International Journal of Pure and Applied Mathematics
Volume 26 No. 1 2006, 1-20

ON THE REPRESENTATION OF τ-CONTINUOUS AND RADON FUNCTIONALS WITH LOWER AND UPPER LOEB INTEGRALS

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Abstract: We consider internal, positive linear functionals $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$, where $\mathcal{D}$ is a Stonian sublattice of all continuous functions on a topological Hausdorff space. Using the lower Loeb integral $\underline{i}$ we construct classical Radon functionals and Radon measures, using the upper Loeb integral $\overline{i}$ we construct classical τ-continuous functionals and τ-continuous measures. We apply these results to get classical representation theorems of τ-continuous functionals and of Radon functionals as integrals with respect to τ-continuous measures and Radon measures respectively.

AMS Subject Classification: 28E05, 26E35
Key Words: Loeb integral, standard part map, τ-continuous functionals, Radon functionals

1. Introduction

In this paper we consider a sufficiently rich superstructure $\widehat{S}$ and work with a nonstandard model for this superstructure, which is polysaturated, i.e. if $\mathcal{C}$ is a system of internal sets with cardinality $|\mathcal{C}|$ smaller or equal to the cardinality $|\widehat{S}|$

Received: September 15, 2005
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we have $\cap_{C\in\mathcal{C}} C \neq \emptyset$, if $\mathcal{C}$ has the finite intersection property. For the general theory of nonstandard analysis see the books of Cutland [2], Hurd et al [4] and Landers et al [8].

Let fin($^*\mathbb{R}$) be the set of finite numbers of $^*\mathbb{R}$. For $a \in \text{fin}(^*\mathbb{R})$, $^\circ a$ denotes the standard number nearest to a in $\mathbb{R}$. If $a \in {}^*\mathbb{R}$ is negative infinite (positive infinite) put $^\circ a = -\infty$ ($^\circ a = \infty$).

$(X, \mathcal{T})$ denotes always a topological Hausdorff space. The set of near-standard points in $^*X$ is denoted by $ns\,^*X$. As X is a Hausdorff space the standard part map $st : ns\,^*X \to X$ is uniquely defined. Let $\mathcal{B}(X)$ be the Borel σ-field of $(X, \mathcal{T})$ and $\nu : {}^*\mathcal{B}(X) \to {}^*[0, \infty[$ be a finite internal content. The following inner and outer Loeb measures

$$\begin{aligned}\underline{\nu}(A) &:= \sup\{^\circ\nu(C) : A \supset C \in {}^*\mathcal{B}(X)\}\,,\\ \overline{\nu}(A) &:= \inf\{^\circ\nu(C) : A \subset C \in {}^*\mathcal{B}(X)\}\end{aligned}$$

have been investigated in several papers, see e.g. Landers et al in [5] and [6]. For Radon measures see also Loeb [9].

For $B \in \mathcal{B}(X)$, the set functions $\underline{\nu}_{st}(B) := \underline{\nu}(st^{-1}B)$ and $\overline{\nu}_{st}(B) := \overline{\nu}(st^{-1}B)$ turned out to be Radon measures and τ-smooth measures respectively under appropriate conditions. These measures could be used to obtain a lot of classical results for topological measure theory, see Landers et al [5].

There is available now a theory of lower and upper Loeb integrals (see Landers et al [7], and Aldaz [1] and Loeb [10] for the Loeb integral). In this paper it is shown that the standard part map is even more powerful if we use lower and upper Loeb integrals instead of lower and upper Loeb measures.

For the sake of completeness we give some definitions. Let

$$\begin{aligned}\mathcal{C}(X) &:= \{f \in \mathbb{R}^X : f \text{ is continuous}\},\\ \mathcal{C}_b(X) &:= \{f \in \mathcal{C}(X) : f \text{ is bounded}\},\\ \mathcal{C}_c(X) &:= \{f \in \mathcal{C}(X) : f \text{ has compact support}\}.\end{aligned}$$

Let furthermore $\mathcal{D} \subset \mathcal{C}(X)$ be a Stonian lattice. A Stonian lattice $\mathcal{D}$ is called completely regular if for every $x \in X$ and $O \in \mathcal{T}$ with $x \in O$ there exists a function $f \in \mathcal{D}$ with $0 \le f \le 1$, $f(x) = 0$ and $f(z) = 1$ for $z \notin O$.

