorithm MK1 is not polynomial because C may include a non-polynomial number of elements. There are two changes that must be made in order to construct a polynomial algorithm. First, notice that it is not necessary to consider explicitly sets $C^{j,j+1}$, for $j = 1, \ldots, m-1$, and we can choose any vector in this set in polynomial time. The second change we must made is similar to that one that transforms Algorithm BA on a polynomial algorithm. Basically, instead of considering all the vectors in C^j, it is only necessary to consider the first and the last vector in each sequence, that is, if using Algorithm BA the vectors $(p_1^{j,k}, p_2^{j,k}) = (k-p) * (p_1^{j,p}, p_2^{j,p}) + (q_1^{j,\ell(k)}, q_2^{j,\ell(k)})$ for $k = p+1, \ldots, p+q$ with $q > 1$ are generated, then, only vectors $(p_1^{j,p}, p_2^{j,p})$ and $(p_1^{j,p+q}, p_2^{j,p+q})$ must be retained. However there is a case where this procedure fails. That is the case where there are several cones, say $C^j, \ldots, C^{j+t}$, intersecting each other, that is, when there are rational, $\frac{a_2^j}{a_1^j}, \cdots, \frac{a_2^{j+t}}{a_1^{j+t}}$, in the same interval $[r, r+1)$, $r \leq \frac{a_2^j}{a_1^j} < \cdots < \frac{a_2^{j+t}}{a_1^{j+t}} < r+1$, where r, t are integers and $t \geq 1$ (see Example 9).

Example 9. Considering $m = 2, a_1^1 = 21, a_2^1 = 76, a_1^2 = 5, a_2^2 = 19$ then $C^1 = \{(3,1), (7,2), (18,5), (48,13), (76,21), (29,8), (11,3), (4,1)\}$, with $3 \leq \frac{3}{1} < \frac{7}{2} < \frac{18}{5} < \frac{47}{13} < \frac{76}{21} < \frac{29}{8} < \frac{11}{3} < \frac{4}{1} \leq 4$, and $C^2 = \{(3,1), (7,2), (11,3), (15,4), (19,5), (4,1)\}$, with $3 \leq \frac{3}{1} < \frac{7}{2} < \frac{11}{3} < \frac{15}{4} < \frac{19}{5} < \frac{4}{1} \leq 4$. Therefore $C = \{(3,1), (7,2), (18,5), (48,13), (76,21), (29,8), (11,3), (15,4), (19,5), (4,1)\}$, with $3 \leq \frac{3}{1} < \frac{7}{2} < \frac{18}{5} < \frac{47}{13} < \frac{76}{21} < \frac{29}{8} < \frac{11}{3} < \frac{15}{4} < \frac{19}{5} < \frac{4}{1} \leq 4$.

Suppose $t = 1$ (for $t > 1$ we can repeat the same procedure). Suppose $\frac{p_1^{j+1,p}}{p_2^{j+1,p}} \leq \frac{a_2^j}{a_1^j} \leq \frac{p_1^{j+1,p+1}}{p_2^{j+1,p+1}} \leq \frac{a_2^{j+1}}{a_1^{j+1}}$ for some integer p, $1 \leq p \leq n_1(j+1) - 1$. Proposition 1 implies that the rationales $\frac{p_1^{j+1,1}}{p_2^{j+1,1}}, \cdots, \frac{p_1^{j+1,p}}{p_2^{j+1,p}}$ occur in the approximation of $\frac{a_2^j}{a_1^j}$ from below. That is, $\frac{p_1^{j+1,p+1}}{p_2^{j+1,p+1}} = \frac{q_1^{j,k}}{q_2^{j,k}}$ for some integer k, $1 \leq k \leq n_2(j) - 1$. Similarly, the rationales $\frac{q_1^{j,s}}{q_2^{j,s}}, \cdots, \frac{q_1^{j,1}}{q_2^{j,1}}$, where $s = \max\{t : q_2^{j,t} < p_2^{j+1,p+1}\}$, occur in the approximation of $\frac{a_2^{j+1}}{a_1^{j+1}}$ from above and $\frac{p_1^{j+1,p+1}}{p_2^{j+1,p+1}} = \frac{q_1^{j,s+1}}{q_2^{j,s+1}}$. Thus, we can consider the rationales corresponding to elements in C^j with value less or equal to $\frac{p_1^{j+1,p+1}}{p_2^{j+1,p+1}}$ and then, consider the rationales corresponding to elements in C^{j+1}. Therefore we apply to the subsets of C^j and C^{j+1} the same procedure applied to the case where only one set of rational is in the interval $[r, r+1]$. Vectors $(p_1^{j+1,p+1}, p_2^{j+1,p+1})$ and $(q_1^{j,k+1}, q_2^{j,k+1})$ can be computed in polynomial time. In Example 9 the vector $(p_1^{j+1,p+1}, p_2^{j+1,p+1})$ is $(11, 3)$. Thus we consider the subset of $C^1 : \{(3,1),(7,2),(18,5),(48,13),(76,21),(29,8),(11,3)\}$, and then, the subset of $C^2 : \{(15,4),(19,5),(4,1)\}$. Let V be the set of vectors obtained using this procedure. We need to consider the set U of the vectors used to obtain those vectors in V and the corresponding set of constants, denoted by Q. Formally, if $(v_1^s, v_2^s), (v_1^{s+1}, v_2^{s+1})$ are two consecutive elements of V, then (v_1^{s+1}, v_2^{s+1}) is obtained as $(v_1^{s+1}, v_2^{s+1}) = (v_1^s, v_2^s) + q(s) \times (u_1^s, u_2^s)$. When $(v_1^{s+1}, v_2^{s+1}) = (p_1^{j,1}, p_2^{j,1}) = (\lfloor a_2^j / a_1^j \rfloor)$, for $j = 2, \ldots, m$, and $p_1^{j,1} > p_1^{j-1,1} - 1$, consider $(u_1^s, u_2^s) = (1, 0)$ and $q(s) = \lfloor a_2^j / a_1^j \rfloor - \lceil a_2^{j-1} / a_1^{j-1} \rceil$.

