International Journal of Pure and Applied Mathematics
Volume 25 No. 4 2005, 481-486

VECTOR BUNDLES ON CURVES, COHERENT SYSTEMS AND THEIR DUALS

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a smooth projective curve. Here we study from the point of view of stability triples (E, f, h), where E is a vector bundle on X and $f : \mathcal{O}_X{}^{\oplus a} \to E$, $h : E \to \mathcal{O}_X{}^{\oplus b}$ are maps such that $f_* : H^0(X, \mathcal{O}_X{}^{\oplus a}) \to H^0(X, E)$ and $\check{h}_* : H^0(X, \mathcal{O}_X{}^{\oplus b}) \to H^0(X, E^*)$ are injective.

AMS Subject Classification: 14H60
Key Words: coherent systems on curves, vector bundles on curves, stable vector bundles

1. Admissible Objects

Let X be a smooth and connected projective curve, a, b positive integers, E a vector bundle on X and morphisms $f : \mathcal{O}_X{}^{\oplus a} \to E$, $h : E \to \mathcal{O}_X{}^{\oplus b}$. We will say that $\tilde{E} := (E, a, b, f, h)$ (or just (E, f, h)) is an admissible object if the induced maps $f_* : H^0(X, \mathcal{O}_X{}^{\oplus a}) \to H^0(X, E)$ and $\check{h}_* : H^0(X, \mathcal{O}_X{}^{\oplus b}) \to H^0(X, E^*)$ at the level of global sections are injective. The parameters of (E, a, b, f, h) are the integers a, b, $r_E := \mathrm{rank}(E)$ and $d_E := \deg(E)$ or a, b, r_E and $\mu_E := \mu(E) := \deg(E)/r_E$. Fix real numbers α, β and set $\mu_{\alpha,-\beta}(\tilde{E}) := \mu_E + a\alpha + b\beta$. The real number $\mu_{\alpha,-\beta}(\tilde{E})$ will be called the $(\alpha, -\beta)$-slope of the admissible object $\tilde{E}$.

Remark 1. Take an admissible object $\tilde{E} := (E, a, b, f, h)$ and set $c_{\tilde{E}} := \mathrm{rank}(h \circ f)$. If $c_{\tilde{E}} > 0$, then $\mathcal{O}_X{}^{\oplus c}$ is a direct factor of E and there is an admissible object $\tilde{G} := (G, a - c, b - c, f', h')$ such that $E \cong G \oplus \mathcal{O}_X{}^{\oplus c}$, and

Received: September 22, 2005

up to an isomorphism of E, f (resp. h) is obtained from f' (resp. h') adding the identity to the factor ${\mathcal{O}_X}^{\oplus c}$. Here we abuse notation, because we allow the cases $c_{\tilde{E}} = a$ or $c_{\tilde{E}} = b$, while our definition of admissible object require $a > 0$ and $b > 0$.

Remark 2. Fix an admissible object $\tilde{E} := (E, a, b, f, h)$ such that $c_{\tilde{E}} = 0$, i.e. such that $h \circ f = 0$. Hence $\mathrm{Im}(f) \subseteq \mathrm{Ker}(h)$. Let A_f denote the saturation of $\mathrm{Im}(f)$ in E. Since $\mathrm{Ker}(h)$ is saturated in E, we have $A_f \subseteq \mathrm{Ker}(h)$. Look the following exact sequence of vector bundles on X:

$$0 \to A_f \to E \to E/A_f \to 0\,. \tag{1}$$

We see that the admissible object (E, f, h) is uniquely determined by the following data:

(a) A coherent system $f : {\mathcal{O}_X}^{\oplus a} \to A_f$ in the sense of [3]; we also assume that $\mathrm{Coker}(f)$ is a torsion sheaf.

(b) A coherent system $\check{h} : {\mathcal{O}_X}^{\oplus b} \to E/A_f$; however,

(c) An extension (1).

Conversely, any datum satisfying (a), (b) and (c) determines a unique admissible object $\tilde{E}$ such that $c_{\tilde{E}} = 0$. However, notice that the coherent system $f : {\mathcal{O}_X}^{\oplus a} \to A_f$ in (a) is very particular, because $a \geq \mathrm{rank}(A_f)$; we have $\mathrm{rank}(A_f) = a$ if and only if f is injective. Adding a factor ${\mathcal{O}_X}^{\oplus c}$ we get also how to reconstruct in this way an arbitrary admissible object $\tilde{F}$. We only use the condition "$c_{\tilde{E}} = 0$" to get the uniqueness of the way to reconstruct $\tilde{F}$. If $c := c_{\tilde{E}} \neq 0$, then we do not have the uniqueness because we may split arbitrarly the direct factor ${\mathcal{O}_X}^{\oplus c}$ between A and B.