Assume that $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ is an internal, positive $^*\mathbb{R}$-linear functional. Then the lower and upper Loeb integrals $\underline{i}$ and $\overline{i}$ are defined for each $g : {}^*X \to {}^*\mathbb{R}$ by

$$\begin{aligned}\underline{i}(g) &= \sup\{^\circ i(e) : {}^*\mathcal{D} \ni e \le g, i(|e|) \in \text{fin}(^*\mathbb{R})\},\\ \overline{i}(g) &= \inf\{^\circ i(e) : g \le e \in {}^*\mathcal{D}, i(|e|) \in \text{fin}(^*\mathbb{R})\}.\end{aligned}$$

The following concept — which uses the standard part map for internal, positive linear functionals in a similar way as for internal contents — is crucial for this paper.

Definition 1. Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a Stonian lattice. Let $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ be an internal, positive ${}^*\mathbb{R}$-linear functional.

For $f : X \to [0, \infty[$ put

$\underline{i}_{st}(f) := \underline{i}((f \circ st) \cdot 1_{ns\,{}^*X}), \quad \overline{i}_{st}(f) := \overline{i}((f \circ st) \cdot 1_{ns\,{}^*X}),$

and

$$\mathcal{D}^{\underline{i}} := \{f \in \mathcal{D} : \underline{i}_{st}(|f|) < \infty\}, \quad \mathcal{D}^{\overline{i}} := \{f \in \mathcal{D} : \overline{i}_{st}(|f|) < \infty\}.$$

For $f : X \to \mathbb{R}$ put, if the right sides are defined,

$$\underline{i}_{st}(f) = \underline{i}_{st}(f^+) - \underline{i}_{st}(f^-), \quad \overline{i}_{st}(f) = \overline{i}_{st}(f^+) - \overline{i}_{st}(f^-).$$

For $T \subset X$ let

$$\underline{i}_{st}(T) := \underline{i}_{st}(1_T), \quad \overline{i}_{st}(T) := \overline{i}_{st}(1_T).$$

In Landers et al [7] we have defined $\underline{i}(A) = \underline{i}(1_A)$ for $A \subset {}^*X$. Hence $\underline{i}_{st}(T) = \underline{i}((1_T \circ st) \cdot 1_{ns\,{}^*X}) = \underline{i}(st^{-1}T)$, and $\overline{i}_{st}(T) = \overline{i}(st^{-1}T)$.

Now we give some definitions of classical concepts as used in this paper.

Definition 2. Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a Stonian lattice. Let $j : \mathcal{D} \to \mathbb{R}$ be a positive linear functional, and $f_\alpha \in \mathcal{D}_+ := \{f \in \mathcal{D} : f \geq 0\}$ be an arbitrary net of functions. Then:

(i) j is a τ-continuous functional if $f_\alpha \downarrow 0 \Rightarrow j(f_\alpha) \to 0$.

(ii) j is a Radon functional if for every $f_0 \in \mathcal{D}_+$ there holds $f_\alpha \leq f_0$ and $f_\alpha \to 0$ uniformly on compact sets $\Rightarrow j(f_\alpha) \to 0$.

If j is Radon then — according to the theorem of Dini — j is τ-continuous and hence especially σ-continuous. If j is Radon then it is easily seen that for every $f_0 \in \mathcal{D}_+$ with $f_0 \leq 1$ and every $\varepsilon \in \mathbb{R}_+$ there exists a compact set K with $j(f_0 - f) < \varepsilon$ for all $f \in \mathcal{D}$ with $0 \leq f \leq f_0$ and $f = f_0$ on K.

Definition 3. Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{B}(X)$ be the Borel σ-field of X. A measure $\mu : \mathcal{B}(X) \to [0, \infty]$ is:

(i) locally finite, if for every $x \in X$ there exists $O \in \mathcal{T}$ with $x \in O$ and $\mu(O) < \infty$;

(ii) outer regular, if

$$\mu(B) = \inf\{\mu(O) : B \subset O \in \mathcal{T}\} \text{ for all } B \in \mathcal{B}(X);$$

(iii) compact approximable, if

$$\mu(B) = \sup\{\mu(K) : K \subset B, K \text{ compact }\} \text{ for all } B \in \mathcal{B}(X);$$

(iv) a Radon measure, if μ is a locally finite and compact approximable;

(v) a τ-continuous measure, if

$$\mathcal{T} \supset \mathcal{T}_0 \uparrow O_1 \Rightarrow \mu(O_1) = \sup\{\mu(O) : O \in \mathcal{T}_0\};$$

(vi) a finitely τ-continuous measure, if

$$\mathcal{T} \supset \mathcal{T}_0 \uparrow O_1 \wedge \mu(O_1) < \infty \Rightarrow \mu(O_1) = \sup\{\mu(O) : O \in \mathcal{T}_0\}.$$