Example 10. In Example 5 instead of considering the set C^1 we only need to consider $\{(1,1), (15,8), (2,1)\}$ because each vector $(2p+1, p+1)$, with $p = 1, \ldots, 6$, can be obtained as $(2p+1, 2p) = (1,1) + p \times (2,1)$. Hence, $V = \{(1,1),(15,8),(2,1),(6,1), (27,4), (7,1),(1,0)\}$, $U = \{(2,1),(-13,-7),(1,0), (7,1),(-20,-3),(-6,-1)\}$, $Q = \{7,1,4,3,1,1\}$. Algorithm MK2 which is the polynomial version of Algorithm MK1 will produce the following output.

s	1	2	2	3	4	5	6	7
(y_1, y_2)	(25,0)	(14,6)	(14,6)	(6,10)	(1,11)	(1,11)	(1,11)	(0,11)
(v_1, v_2)	(1,1)	(15,8)	(15,8)	(2,1)	(6,1)	(27,4)	(7,1)	(1,0)
(u_1, u_2)	(2,1)	(-13,-7)	(-13,-7)	(1,0)	(7,1)	(-20,-3)	(-6,-1)	
(γ^1, γ^2)	(2,202)	(0,84)	(0,84)	(4,8)	(29,1)	(29,1)	(29,1)	(37,5)
$(r_\leq^1, r_\leq^2)$	(23,7)	(156,0)	(156,0)	(19,-1)	(3,-33)	(0,-156)	(-1,-41)	(-4,-8)
$(r_\geq^1, r_\geq^2)$	(-19,1)	(137,1)	(137,1)	(4,8)	(1,41)	(1,-115)	(3,-33)	
p	5	0	0	3	0	0	1	
(c, d)	(6,11)	(15,8)	(2,1)	(5,1)	(27,4)	(7,1)	(1,0)	

Algorithm MK2

(a) *Compute* V, U, Q *using the Euclidean algorithm.*

(b) *Initialization:* $j \leftarrow 1$; $t \leftarrow 1$; $s \leftarrow 1$; $J \leftarrow \{1, \ldots, m\}$; $(y_1^t, y_2^t) \leftarrow (\lfloor \frac{D^1}{a_1^1} \rfloor, 0)$; $(c, d) \leftarrow (v_1^s, v_2^s)$; $\gamma^j(y_1^t, y_2^t) \leftarrow D^j - y_1^t a_1^j - y_2^t a_2^j$; $r_\leq^j(c, d) \leftarrow -c a_1^j + d a_2^j$.

(c) *While* $y_1^t > 0$ *do*

(c.1) *Compute the next extreme point:*
(c.1.i) Set $n \leftarrow \min\{\lfloor y_1^t / c \rfloor, \min_{j \in J}\{\lfloor \gamma^j(y_1^t, y_2^t) / r_\leq^j(c, d) \rfloor\}\}$.
(c.1.ii) If $n > 0$ set $(y_1^{t+1}, y_2^{t+1}) \leftarrow (y_1^t, y_2^t) + n(-c, d), t \leftarrow t + 1$.