Let $u : {\mathcal{O}_X}^{\oplus a} \to A$ (or just (A, u)) be a coherent system on X. For any $\alpha \in \mathbb{R}$ set $\mu_\alpha(A, u) := \mu(A) + \alpha \cdot a = \deg(A)/\mathrm{rank}(A)$ and use the α-slope μ_α to define the notion of stability and semistability for coherent systems (see [3]).

Remark 3. Assume $g := p_a(X) \geq 2$ and fix integers $a > 0$, $b > 0$ and stable vector bundles A, B on X. Let $\tilde{E} = (E, f, h)$ be any admissible object with $c_{\tilde{E}} = 0$ constructing from data of Remark 2 satisfying (a), (b) and (c) with $A_f := A$ and $E/A_f := B$. Hence E is an arbitrary extension of B by A. Since $a > 0$ and A is stable, either $\deg(A) > 0$ or $a = 1$ and $A \cong \mathcal{O}_X$. Since $b > 0$ and B is stable, either $\deg(B) < 0$ or $b = 1$ and $B \cong \mathcal{O}_X$. Hence (except in the trivial case $a = b = 1$, $A \cong B \cong \mathcal{O}_X$) the extension (1) is the Harder-Narasimhan filtration of E. Hence A and B are uniquely determined by E. Furthermore, again by the uniqueness property of the Harder-Narasimhan filtration, even the extension (1) is, for fixed A, B, unique (up to a non-zero multiplicative constant. Hence the set of all such bundles E contains the decopomposable bundle $A \oplus B$, while, outside, its point $A \oplus B$, this set is uniquely parametrized by a projective space of dimension $h^0(X, \mathrm{Hom}(B, A)) - 1$. Now we drop the assumptions "$p_a(X) \geq 2$" and that A and B are stable. We fix non-negative real numbers α, β. We assume that the coherent system $f : \mathcal{O}_X{}^{\oplus a} \to A$ is α-stable and that the coherent system $\check{h} : \mathcal{O}_X{}^{\oplus b} \to B$ is β-stable. In this case we will say that any associated admissible object $\tilde{E}$ is separately $(\alpha, -\beta)$-stable; similarly, we define separate $(\alpha, -\beta)$-semistablity of $(A, B, f, \check{h})$. If $\alpha = 0$ (resp. $\beta = 0$) this assumption is equivalent to the stability of A (resp. B^*, i.e. of B). Set $\mu_{\alpha,-\beta}(\tilde{E}) = \mu_\alpha(A, f) + \mu_\beta(B^*, \check{h})$. If $c := c_{\tilde{E}} \neq 0$, then there is a uniquely determined admissible object $\tilde{F}$ with $c_{\tilde{F}} = 0$ and associated invariants $(a - c, b - c)$. In this case (assume $a > c$ and $b > c$) we will set $\mu_{\alpha,-\beta}(\tilde{E}) := \mu_{\alpha,-\beta}(\tilde{F})$. If $a = b = c$, then we set $\mu_{\alpha,-\beta}(\tilde{E}) := 0$. Similarly, if $a = c$ (resp. $b = c$) we use only the coherent system $(B^*, \check{h})$ (resp. (A, f)) to define the $(\alpha, -\beta)$-slope $\mu_{\alpha,-\beta}(\tilde{E})$ of $\tilde{E}$. Using the $(\alpha, -\beta)$-slope $\mu_{\alpha,-\beta}$ we get the notion of joint $(\alpha, -\beta)$-stability and joint $(\alpha, -\beta)$-semistability.

Here we will prove the following result.

Theorem 1. *Fix a smooth and connected genus g projective curve X, $P \in X$, positive integers a, b, r, $\alpha, \beta + +\mathbb{R}$, $\alpha \geq 0$, a vector bundle A on X such that $a' := \mathrm{rank}(A) \leq a$ and a map $f : \mathcal{O}_X{}^{\oplus a} \to A$ such that $f_* : H^0(X, \mathcal{O}_X{}^{\oplus a}) \to H^0(X, E)$ is injective and* $\mathrm{Coker}(f)$ *is a torsion sheaf. Assume $r \geq a' + b$ and take any rank $r - a'$ vector bundle D on X. Set $B_t := D(-tP)$. Then there exists an integer t_0 (depending only on X, A, D, but not from P) and $\beta_0 \in \mathbb{R}$ (depending on α, X, A and D, but not from t) such that for all integers $t \geq t_0$ the vector bundle B_t has the following properties:*

(i) $h^0(X, B_t{}^*) \geq b$;

(ii) fix a general b-dimensional linear subspace of $h^0(X, B_t{}^*)$ and call $\check{h} : \mathcal{O}_X{}^b \to B_t{}^*$ and let E denote any extension of B by A. Let $\tilde{E} := (E, a, b, f, h)$ the admissible object associated by Remark 2; then $\tilde{E}$ is $\mu_{\alpha,-\beta}$-stable for all $\beta \geq \beta_0$ and $c_{\tilde{E}} = 0$.