2. The Main Results

Theorem 4. *Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a Stonian lattice. If $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ is a positive ${}^*\mathbb{R}$-linear internal functional, then:*
(i) $\mathcal{D}^{\overline{i}}$ is a Stonian lattice;
(ii) $\overline{i}_{st} : \mathcal{D}^{\overline{i}} \to \mathbb{R}$ is a positive linear and τ-continuous functional;
(iii) Let moreover $\mathcal{D}$ be completely regular, then $\overline{i}_{st} : \mathcal{B}(X) \to [0, \infty]$ is a finitely τ-continuous outer regular measure with

$$\overline{i}_{st}(f_0) = \int f_0 \, d\,\overline{i}_{st} \quad \text{for all } 0 \leq f_0 \in \mathcal{D}$$

and hence

$$\overline{i}_{st}(f_0) = \int f_0 \, d\,\overline{i}_{st} \quad \text{for all } f_0 \in \mathcal{D}^{\overline{i}}.$$

Theorem 5. *Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. If $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ is a positive ${}^*\mathbb{R}$-linear internal functional, then:*
(i) $\mathcal{D}^{\underline{i}}$ is a Stonian lattice;
(ii) $\underline{i}_{st} : \mathcal{D}^{\underline{i}} \to \mathbb{R}$ is a positive linear Radon functional;
(iii) $\underline{i}_{st} : \mathcal{B}(X) \to [0, \infty]$ is a compact approximable measure with $\underline{i}_{st}(f_0) = \int f_0 \, d\,\underline{i}_{st}$ for all $0 \leq f_0 \in \mathcal{D}$ and hence

$$\underline{i}_{st}(f_0) = \int f_0 \, d\,\underline{i}_{st} \quad \text{for all } f_0 \in \mathcal{D}^{\underline{i}}.$$

Theorem 6. *Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. If $j : \mathcal{D} \to \mathbb{R}$ is a τ-continuous positive linear functional then $\overline{{}^*j}_{st} : \mathcal{B}(X) \to [0, \infty]$ is a finitely τ-continuous outer regular measure with*

$$j(f) = \int f \, d\,\overline{{}^*j}_{st} \quad \text{for all } f \in \mathcal{D}.$$

For a classical proof of some of the following results see Elstrodt [3].

Theorem 7. *Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. If $j : \mathcal{D} \to \mathbb{R}$ is a positive linear Radon functional, then $\underline{{}^*j}_{st} : \mathcal{B}(X) \to [0, \infty]$ is a Radon measure with*

$$j(f) = \int f \, d\,\underline{{}^*j}_{st} \quad \text{for all } f \in \mathcal{D}.$$

Corollary 8. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Let $\mathcal{D} = \mathcal{C}_c(X)$ and $j : \mathcal{C}_c(X) \to \mathbb{R}$ be a positive linear functional. Then $\underline{{}^*j}_{st}$ is a Radon measure on $\mathcal{B}(X)$ with*

$$j(f) = \int f \; d\underline{{}^*j}_{st} \quad \text{for all } f \in \mathcal{C}_c(X).$$

Corollary 9. *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Let $\mathcal{D} = \mathcal{C}_c(X)$ and $j : \mathcal{C}_c(X) \to \mathbb{R}$ be a positive linear functional. Then $\overline{{}^*j}_{st}$ is a τ-continuous outer regular measure on $\mathcal{B}(X)$ with*

$$j(f) = \int f \; d\overline{{}^*j}_{st} \quad \text{for all } f \in \mathcal{C}_c(X).$$

Corollary 10. *Let $(X, \mathcal{T})$ be a completely regular Hausdorff space. Let $\mathcal{D} = C_b(X)$ and $j : \mathcal{C}_b(X) \to \mathbb{R}$ be a positive linear and τ-continuous functional. Then $\overline{{}^*j}_{st}$ is a τ-continuous outer regular, finite measure on $\mathcal{B}(X)$ with*

$$j(f) = \int f \; d\overline{{}^*j}_{st} \quad \text{for all } f \in C_b(X).$$

Corollary 11. *Let $(X, \mathcal{T})$ be a completely regular Hausdorff space. Let $j : C_b(X) \to \mathbb{R}$ be a positive linear and Radon functional. Then $\underline{{}^*j}_{st}$ is a finite Radon measure with*

$$j(f) = \int f \; d\underline{{}^*j}_{st} \quad \text{for all } f \in C_b(X).$$

3. Proof of the Results and Some Auxiliary Lemmata

We prove at first that $\overline{i}_{st}$ is outer regular on $\mathcal{P}(X)$ for a completely regular Stonian lattice $\mathcal{D}$.