(c.2) *Compute the next vector* (c, d)*:*
(c.2.i) $J \leftarrow \{j \in \{1, \ldots, m\} : r_\leq^j(v_1^s, v_2^s) > 0\}, J' \leftarrow \{j \in J : r_\geq^j(u_1^s, u_2^s) > 0\}$, $p \leftarrow \max\{0, \max_{j \in J'}\{\lceil \frac{r_\leq^j(v_1^s, v_2^s) - \gamma^j(y_1^t, y_2^t)}{r_\geq^j(u_1^s, u_2^s)} \rceil\}\}$.

If $y_1^t - v_1^s - pu_1^s < 0$ and $u_1^s < 0$ then $p \leftarrow \lceil \frac{(v_1^s - y_1^t)}{(-u_1^s)} \rceil$, go to $(c.2.ii)$.
If $y_1^t - v_1^s - pu_1^s < 0$ and $u_1^s \geq 0$ then $s \leftarrow s+1$, return to $(c.2.i)$.
If $r_{\leq}^j(v_1^s, v_2^s) - pr_{\geq}^j(u_1^s, u_2^s) > \gamma^j(y_1^t, y_2^t)$ for some $j \in J \setminus J'$ then $s \leftarrow s+1$. Return to (c.2.i).
(c.2.ii) If $p = 0$ then $(c,d) \leftarrow (v_1^s, v_2^s)$. Go to (c.1).
If $0 < p \leq q(s)$ then $(c,d) \leftarrow (v_1^s, v_2^s) + p(u_1^s, u_2^s)$, $s \leftarrow s+1$. Go to (c.1).
If $p > q(s)$ then $s \leftarrow s+1$. Return to (c.2.i).

Lemma 11. *Step (c) of Algorithm MK2 has $\mathcal{O}(|V|)$ iterations.*

Proof. It suffices to prove that case $p = 0$ can only occur once for each s. Let $(y_1^{t+1}, y_2^{t+1}) = (y_1^t, y_2^t) + n(v_1^s, v_2^s)$.

If $n = \lfloor \frac{y_1^t}{v_1^s} \rfloor$, that is, $\lfloor \frac{y_1^t}{v_1^s} \rfloor < \min_{j \in J}\{\lfloor \frac{\gamma^j(y_1^t, y_2^t)}{r_{\leq}^j(v_1^s, v_2^s)} \rfloor\}$ then, in Step (c.2.ii), occurs $p = 0$ and, as $y_1^{t+1} - v_1^s - pu_1^s = y_1^{t+1} - v_1^s = y_1^t - nv_1^s - v_1^s < 0$ we have $s \leftarrow s+1$ if $u_1^s \geq 0$ or $p \leftarrow \lceil (v_1^s - y_1^{t+1})/(-u_1^s) \rceil$ otherwise (notice that $\lceil (v_1^s - y_1^{t+1})/(-u_1^s) \rceil > 0$), hence $p = 0$ can only occur once for each s. If n is obtained at $\lfloor \gamma^{j^*}(y_1^t, y_2^t)/r_{\leq}^{j^*}(v_1^s, v_2^s) \rfloor$ for some $j^* \in J$ then, in the next Step (c.2.ii), we have $\gamma^{j^*}(y_1^{t+1}, y_2^{t+1}) < r_{\leq}^{j^*}(v_1^s, v_2^s)$. This inequality implies $p \geq 1$. □

Lemma 12. *(y_1, y_2) is generated by Algorithm MK2 if and only if it is generated by Algorithm MK1.*