Proof. The equality $c_{\tilde{E}} = 0$ follows from the construction of $\tilde{E}$ given in Remark 2. For any integer z such that $1 \leq z \leq r - a' - 1$ there is $\gamma_z \in \mathbb{R}$ such that $\mu(D) \leq \mu(B^*) + \gamma_z$ for all rank z subsheaves D of B^*. Notice that $\mu(K) \leq \mu(B^*(tP)) + \gamma_z$ for all rank z subsheaves K of $B^*(tP)$. set $\gamma := \max_{1 \leq z \leq r - a' - 1}\{\gamma_z\}$. For any integer c such that $1 \leq c \leq a'$ let Δ_c denote the maximal α-slope of a coherent subsytem (D, W) of A with $\mathrm{rank}(D) = c$. let Δ be the maximum between 0 and all Δ_c, $1 \leq c \leq a'$.

(a) Here we assume $r = a' + b$. We will follow the proof of [2], Th. 1, with only notational modifications. Since $\mathcal{O}_X(P)$ is ample, there is an integer

t_1 (depending only from g and the constant γ just introduced) such that for all integers $t \geq t_1$ the sheaf $B^*(tP)$ is spanned by its global sections. Fix any integer $t \geq t_1$. Since $B^*(tP)$ is spanned, the evaluation map $i_{t,W} : \mathcal{O}_X \otimes W \to B^*(tP)$ is an injective map of sheaves when W is a general n-dimensional linear subspace of $H^0(X, E(tH))$. Since $\dim(X) = 1$, we obtain an exact sequence

$$0 \to \mathcal{O}_X \otimes W \to B^*(tP) \to G \to 0 \tag{2}$$

in which G is a skyscraper sheaf. Since X is a smooth curve, we may also assume that $i_{t,W}$ has rank at least $b-1$ at each point of X, i.e. for every $Q \in \mathrm{Supp}(G)$ there is a one dimensional linear subspace J_Q of W such that J_Q is the kernel of the composition of the map $\mathcal{O}_X \otimes W \to B^*(tP)$ with the restriction map $H^0(X, B^*(tP)) \to H^0(\{Q\}, B^*(tP)|\{Q\})$. Hence we obtain a finite family $\{J_Q\}_{Q \in \mathrm{Supp}(G)}$ of one-dimensional linear subspaces. If $t \geq t_1 + 2g + 1$, we may even assume that $H^0(X, B^*(tP))$ spans the jet sheaf of B^* at each point $Q \in X$. Hence for general W we may also assume $h^0(X, D) = 1$ for every non-zero subsheaf D of G supported by a point of X. Varying W among the sufficiently general b-dimensional linear subspaces of $H^0(X, B^*(tP))$ we obtain a group $J(t)$ of permutations of the set $\Sigma := \mathrm{Supp}(G)$. There is an integer $t_2 \geq t_1 + 2g + 1$ (depending only from g and B^*) such that $h^0(X, B^*(tP)) \geq 6b^2$ and $\deg(B^*(tP)) \geq \Delta + 6 + \gamma$ for every $t \geq t_2$.

First Claim. *For $t \geq t_2 + 24g + 2$ the permutation group $J(t)$ is either the full symmetric group of Σ or the alternating group of Σ.*

Proof of First Claim. By the classification of 6-transitive finite permutation groups ([4], Theorem 2.4), it is sufficient to show that $J(t)$ is 6-transitive. For any $Q \in X$ set $I(t, Q) := \{f \in H^0(X, B^*(tP)) : f|\{Q\} = 0\}$. Since $B^*(tP)$ is spanned, $I(t, Q)$ is a codimension b linear subspace of $H^0(X, B^*(tP))$. We fix an integer c such that $0 \leq c \leq 5$ and assume that Σ is c-transitive. To prove First Claim in six steps it is sufficient to prove that $J(t)$ is $(c+1)$-transitive. Fix $c+1$ general points $Q_1, \dots, Q_c, Q$ of X_{reg}. Since $(2g+1)P$ is very ample and $B^*((t-24g)P)$ is spanned, we see that for general W the $(c+1)$ codimensiona b linear spaces $I(t, Q_1) \cap W, \dots, I(t, Q_c) \cap W, I(t, Q) \cap W$ are general in W. Move around Q in X and use that X is irreducible. □