Lemma 1. *Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice and let $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ be an internal ${}^*\mathbb{R}$-linear functional. Then for each $X_0 \subset X$*

$$\overline{i}_{st}(X_0) = \inf\{\overline{i}_{st}(O) : X_0 \subset O \in \mathcal{T}\}.$$

Proof. It suffices to prove "$\geq$". Moreover assume w.l.o.g. that $\overline{i}_{st}(X_0) < \infty$. Let $g \in {}^*\mathcal{D}$ with

$$i(|g|) \in \text{fin}({}^*\mathbb{R}) \text{ and } 1_{st^{-1}X_0} \leq g \tag{1}$$

be given. It suffices to find $O \in \mathcal{T}$ with

$$1_{st^{-1}X_0} \leq 1_{st^{-1}O} \leq 1_C\,, \tag{2}$$

where $C := \{g \geq 1\}$ is an internal set.

For each $x_0 \in X_0$ we have $m_{\mathcal{T}}(x_0) \subset C$ by (1). As our model is polysaturated we obtain for each $x_0 \in X_0$ a set $O_{x_0} \in \mathcal{T}_{x_0}$ with

$${}^*O_{x_0} \subset C. \tag{3}$$

As $\mathcal{D}$ is a completely regular Stonian lattice for each $x_0 \in X_0$ there exists $f_{x_0} \in \mathcal{D}$ with

$$f_{x_0}(x_0) = 1 \text{ and } f_{x_0}(x) = 0 \text{ for } x \notin O_{x_0}. \tag{4}$$

Put

$$O := \bigcup_{x_0 \in X_0} \{f_{x_0} > 0\}.$$

Then O is open and $X_0 \subset O$. Hence $st^{-1}X_0 \subset st^{-1}O$. For (2) it suffices to prove

$$st^{-1}O \subset C. \tag{5}$$

Let now $y \in st^{-1}O$. Then there exists $x \in O$ with $y \approx x$. Hence, by definition of O, there exists $x_0 \in X_0$ with $f_{x_0}(x) > 0$. Therefore by (4) we have $x \in O_{x_0}$. Hence $y \in {}^*O_{x_0}$ and by (3) we have $y \in C$. This proves (5) and therefore the assertion. □

To prove that $\underline{i}_{st}$ is a compact approximable measure we need the following auxiliary result. Denote by $\mathcal{K}$ the system of all compact sets of a topological space.

Lemma 2. *Let $(X, \mathcal{T})$ be a regular topological space. Assume that $\mathcal{D} \subset \mathbb{R}^X$ is a Stonian lattice. Let $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ be a positive ${}^*\mathbb{R}$-linear internal functional and $0 \leq f_0 \leq 1$. Then*

$$\underline{i}[(f_0 \circ st) \cdot 1_{ns\,{}^*X}] = \sup_{K \in \mathcal{K}} \underline{i}[(f_0 \circ st) \cdot 1_{st^{-1}K}]. \tag{+}$$

Proof. It suffices to prove "$\leq$" in (+). Let

$${}^*\mathcal{D}_+^{\text{fin}} = \{0 \leq f \in {}^*\mathcal{D} : i(|f|) \in \text{fin}({}^*\mathbb{R})\}.$$

We have, according to Lemma 16 (i) of [7],

$$\underline{i}[(f_0 \circ st)\, 1_{ns\,{}^*X}] = \sup\{\underline{i}(g) : g \in {}^*\mathcal{D}_+^{\text{fin}},\ g \le (f_0 \circ st) 1_{ns\,{}^*X}\}$$
$$\le \sup\{\underline{i}[(f_0 \circ st) 1_{\{g>0\}}] : g \in {}^*\mathcal{D}_+^{\text{fin}}, g \le 1_{ns\,{}^*X}\}$$
$$\le \sup\{\underline{i}[(f_0 \circ st) 1_C] : C \subset ns\,{}^*X, C \text{ intern}\}$$
$$\le \sup\{\underline{i}[(f_0 \circ st) 1_{st^{-1}(st\,C)}] : C \subset ns\,{}^*X, C \text{ intern}\}$$
$$\le \sup\{\underline{i}[(f_0 \circ st) 1_{st^{-1}(K)}] : K \in \mathcal{K}\}.$$

In the last inequality we used the fact, that in a regular space $st\,C$ is compact for internal sets $C \subset ns\,{}^*X$ (see for instance 28.8 in Landers et al [8]). □

Let $\rho : \mathcal{P}({}^*X) \to [0, \infty]$ be a monotone function with $\rho(\emptyset) = 0$. Then it is well known that

$$\mathcal{M}(\rho) := \{M \subset {}^*X : \forall A \in \mathcal{P}({}^*X)(\rho(A) = \rho(A \cap M) + \rho(A \setminus M))\}$$

is an algebra and ρ is an additive function on $\mathcal{M}(\rho)$. We apply this result in the following to the set functions $\rho = \underline{i}$ or $\rho = \overline{i}$, where $\underline{i}(A) := \underline{i}(1_A)$, $\overline{i}(A) := \overline{i}(1_A)$ for $A \in \mathcal{P}({}^*X)$. Furthermore

$$\mathcal{M}_0 := \{A \in \mathcal{P}({}^*X) : 1_A \in \mathcal{L}(i)\} = \{A \in \mathcal{P}({}^*X) : \underline{i}(A) = \overline{i}(A) \in \mathbb{R}\}$$

is called the system of Loeb-integrable sets. For $M_0 \in \mathcal{M}_0$ we also write $i^L(M_0)$ instead of $\underline{i}(M_0)(= \overline{i}(M_0))$.