Proof. We show that those vectors $(c,d) = (v_1^s, v_2^s) + k(u_1^s, u_2^s)$ that are not tested in (c.1) can be excluded because they cannot be used to obtain any feasible solution, that is: if $0 \leq k < p$ then $(y_1^t - c, y_2^t + d)$ is not feasible; if $p < k < q(s)$ then $(y_1^{t+1} - c, y_2^{t+1} + d)$ is not feasible. The value of $0 < p < q(s)$ can be obtained from one of the following two ways: (i) $p \leftarrow \max_{j \in J'}\{\lceil (r_{\leq}^j(v_1^s, v_2^s) - \gamma^j(y_1^t, y_2^t))/r_{\geq}^j(u_1^s, u_2^s) \rceil\}$, (ii) $p \leftarrow \lceil (v_1^s - y_1^t)/(-u_1^s) \rceil$, with $u_1^s < 0$. Consider Case (i). Let $h = arg\max_{j \in J'}\{\lceil (r_{\leq}^j(v_1^s, v_2^s) - \gamma^j(y_1^t, y_2^t))/r_{\geq}^j(u_1^s, u_2^s) \rceil\}$. Therefore, $(p-1)r_{\geq}^h(u_1^s, u_2^s) < r_{\leq}^h(v_1^s, v_2^s) - \gamma^h(y_1^t, y_2^t)) \leq pr_{\geq}^h(u_1^s, u_2^s) \Leftrightarrow \gamma^h(y_1^t, y_2^t) - r_{\geq}^h(u_1^s, u_2^s) < r_{\leq}^h(v_1^s, v_2^s) - pr_{\geq}^h(u_1^s, u_2^s) \leq \gamma^h(y_1^t, y_2^t)$. Considering Case (ii) we have $y_1^t + u_1^s < v_1^s + pu_1^s \leq y_1^t$. First we consider $0 \leq k < p$. In Case (i), as $(c,d) = (v_1^s, v_2^s) + k(u_1^s, u_2^s)$ it follows that $r_{\leq}^h(c,d) = r_{\leq}^h(v_1^s, v_2^s) - kr_{\geq}^h(u_1^s, u_2^s) \geq r_{\leq}^h(v_1^s, v_2^s) - (p-1)r_{\geq}^h(u_1^s, u_2^s) > \gamma^h(y_1^t, y_2^t)$. In Case (ii) we have $y_1^t < v_1^s + (p-1)u_1^s \leq v_1^s + ku_1^s = c$ because $u_1^s < 0$. In both cases $(y_1^t, y_2^t) + (c,d)$ is not feasible. Now suppose $0 < p < k < q(s)$. As $k > p$, this case implies that a new point $(y_1^{t+1}, y_2^{t+1}) = (y_1^t, y_2^t) - n[(v_1^s, v_2^s) + p(u_1^s, u_2^s)]$ has been obtained. We need to prove that $(y_1^{t+1}, y_2^{t+1}) + (-c, d)$ is not feasible, where

$(c,d) = (v_1^s, v_2^s) + k(u_1^s, u_2^s)$ with $0 < p < k < q(s)$. If p was obtained from (i) then $\gamma^h(y_1^t, y_2^t) - r^h_{\geq}(u_1^s, u_2^s) < r^h_{\leq}(v_1^s, v_2^s) - pr^h_{\geq}(u_1^s, u_2^s) \leq \gamma^h(y_1^t, y_2^t)$. The second inequality implies $n \geq 1$. Hence, $\gamma^h(y_1^{t+1}, y_2^{t+1}) = \gamma^h(y_1^t, y_2^t) - n[r^h_{\leq}(v_1^s, v_2^s) - pr^h_{\geq}(u_1^s, u_2^s)] \leq \gamma^h(y_1^t, y_2^t) - [r^h_{\leq}(v_1^s, v_2^s) - pr^h_{\geq}(u_1^s, u_2^s)] < r^h_{\geq}(u_1^s, u_2^s)$. On the other hand $r^h_{\leq}(c,d) = r^h_{\leq}(v_1^s, v_2^s) - kr^h_{\geq}(u_1^s, u_2^s) = r^h_{\leq}(v_1^{s+1}, v_2^{s+1}) + (q(s)-k)r^h_{\geq}(u_1^s, u_2^s) \geq r^h_{\geq}(u_1^s, u_2^s)$. therefore, $\gamma^h(y_1^{t+1}, y_2^{t+1}) < r^h_{\leq}(c,d)$. If p was obtained from (ii) then $y_1^t + u_1^s < v_1^s + pu_1^s \leq y_1^t$. Thus $n \geq 1$. Hence $y_1^{t+1} = y_1^t - n(v_1^s + pu_1^s) \leq y_1^t - (v_1^s + pu_1^s) < -u_1^s$ (notice that $v_1^s + pu_1^s \geq 0$). On the other hand, $c = v_1^s + ku_1^s = v_1^{s+1} - (q(s)-k)u_1^s \geq 0 - u_1^s$ because $v_1^{s+1} \geq 0$ and $u_1^s < 0$. Thus $y_1^{t+1} < c$. □

Proposition 13. *$(y_1, y_2) \neq (0,0)$ is an extreme point of Q_M if and only if (y_1, y_2) is generated by Algorithm MK2.*

