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# REAL CURVES COVERING IN TWO WAYS A GIVEN REAL CURVE

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**Abstract:** Let $Y$ be a smooth and connected projective curve defined over $\mathbb{R}$. Here we construct real smooth projective curves with two morphisms onto $Y$.

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## 1. Real Curves Covering in Two Ways a Given Real Curve

For all smooth and connected complex projective curves $Y$ let $\bar{Y}$ denote the complex dual of $Y$. If $Y$ is defined over $\mathbb{R}$, then $Y \cong \bar{Y}$ (as complex algebraic schemes), but the converse is not always true. It is true if $\mathrm{Aut}(Y)$ has no order two element. For any $M \in \mathrm{Pic}^d(Y)$ let $\bar{M} \in \mathrm{Pic}^d(\bar{Y})$ denote the conjugate line bundle. Notice that $\bar{\bar{Y}} \cong Y$, $\bar{\bar{M}} \cong M$, $h^i(\bar{Y}, \bar{M}) = h^0(Y, M)$, $i = 0, 1$, and $\bar{\omega_Y} = \omega_{\bar{Y}}$. Furthermore, $M$ is spanned if and only if $\bar{M}$ is spanned.

**Remark 1.** Fix the smooth curve $Y$ and set $q := p_a(Y)$ and $S := Y \times \bar{Y}$. Let $\pi_1 : T \to Y$ and $\pi_2 : T \to \bar{Y}$ denote the projections. For all $M \in \mathrm{Pic}(Y)$ and $N \in \mathrm{Pic}^d(\bar{Y})$ set $\mathcal{O}T(M, N) := \pi_1^*(M) \otimes \pi_2^*(N)$. We have $h^0(T, \mathcal{O}_T(M, N) = h^0(Y, M) \cdot h^0(\bar{Y}, N)$ and $h^1(T, \mathcal{O}_T(M, N)) = h^1(Y, M) \cdot h^0(\bar{Y}, N) + h^0(Y, M) \cdot h^1(\bar{Y}, N)$ (Knünneth). From no on, we assume that we may identify $\bar{\bar{Y}}$, not just that these two curves are isomorphic. For instance, this assumption is satisfied if $Y$ is defined over $\mathbb{Y}$. Under this assumption complex conjugation

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acts on $T$. Furthermore, we may identify $\bar{\bar{M}}$ and $M$ and complex conjugation acts $H^0(T, \mathcal{O}_T(M, \bar{M})$ and hence it induces a real structure on this complex vector space. We will denote with $\sigma$ or with $^-$ the complex conjugation both on $T$, on any complex vector space $H^0(T, \mathcal{O}_T(M, \bar{M})$, on any $\sigma$-invariant complex linear subspace of $H^0(T, \mathcal{O}_T(M, \bar{M})$, and on the associated projective spaces. Let $Z$ be any $\sigma$-invariant zero-dimensional subscheme. The scheme $Z$ is defined over $\mathbb{R}$ and the projective space $|\mathbb{I}_Z(M, \bar{M})|$ (if non-empty) has a real structure. Hence any $D \in |\mathbb{I}_Z(M, \bar{M})|$ is a pure one-dimensional algebraic scheme defined over $\mathbb{R}$.

To state a part of Theorem 1 we need the following notation.

**Notation 1.** Fix smooth and connected projective curve $Y_i$, $i = 1, 2$, and set $S := Y_1 \times Y_2$. Let $\pi_i : S \to Y_i$, $i = 1, 2$, denote the projections. For all integers $a \geq 0$, $b \geq 0$, and any $P = (P_1, P_2) \in S$ set $(a, b)P := \{aP_1, Y_1\} \times \{bP_2, Y_2\} \subset S$. Hence $(a, b)P$ is a zero-dimensional subscheme of $T$ and length$((a, b)P) = ab$. We will say that $Z$ is a cubical fat point of type $(a, b)$. Set $(0, b)P = (a, 0)P = \emptyset$. When $Y_2 = \bar{Y}_1$ as in Remark 1 we will only use the case $a = b$ and $P \in T(\mathbb{R})$.