The systems $\mathcal{M}(\underline{i})$ and $\mathcal{M}(\overline{i})$ respectively are called the systems of lower Loeb-measurable sets and upper Loeb-measurable sets respectively. It was shown in Landers et al [7], that $\mathcal{M}(\underline{i})$ and $\mathcal{M}(\overline{i})$ are σ-fields and $\underline{i}|\mathcal{M}(\underline{i})$ and $\overline{i}|\mathcal{M}(\overline{i})$ are measures.

Lemma 3. *Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. If $i : {}^*\mathcal{D} \to {}^*\mathbb{R}$ is a positive ${}^*\mathbb{R}$-linear internal functional, then*

(i) st is $\mathcal{M}(\underline{i}) \cap ns({}^*X), \mathcal{B}(X)$-measurable;

(ii) $\underline{i}_{st}|\mathcal{B}(X)$ is a measure with
$$\underline{i}_{st}(T) = \sup\{\underline{i}_{st}(K) : T \supset K \in \mathcal{K}\}, T \subset X;$$
(iii) $\overline{i}_{st}|\mathcal{B}(X)$ is a finitely τ-continuous measure;

(iv) $i({}^*f)$ finite for all $f \in \mathcal{D} \Rightarrow \overline{i}_{st}|\mathcal{B}$ is locally finite.

Proof. (i) Let $O \in \mathcal{T}$ be given. As $\mathcal{D}$ is a completely regular Stonian lattice, we have
$$1_O = \sup\{f \in \mathcal{D}_+ : f \le 1_O\}. \tag{1}$$
$\mathcal{D} \subset \mathcal{C}(X)$ implies for $f \in \mathcal{D}$
$$^\circ({}^*f) 1_{ns\,{}^*X} = (f \circ st) \cdot 1_{ns\,{}^*X}. \tag{2}$$

According to Theorem 6 (i) of [7] we have

$$\sup_{\mathcal{D}_+ \ni f \le 1_O} {}^\circ({}^*f) \text{ is } \mathcal{M}(\underline{i})\text{-measurable.} \tag{3}$$

As $1_{st^{-1}O}|_{ns\,{}^*X} = \sup_{\mathcal{D}_+ \ni f \le 1_0} {}^\circ({}^*f)|_{ns\,{}^*X}$ by (1) and (2), we obtain $st^{-1}O \in \mathcal{M}(\underline{i}) \cap ns({}^*X)$ by (3).

(ii) $\underline{i}_{st}|\mathcal{B}(X)$ is a measure according to Theorem 4(i) of [7] and (i). Let now $T \subset X$ be given. As $\mathcal{D}$ is a completely regular Stonian lattice, $(X, \mathcal{T})$ is regular and hence $st\, C \in \mathcal{K}$ for internal $C \subset ns\,{}^*X$ (see [8]). Moreover $st(st^{-1}T) = T$, see 32.1 (ii) in [8]. Hence, according to Lemma 16 (i) of [7],

$$\begin{aligned} \underline{i}(T) &= \underline{i}(st^{-1}T) = \sup\{\underline{i}(g) :\ {}^*\mathcal{D}_+^{\text{fin}} \ni g \le 1_{st^{-1}T}\} \\ &\le \sup\{\underline{i}(g > 0) :\ {}^*\mathcal{D}_+^{\text{fin}} \ni g \le 1_{st^{-1}T}\} \\ &\le \sup\{\underline{i}(C) : C \subset st^{-1}T,\ C \text{ intern}\} \\ &\le \sup\{\underline{i}(st^{-1}(st\ C)) : C \subset st^{-1}T,\ C \text{ intern}\} \\ &\le \sup\{\underline{i}(st^{-1}(K)) : K \subset st(st^{-1}T),\ K \in \mathcal{K}\} \\ &= \sup\{\underline{i}_{st}(K) : T \supset K \in \mathcal{K}\}. \end{aligned}$$