Proof. The result follow from Proposition 7, Proposition 8 and Lemma 12. □

Proposition 14. *Algorithm MK2 has complexity*

$$\mathcal{O}(m\sum_{i=1}^{m}\log(\frac{D^i}{\min\{a_1^i, a_2^i\}})).$$

Proof. The number of iterations is given by the number of times that occurs step (c.2) plus the number of times that s is not increased, that is the number of times that occurs $p = 0$. As we saw in the proof of Lemma 11, the case $p = 0$ can only occur once for each s. Each iteration requires $\mathcal{O}(m)$ operations. Therefore this algorithm has complexity $\mathcal{O}(m\times \mid V \mid)$. We can write $\mid V \mid = \sum_{i=1}^m v^i$, where v^i is the number of coefficients computed by the Euclidean algorithm considering a_1^i and a_2^i. Since $v^i = \mathcal{O}(\log(\frac{D^i}{\min\{a_1^i, a_2^i\}}))$ the result follows. □

We denote by NT the number of extreme points generated by Algorithm MK2. The following intuitive results, stated without proof, follows.

Proposition 15. *Inequalities $\alpha_1^t y_1 + \alpha_2^t y_2 \leq \alpha^t$ for $t = 1, \ldots, NT-1$, where $\alpha_1^t = y_2^{t+1} - y_2^t$, $\alpha_2^t = y_1^t - y_1^{t+1}$, $\alpha^t = \alpha_1^t y_1^t + \alpha_2^t y_2^t$, are valid for Q_M and define facets of Q_M.*

Theorem 16. *Inequalities $y_1 \geq 0, y_2 \geq 0$ and $\alpha_1^t y_1 + \alpha_2^t y_2 \leq \alpha^t$ for $t = 1, \ldots, NT-1$, where $\alpha_1^t = y_2^{t+1} - y_2^t$, $\alpha_2^t = y_1^t - y_1^{t+1}$, $\alpha^t = \alpha_1^t y_1^t + \alpha_2^t y_2^t$, suffice to describe Q_M.*

Remark 17. From the Example 5 we can verify that only inequality $6y_1 + 11y_2 \le 150$ is valid considering constraint R_2. All other inequalities defining non-trivial inequalities are valid only in the presence of both constraints. It can be checked that $y_1 + 5y_2 \le 56$ cannot be obtained as a Chvatal-Gomory rank one inequality. Observe that there are no non-negative multipliers α and β such that $\lfloor 8\alpha + 4\beta \rfloor y_1 + \lfloor 15\alpha + 27\beta \rfloor y_2 \le \lfloor 202\alpha + 302\beta \rfloor$.

4. Multiple Knapsack Sets with $n > 2$ Variables

In this section we discuss the generation of valid inequalities for multiple knapsack models with $n > 2$ variables from the facet defining inequalities of Q_M. Consider the general multiple knapsack set $Y = \{(y_1, \ldots, y_n) \in \mathbb{N}_0^n : \sum_{j=1}^n a_j^i y_j \le D^i, i = 1, \ldots, m\}$. Setting all but two variables to zero we obtain the set Y_M. We suppose w.l.o.g. that these two variables are y_1 and y_2. Then we generate facet defining inequalities for Q_M and lift the remaining variables in order to obtain a valid inequality for Y.

Let $\phi^M(z_1, \ldots, z_m)$ be the lifting function associated with

$$\alpha_1 y_1 + \alpha_2 y_2 \le \alpha\,, \tag{1}$$

that is, $\phi^M(z_1, \ldots, z_m) = \min\{\alpha - \alpha_1 y_1 - \alpha_2 y_2 : a_1^i y_1 + a_2^i y_2 \le D^i - z_i,\ i = 1, \ldots, m, y_1, y_2 \in \mathbb{N}_0\}$. The lifting problem associated with (1) is to find $\alpha_j, j \in \{3, \ldots, n\}$ such that $\alpha_1 y_1 + \alpha_2 y_2 + \sum_{j=3}^n \alpha_j y_j \le \alpha$ is valid for Y. In order to lift all variables simultaneously we may use superadditive functions (see [6]).

Definition 18. A function $\psi : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is superadditive on A if $\psi(x_1) + \psi(x_2) \le \psi(x_1 + x_2)$ for all $x_1, x_2, x_1 + x_2 \in A$.

Using superadditivity we have the following result.