Second Claim. *Use the notation introduced in the proof of First Claim. Fix a general b-dimensional linear subspace W of $H^0(X, B^*(tP))$ and take the associated $\Sigma := \mathrm{Supp}(G)$. Then for any subset S of Σ the linear span of $\cup_{Q \in S}(I(t, Q) \cap W)$ has dimension $\min\{b, \sharp(S)\}$.*

Proof of Second Claim. The last part of the proof of First Claim proved Second Claim when $\sharp(S) \leq 6$. To get the general case it is sufficient to prove

Second Claim for all S such that $\sharp(S) \leq b$. To get the general case use that $J(t)$ is at least $(\sharp(\Sigma) - 2)$-transitive. □

Third Claim. *Fix any integer $t \geq t_2 + 24g$ and any general b-dimensional linear subspace W of $H^0(X, B^*(tP))$. Then for any integer x such that $1 \leq x < b$ and any rank x subsheaf F of $B^*(tP)$ we have $\dim(W \cap H^0(X, F)) \leq x$.*

Proof of Third Claim. Just use the injectivity of the map of sheaves $\mathcal{O}_X \otimes W \to B^*(tP)$. □

Fourth Claim. *Fix any integer $t \geq t_2 + 24g$ and any general x-dimensional linear subspace W of $H^0(X, B^*(tP))$. Then for any integer x such that $1 \leq x < n$ and any x-dimensional linear subspace M of W, the saturation in $B^*(tP)$ of the image $A(M)$ of the evaluation map $u : \mathcal{O}_X \otimes M \to B^*(tP)$ has saturation with degree at most x.*

Proof of Fourth Claim. By Second Claim there are at most x points $Q \in \mathrm{Supp}(G)$ such that $J(t, Q) \cap W \subseteq M$. □

Fix any integer $t \geq t_2 + 24g$, any integer x such that $1 \leq x < n$ and any general b-dimensional linear subspace W of $H^0(X, B^*(tP))$. Let (F, M) be a coherent subsystem of $(B^*(tP), W)$ such that $\mathrm{rank}(F) = x$. By Fourth Claim either $\dim(M) \leq x - 1$ or $\deg(F) \leq x + \gamma_x + \Delta$. First, assume $\dim(M) \leq x - 1$. Since $\mu(F) \leq \mu(B^*(tP)) + \gamma$, we have $\mu_\beta(F, M) + \Delta < \mu_\beta(B^*(tP))$ if $\alpha > (x-1)(\gamma + \Delta)/x$. Notice that $(x-1)(\gamma+\Delta)/x \leq (r-1)(\gamma+\Delta)/r$ depends only from E, not from t. Now assume $\deg(F) \leq x + \gamma + \Delta + \mu(E)$. Since $\mu(B^*(tP)) = \mu(B^*) + t$, there is an integer $t_0 \geq t_2 + 24g$, such that $\mu(E(tP)) > 1 + \gamma + \Delta$. By Third Claim we have $\mu_\beta(F, M) < \mu_\beta(B^*(tP)) - \Delta$ for every $\beta \gg 0$. Use the definition of Δ and of $(\alpha, -\beta)$-slope.

(b) Here we assume $r > a' + b$. The proof is a straightforward (but much easier) modification of the proof of part (a). For more details, see the proof of [1], Theorem 1. □

Remark 4. The proof of Theorem 1 shows that in the statement of Theorem 1 instead of $B(-tP)$ we may take the vector bundle $B \otimes L_t$, where L_t is any degree $-t$ line bundle on X.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Stable coherent systems on integral projective curves: an asymptotic existence theorem, *Int. J. Pure Appl. Math.*, **18**, No. 4 (2005), 495-497.

[2] E. Ballico, Stable coherent systems (E, V) with $\mathrm{rank}(E) = \dim(V)$ on projective varieties, *Int. J. Pure Appl. Math.*, **18**, No. 4 (2005), 499-505.

[3] S. B. Bradlow, O. Garcia-Prada, V. Muñoz, P.E. Newstead, Coherent systems and Brill-Noether theory, *Internat. J. Math.*, **14**, No. 7 (2003), 683-733.

[4] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.