(iii) $\bar{i}_{st}|\mathcal{B}(X)$ is a measure according to Theorem 3 (i) of [7] and (i). We prove now that $\bar{i}_{st}$ is finitely τ-continuous. Let

$$\mathcal{T}_0 \subset \mathcal{T} \quad \text{with} \quad \mathcal{T}_0 \uparrow O \quad \text{and} \quad \bar{i}_{st}(O) < \infty \tag{4}$$

be given. We have to prove

$$\bar{i}_{st}(O) \le \sup_{T_0 \in \mathcal{T}_0} \bar{i}_{st}(T_0). \tag{5}$$

Let $x_0 \in O$ be given. Then there exists $T_0 \in \mathcal{T}_0$ with $x_0 \in T_0$ according to (4). Since $\mathcal{D}$ is a completely regular Stonian lattice there exists $f_{x_0} \in \mathcal{D}$ with $0 \le f_{x_0} \le 1, f_{x_0}(x_0) = 1$ and $f_{x_0}(x) = 0$ for $x \notin T_0$. Put

$$\mathcal{F} := \{f_{x_1} \vee \ldots \vee f_{x_n} : x_1, \ldots, x_n \in O, n \in \mathbb{N}\}.$$

Then $\mathcal{F} \uparrow 1_O$, $\mathcal{F} \subset \mathcal{D}$. Hence $\sup_{f \in \mathcal{F}} {}^\circ({}^*f)1_{ns\,{}^*X} \underset{(2)}{=} \sup_{f \in \mathcal{F}}(f \circ st) \cdot 1_{ns\,{}^*X} = 1_O \circ st$, and we obain by Theorem 8(ii) of [8]

$$\bar{i}_{st}(O) = \bar{i}(st^{-1}O) = \sup_{f \in \mathcal{F}} \bar{i}((f \circ st)1_{ns\,{}^*X}). \tag{6}$$

Let now $f \in \mathcal{F}$ be given. Then by definition of $\mathcal{F}$ there exists $T_0 \in \mathcal{T}_0$ with $f(x) = 0$ if $x \notin T_0$ (use $\mathcal{T}_0 \uparrow$). Hence

$$\bar{i}((f \circ st)1_{ns\,{}^*X}) \le \bar{i}((1_{T_0} \circ st)1_{ns\,{}^*X}) \le \bar{i}_{st}(T_0). \tag{7}$$

Now (6) and (7) imply (5).

(iv) Let $x_0 \in X$ be given. As $\mathcal{D}$ is a completely regular Stonian lattice there exists $f \in \mathcal{D}$ with $0 \le f \le 1$ and $f(x_0) = 1$. Put $O = \{x \in X : f(x) > 1/2\}$. Then $x_0 \in O \in \mathcal{T}$ and $1_O \le 2f$. As f is continuous we obtain $1_{st^{-1}O} \le (2f \circ st)1_{ns\,{}^*X} \le 3\,{}^*f$. Hence $\overline{i}_{st}(O) = \overline{i}(st^{-1}O) \le \overline{i}(3\,{}^*f) < \infty$. □

Proof of Theorem 4. (i) $\mathcal{D}^{\overline{i}}$ is a lattice according to Lemma 1 of [7]. As $|1 \wedge f| \le |f|$, $\mathcal{D}^{\overline{i}}$ is Stonian.

(ii) $\overline{i}$ is trivially positive. We show at first that

$$\overline{i}_{st}(f) = \int {}^\circ({}^*f)|_{ns\,{}^*X}\, d\overline{i}|_{ns\,{}^*X} \text{ for } f \in \mathcal{D}^{\overline{i}}. \tag{1}$$

${}^\circ({}^*f)$ is $\mathcal{M}(\underline{i})$-measurable according to Lemma 15 (ii) of [7]. As f is continuous,

$${}^\circ({}^*f) \cdot 1_{ns\,{}^*X} = (f \circ st)1_{ns\,{}^*X}. \tag{2}$$

It suffices to prove (1) for $f \ge 0$. For $f \ge 0$ we have according to Theorem 7 (iii) of [7]

$$\int {}^\circ({}^*f)|_{ns\,{}^*X}\, d\overline{i}|_{ns\,{}^*X} = \overline{i}({}^\circ({}^*f)1_{ns\,{}^*X}) \underset{(2)}{=} \overline{i}((f \circ st)1_{ns\,{}^*X}) = \overline{i}_{st}(f).$$

From (1) it follows that $\overline{i}_{st}$ is linear on $\mathcal{D}^{\overline{i}}$. To prove that $\overline{i}_{st}$ is τ-continuous, let

$$\mathcal{F} \subset \mathcal{D}^{\overline{i}} \text{ with } \mathcal{F} \uparrow g \in \mathcal{D}^{\overline{i}} \text{ and } f \ge 0 \text{ for } f \in \mathcal{F}. \tag{3}$$