Proposition 19. *Consider the valid inequality (1) for Y_M. If $\psi : A = [0, D^1] \times \cdots \times [0, D^m] \to \mathbb{R}$ is superadditive on A and $\psi(z_1, \ldots, z_m) \le \phi(z_1, \ldots, z_m)$ for all $(z_1, \ldots, z_m) \in A$ then $\alpha_1 y_1 + \alpha_2 y_2 + \psi(a_3^1, \ldots, a_3^m) y_3 + \cdots + \psi(a_n^1, \ldots, a_n^m)\, y_n \le \alpha$ is valid to Y.*

We consider two types of facet-defining inequalities (1) for Q_M : Type I. Inequality (1) is valid for the model obtained when we consider only one of the m knapsack constraints. Thus defining a facet of a single knapsack constraint polyhedra. Type II. Inequality (1) in not valid for each knapsack constraint considered alone with the non-negativity constraints. For inequalities of Type I we use superadditive valid lifting functions developed for the single constraint case. The lifting function when only constraint i is considered and (1) is a valid inequality for that single constraint knapsack set, is given by, $\phi(z_i) = \min\{\alpha - \alpha_1 y_1 - \alpha_2 y_2 : a_1^i y_1 + a_2^i y_2 \leq D^i - z_i, y_1, y_2 \in \mathbb{N}_0\}$.

If $\psi' : [0, D^i] \to \mathbb{R}$ is superadditive on $[0, D^i]$ then $\psi(z_1, \ldots, z_m) = \psi'(z_i)$ is superadditive on $[0, D^1] \times \cdots \times [0, D^m]$. If $\psi'(z) \leq \phi(z)$ for all $z \in [0, D^i]$, then $\phi^M(z_1, \ldots, z_m) \geq \phi^M(0, \ldots, 0, z_i, 0, \ldots, 0) \geq \phi(z_i) \geq \psi'(z_i) = \psi(z_1, \ldots, z_m)$, for all $(z_1, \ldots, z_m) \in [0, D^1] \times \cdots \times [0, D^m]$. Therefore, for inequalities of Type I, we can use the superadditive valid lifting functions given in [1] for the single constraint knapsack set to generate the coefficients of all the remaining variables. For inequalities of Type II, as (1) is not valid when we consider the single knapsack constraint sets, we consider several strategies:

a) We can use the lower convex envelope of $\phi(z_i)$. This is the function ψ_1 presented in [1]. Next we explain how to construct ψ_1. Given a knapsack constraint $a_1^k y_1 + a_2^k y_2 \leq D^k$, let us define $\gamma(y_1, y_2) = D^k - a_1^k y_1 - a_2^k y_2$ and, for a valid inequality, $\alpha_1 y_1 + \alpha_2 y_2 \leq \alpha$ define $\tau(y_1, y_2) = \alpha - \alpha_1 y_1 - \alpha_2 y_2$. If (Y_1, Y_2) belongs to an ordered set $E = \{(Y_1^t, Y_2^t), t = 1, \ldots, j\}$, then we use the notation $\gamma^t = \gamma(Y_1^t, Y_2^t)$ and $\tau^t = \tau(Y_1^t, Y_2^t)$. The function ψ_1 is a piecewise linear function whose graph contains the points $(\gamma(Y_1^1, Y_2^1), \tau(Y_1^1, Y_2^1)), \ldots, (\gamma(Y_1^j, Y_2^j), \tau(Y_1^j, Y_2^j))$, where $(Y_1^1, Y_2^1), \ldots, (Y_1^j, Y_2^j) \in E$. The set E is in our case a subset of the set of extreme points of Q_M. The function ψ_1, depends on the set E (set of ordered points), $\alpha_1, \alpha_2, \alpha$ (coefficients of the valid inequality) and a_1^k, a_2^k, D^k (coefficients of the knapsack constraint). It is defined by

$$\psi_1(z) = \begin{cases} 0, & 0 \leq z \leq \gamma^{j-1}, \\ \tau^t + \frac{\tau^{t-1} - \tau^t}{\gamma^{t-1} - \gamma^t}(z - \gamma^t), & \gamma^t < z \leq \gamma^{t-1},\ t = 2, \ldots, j, \\ \tau^1 + \frac{\alpha_1}{a_1^k}(z - \gamma^1), & \gamma^1 < z \leq D^k, \end{cases}$$

The following results were established in [2].