**Remark 2.** Take $S$ as in Notation 1. Fix integers $a > 0$ and $b > 0$ and $P \in S$. Set $Z := (a, b)P$ and let $C \subset S$ a sufficiently general curve containing $Z$. Let $u : \tilde{C} \to C$ be the partial normalization of $C$ in which we normalize only the point $P$. Hence $u_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C$ is a torsion sheaf whose support is contained in $P$. Set $\epsilon_{a,b} :=$ length$(u_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C)$. It is esay to see that $\epsilon_{a,b}$ is a symmetric function of $a$ and $b$ which vanishes if either $a = 1$ or $b = 1$. Set $c := \min\{a, b\}$. Since $\{cP\} \subseteq Z \subset \{(a + b)P\}$ and $m(m - 1)/2$ is the arithmetic genus of an ordinary planar singularity with multiplicity $m$, we have $c(c - 1)/2 \leq (a + b)(a + b - 1)/2$. Taking $a = 2$ and looking at the arithmetic genus of a tacnode, we get $\epsilon_{a,b} = (a - 1)(b - 1)$. We will say that $P$ is an ordinary $(a, b)$ point of $C$.

It is easy (and omitted) to adapt the proof of [1], Theorem 4, to get the following result.

**Theorem 1.** *Fix integers $g$, $q \geq 0$, $d > u > 0$, $u > 0$ and a smooth and connected projective curve $Y$ defined over $\mathbb{R}$ such that $p_a(Y) = q$, and there are $N \in Pic^u(Y_i)$, $M_\in Pic^d(Y)$ with $N$ spanned and $h^1(Y, M \otimes N^*) = 0$. Assume $g \geq (u+2q)d+(u+2q)q+qd-u-2q-d+1$ and $d^2+2dq-2d-2q+1-(d+q) \leq g \leq d^2+2dq-2d-2d+1$. Hence $d-u \geq 2q$. Set $\epsilon := d^2+2dq-2d-2d+1-g$. Hence $0 \leq \epsilon \leq d+q$. There is an integer $a$ such that $1 \leq a \leq d-u+1-q$ and $(a-1)^2 \leq \epsilon \leq (a-1)^2 + \lfloor (d-u+1-q-a-2)^2/3 \lfloor$. Set $s := \epsilon - (a-1)^2$.*

*Then there exist a smooth and connected genus $g$ curve $X$ defined over $\mathbb{R}$ and degree $d$ morphisms $f_i : X \to Y$, $i = 1, 2$, such that the induced map $(f_1, f_2) : X \to Y \times \bar{Y}$ is birational onto its image and $f_2$ is the complex conjugate of $f_1$. Furthermore, $(f_1, f_2)(X)$ has an ordinary real $(a, a)$ point and $s$ ordinary double points (all of them real) as its only singularities.*

**Remark 3.** When $q = 0$ Theorem 1 gives an existence theorem for real curves $X$ admitting two degree $d$ pencils $X \to \mathbf{P}^1$ which are not real, but exchanged by the complex conjugation. Now we will see that we may take $X$ with the additional condition that $d$ is the gonality of $X$ (as a complex curve). Assume the existence of an integer $m < d$ and of a degree $m$ morphism $h : X \to \mathbf{P}^1$. Since $2g - 2 > d(d-3) + (d-1)(d-2)$, the genus formula for curve in $\mathbf{P}^1 \times \mathbf{P}^1$ implies that the induced map $(f, h) : X \to \mathbf{P}^1 \times \mathbf{P}^1$ is not birational onto its image. This is impossible if there is $P \in \mathbf{P}^1$ such that $\sharp(f^{-1}(P)_{red}) = d - 2$, because in this case $f$ has exactly one ramification point and at least one non-ramification point over $P$. The proof of [1], Theorem 4, gives that we may find $X$ with this additional condition.

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## References

[1] E. Ballico, Pencils on curves covering a given curve, curves covering two given curves and cubical fat points, Preprint.