We have

$$\begin{aligned} \infty > \overline{i}_{st} &= \overline{i}((g \circ st)1_{ns\,{}^*X}) \underset{(3)}{=} \overline{i}([(\sup_{f \in \mathcal{F}} f) \circ st]1_{ns\,{}^*X}) \\ &\underset{(2)}{=} \overline{i}(\sup_{f \in \mathcal{F}} {}^\circ({}^*f) \cdot 1_{ns\,{}^*X}). \end{aligned} \tag{4}$$

Now we obtain the τ-continuity, using Theorem 8 (ii) of [7], by

$$\begin{aligned} \sup_{f \in \mathcal{F}} \overline{i}_{st}(f) &= \sup_{f \in \mathcal{F}} \overline{i}((f \circ st)1_{ns\,{}^*X} \underset{(2)}{=} \sup_{f \in \mathcal{F}} \overline{i}({}^\circ({}^*f)1_{ns\,{}^*X}) \\ &= \overline{i}(\sup_{f \in \mathcal{F}} {}^\circ({}^*f)1_{ns\,{}^*X}) \underset{(4)}{=} \overline{i}_{st}(g). \end{aligned}$$

(iii) According to Lemma 1 the set function $\underline{i}_{st}$ is outer regular and according to Lemma 3 (iii) it is a finitely τ-continuous measure. Let $0 \le f_0 \in \mathcal{D}$ be given. Then $f_0 \circ st$ is according to Lemma 3 (i) $\mathcal{M}(\overline{i}) \cap ns({}^*X)$-measurable. It is easy to see that there exists a $\mathcal{M}(\bar{i})$-measurable function f with $f|_{ns\,{}^*X} = f_0 \circ st$. Hence according to Definition 2 and Theorem 7 (iii) of [7]

$$\begin{array}{rcccl} \bar{i}_{st}(f_0) & \underset{\text{Def. 2}}{=} & \bar{i}((f_0 \circ st)1_{ns\,{}^*X}) & = & \bar{i}(f\ 1_{ns\,{}^*X}) \\ & \underset{7\text{(iii)}}{=} & \int f|_{ns\,{}^*X}\, d\bar{i}|_{ns\,{}^*X} & = & \int f_0 \circ st\ d\bar{i}|_{ns\,{}^*X} \\ & = & \int f_0\ d\bar{i}_{st}. & & \end{array}$$

□

Proof of Theorem 5. (i) We show at first that

$$\underline{i}_{st}(f) = \int {}^\circ({}^*f)|_{ns\,{}^*X}\, d\underline{i}|_{ns\,{}^*X} \text{ for } f \in \mathcal{D}^{\underline{i}}. \tag{1}$$

${}^\circ({}^*f)$ is $\mathcal{M}(\underline{i})$-measurable according to Lemma 15 (ii) of [7]. As f is continuous we have

$$^\circ({}^*f) \cdot 1_{ns\,{}^*X} = (f \circ st)1_{ns\,{}^*X}. \tag{2}$$

It suffices to prove (1) for $f \geq 0$. For $f \geq 0$ we have according to Theorem 7 (ii) of [7]

$$\int {}^\circ({}^*f)|_{ns\,{}^*X}\, d\underline{i}|_{ns\,{}^*X} = \underline{i}({}^\circ({}^*f)1_{ns\,{}^*X}) \underset{(2)}{=} \underline{i}((f \circ st)1_{ns\,{}^*X}) = \underline{i}_{st}(f).$$

From (1) we obtain that $\mathcal{D}^{\underline{i}}$ is a linear space with $|f|, 1 \wedge f \in \mathcal{D}^{\underline{i}}$ for $f \in \mathcal{D}^{\underline{i}}$.

(ii) From (1) we obtain that $\underline{i}_{st}$ is a positive linear functional. It remains to prove that $\underline{i}_{st}$ is a Radon functional. As $\underline{i}_{st}$ is σ-continuous according to (1), we may w.l.o.g. assume that $f_0 \in (\mathcal{D}^{\underline{i}})_+$ in Definition 2 (ii) fulfills $f_0 \leq 1$. Let now $\varepsilon > 0$ be given. According to Lemma 2 there exists $K_\varepsilon \in \mathcal{K}$ with

$$\underline{i}[(f_0 \circ st)1_{ns\,{}^*X}] \leq \underline{i}[(f_0 \circ st)1_{K_\varepsilon} \circ st] + \varepsilon. \tag{3}$$

Assume that $f_\alpha \in \mathcal{D}^{\underline{i}}$, with $0 \leq f_\alpha \leq f_0$, converges uniformly to zero on all $K \in \mathcal{K}$. According to (1) and (2) we have to show

$$\underline{i}_{st}(f_\alpha) = \int f_\alpha \circ st|_{ns\,{}^*X}\, d\underline{i}|_{ns\,{}^*X} \to 0. \tag{4}$$