Proposition 20. *Let $E = \{(Y_1^t, Y_2^t), t = 1, \ldots, j\}$ be a set of points satisfying $0 \leq Y_1^j \leq \cdots \leq Y_1^1$, $Y_2^j > \cdots > Y_2^1 \geq 0$, $\gamma^t > \gamma^{t+1}$ and $\tau^t > \tau^{t+1}$ for $t = 1, \ldots, j-1$. If $\frac{Y_1^1 - Y_1^2}{Y_2^2 - Y_2^1} < \ldots < \frac{Y_1^{j-1} - Y_1^j}{Y_2^j - Y_2^{j-1}} \leq \frac{\alpha_2}{\alpha_1} \leq \frac{a_2^k}{a_1^k}$, then ψ_1 is convex on $[0, D^k]$.*

In fact, this result is enough to conclude that ψ_1 is superadditive.

Proposition 21. *Consider a convex function $f : I = [0, D] \subseteq \mathbb{R}_0^+ \to \mathbb{R}$ satisfying $f(0) = 0$. Then f is superadditive on I.*

Proposition 22. *Let $E, \alpha_1, \alpha_2, \alpha, a_1^k, a_2^k, D^i$ be the parameters satisfying the conditions of Proposition 20 and suppose they satisfy, additionally, the following conditions: (i) $a_2^1 = 0$, (ii) for all $t = 2, \ldots, j$, the inequality $\alpha_1^t y_1 +$*

$\alpha_2^t y_2 \le \alpha_1^t Y_1^t + \alpha_2^t Y_2^t$, where $\alpha_1^t = Y_2^t - Y_2^{t-1}, \alpha_2^t = Y_1^{t-1} - Y_1^t$, is valid for Y_M. Then, $\psi_1(z^k) \le \phi^M(0,\ldots,0,z^k,0,\ldots,0)$ for all $z^k \in [0, D^k]$.

In order to be able to use ψ_1 we consider two choices. Let (Y_1^{j-1}, Y_2^{j-1}) and (Y_1^j, Y_2^j) be the two extreme points satisfied at equality by $\alpha_1 y_1 + \alpha_2 y_2 \le \alpha$. The first option consists in considering $E = \{(Y_1^1, Y_2^1), \ldots, (Y_1^j, Y_2^j)\}$ and choose the inequality k that satisfies $k = \min\{i = 1,\ldots,m : \frac{a_2^i}{a_1^i} \ge \frac{\alpha_2}{\alpha_1}\}$. The second option consists in considering $E = \{(Y_1^{j-1}, Y_2^{j-1}), \ldots, (Y_1^n, Y_2^n)\}$ and choose inequality k satisfying $k = \max\{i = 1,\ldots,m : \frac{a_2^i}{a_1^i} \le \frac{\alpha_2}{\alpha_1}\}$. In this last case Proposition 20 and Proposition 22 are valid (it suffices to exchange coordinates and exchange α_1, a_1 by α_2 and a_2, respectively). We choose the option that minimizes the value of z_0.

b) Another approach consists in the introduction of Chvatal-Gomory cuts. As we mentioned in Remark 17 it seems that it is not trivial to write the valid inequality (1) as a sequence of Chvatal-Gomory cuts. However, it can be easily checked that for each facet-defining inequality for the 2-dimensional multiple knapsack set, $\alpha_1 y_1 + \alpha_2 y_2 \le \alpha$, the coefficients α_1, α_2 can be obtained either as $\lfloor \lambda a_1^i \rfloor$, $\lfloor \lambda a_2^i \rfloor$, respectively, for some $i \in \{1,\ldots,m\}$ and $\lambda > 0$ or, they can be obtained as $\lfloor \lambda a_1^i + (1-\lambda) a_1^{i+1} \rfloor$, $\lfloor \lambda a_2^i + (1-\lambda) a_2^{i+1} \rfloor$, for some $i \in \{1,\ldots,m-1\}$ and $\lambda > 0$. The value of λ can be easily obtained. Then, in order to lift the remaining variables, apply the mixed integer rounding function $F_\alpha(d) = \begin{cases} \lfloor d \rfloor & \text{if } d - \lfloor d \rfloor \le \alpha \\ \lfloor d \rfloor + \frac{d - \lfloor d \rfloor - \alpha}{\alpha} & \text{if } d - \lfloor d \rfloor > \alpha \end{cases}$ either to the knapsack constraint considered in the first case or to the knapsack constraint obtained form the aggregation of two knapsack sets. Thus, we obtain a valid inequality with coefficients α_1 and α_2 on the variables y_1 and y_2 and with coefficients either $F_\alpha(\lfloor \lambda a_k^i \rfloor)$ or $F_\alpha(\lfloor \lambda a_k^i + (1-\lambda) a_k^{i+1} \rfloor)$ on each other variable y_k, $k \notin \{1,2\}$. Observe that the RHS is greater or equal to α.

c) A third approach is based on the exact lifting of a third variable. In order obtain a valid inequality, the lifting coefficient of variable k, α_k, must satisfy $\alpha_k \le \min_{t \in \{1,\ldots,T\}} \frac{\phi^M(t a_k^1,\ldots,t a_k^m)}{t}$, where $T = \min\{D^i / a_k^i : a_k^i > 0, i \in \{1,\ldots,m\}\}$. Notice that the problem associated with ϕ^M can be solved in polynomial time, however, the computation of α_k involves a non-polynomial number of operations.

5. Computational Tests

In this section we report some computational tests. We consider two different tests with $n = 100$ variables, one where the instances have $m = 2$ constraints and the other with $m = 10$ constraints. The coefficients were randomly generated in predefined intervals. All the computational tests were performed on a PC, Pentium IV, 2.4Ghz with 768 Mb RAM and using the optimization package *Xpress-MP*, Version 14.10 with the modeler *MOSEL*. For each instance we solve the linear relaxation without any additional inequality. Then, for each pair of variables, such that at least one of them has a strictly positive value in the optimal solution, we consider a restricted knapsack problem. Finally, we lift each one of the corresponding facet-defining inequalities and add those inequalities that are violated by solution of the linear relaxation. This procedure is repeated until the last linear solution satisfies all the inequalities generated.

In the first table we report the results for the case $m = 2$. In column Opt the value of the optimal solution is given. In the second column we present the value of the linear relaxation and in the third column the linear gap, $\frac{LP-Opt}{Opt} \times 100$, is given. In the fourth column, the percentage of the linear gap reduced using the cuts introduced by *Xpress*, using the default options, is given. In the fifth column, we present the percentage of the gap reduced by the inclusion of cuts using the three strategies given above. Finally, in the last column we consider all our strategies together with the *Xpress* cuts. In the second table we report the results obtained considering $m = 10$. In column Opt/BFS we present either the value of the optimal solution or the value of the best lower bound obtained. In this last case the value is followed by *. In the second column the value of the linear relaxation is given. In the third column we present the corresponding linear gap. In the fourth column the number of initial variables in the optimal linear solution is given. In the last three columns is given the percentage of the linear gap reduced using the respective cuts. For the case $m = 2$ the strategies followed allowed us to improve the upper bound significantly. However when we consider $m = 10$ only in few instances our strategies contributed significantly for the reduction of the linear gap. As the number of active inequalities in the optimal linear relaxation is increased the effect of the strategies followed seams to be reduced. Notice that these strategies are mainly based on the single constrained case.

Opt	LP	LPgap	Xcuts (%)	Ocuts (%)	Xcut+Ocuts (%)
6393	6506.1	1.8	29	26	98
6597	6677.4	1.2	32	65	84
5297	5342.4	0.9	20	28	28
7786	7934.5	1.9	13	34	43
5928	5946.1	0.3	14	57	77
4784	4816.1	0.7	17	44	67
4922	5023.2	2.1	14	99	100
5828	5995.6	2.9	12	96	98
6253	6262.4	0.2	79	100	100

Opt/BFS	LP	LPgap	NBC	Xcuts (%)	Ocuts (%)	Xcut + Ocuts (%)
4421^*	4557.5	3.1	3	11.2	7.4	13.6
4969	5203.9	4.7	5	53.2	46.3	53.4
4753	4815.9	1.3	4	42.9	67.4	83.4
4446	4550	2.3	4	2.6	63.4	69.6
4639	4815	3.8	3	17.3	24.2	33.5
4505^*	4630	2.8	5	30.7	8.4	30.7
4402	4657.7	5.8	3	21.1	26.6	35.3
4308^*	4591.1	6.6	6	5.8	0.2	5.8
3956^*	4173.5	5.6	4	1.3	8.5	11.9

6. Conclusions

We gave a complete characterization of the convex hull of the multiple integer knapsack sets with two integer variables. The study of the properties of the coefficients of the faces of this polyhedra may be important for the generalization of the valid inequalities for the general case with $n > 2$ variables. The generalization of valid inequalities based on the superadditive valid lifting functions developed for the single constrained case had contributed to reduce the gap between the linear and the integer formulation. To improve these valid inequalities it would be interesting to study the lifting function considering all constraints simultaneously. However this approach appears to be significantly harder than in the single constraint case.

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