For each $K \in \mathcal{K}$ we have according to Lemma 1(v) of [7]

$$\begin{array}{rcl} \underline{i}_{st}(f_\alpha) & = & \underline{i}((f_\alpha \circ st)1_{ns\,{}^*X}) \\ & \leq & \bar{i}((f_\alpha \circ st)1_K \circ st) + \underline{i}((f_\alpha \circ st)1_{ns\,{}^*X - st^{-1}K}) \\ & \leq & \bar{i}((f_\alpha \circ st)1_K \circ st) + \underline{i}((f_0 \circ st)1_{ns\,{}^*X - st^{-1}K}). \end{array}$$

Hence it suffices to find a compact set K and α_0 with

$$\underline{i}((f_0 \circ st)1_{ns\,{}^*X - st^{-1}K}) \leq \varepsilon. \tag{5}$$

and

$$\bar{i}((f_\alpha \circ st)1_K \circ st) \leq \varepsilon \;\text{ for all } \alpha \geq \alpha_0. \tag{6}$$

By Lemma 1 (iv) of [7] and (3) we have

$$\begin{aligned} &\underline{i}((f_0 \circ st)1_{ns\,{}^*X - st^{-1}K_\varepsilon}) \\ &\quad\leq \;\; \underline{i}((f_0 \circ st)1_{ns\,{}^*X}) - \underline{i}((f_0 \circ st)\; 1_{st^{-1}-K_\varepsilon}) \;\;\leq\;\; \varepsilon. \end{aligned} \tag{7}$$

Now $K_n := \{f_0 \geq \frac{1}{n}\} \cap K_\varepsilon \in \mathcal{K}$ with $K_n \uparrow \{f_0 > 0\} \cap K_\varepsilon$. As $st^{-1}K_n \in \mathcal{M}(\underline{i}) \cap ns\,{}^*X$ by Lemma 3 (i) we obtain from Theorem 7 (ii) of [7]

$$\begin{aligned} \underline{i}(f_0 \circ st)1_{st^{-1}K_n}) \quad &= \quad \underline{i}([{}^\circ({}^*f_0)1_{st^{-1}K_n}]1_{ns\,{}^*X}) \\ \uparrow \;\; \underline{i}({}^\circ({}^*f_0)1_{st^{-1}K_\varepsilon}) \quad &= \quad \underline{i}((f_0 \circ st)1_{st^{-1}K_\varepsilon}). \end{aligned} \tag{8}$$

From (7) and (8) we obtain (5) with $K = K_{n_0}$ for an appropriate n_0. We show that (6) holds for $K := K_{n_0}$. As f_α converges uniformly to zero on K and $f_0 \geq 1/n_0$ on K there exists α_0 such that

$$(f_\alpha \circ st)1_K \circ st \leq \frac{\varepsilon(f_0 \circ st)1_K \circ st}{\bar{i}((f_0 \circ st)1_{ns\,{}^*X}) + 1}$$

for all $\alpha \geq \alpha_0$.

Hence $\bar{i}((f_\alpha \circ st)1_K \circ st) \leq \varepsilon$ for all $\alpha \geq \alpha_0$, i.e. (6) holds.

(iii) $\underline{i}_{st} : \mathcal{B}(X) \to [0, \infty]$ is a compact approximable measure according to Lemma 3 (ii). Let $0 \leq f_0 \in \mathcal{D}$ be given. Then $f_0 \circ st$ is $\mathcal{M}(\underline{i}) \cap ns({}^*X)$-measurable. It is easy to see that there exists a $\mathcal{M}(\underline{i})$-measurable function f with $f|_{ns({}^*X)} = f_0 \circ st$. Hence according to Definition 2 and Theorem 7 (ii)

$$\begin{aligned} \underline{i}_{st}(f_0) \;&\underset{\text{Def. 2}}{=}\; \underline{i}((f_0 \circ st) \cdot 1_{ns\,{}^*X}) \;=\; \underline{i}(f\; 1_{ns\,{}^*X}) \\ &\underset{7\text{(ii)}}{=}\; \int f|_{ns\,{}^*X}\, d\underline{i}|_{ns\,{}^*X} \;=\; \int f_0 \circ st\; d\underline{i}|_{ns\,{}^*X} \\ &=\; \int f_0\, d\underline{i}_{st}. \end{aligned} \qquad \square$$

Lemma 4. *Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice and $j : \mathcal{D} \to \mathbb{R}$ be a positive linear and σ-continuous functional. Then ${}^\circ({}^*f) \in \mathcal{L}({}^*j)$ and*

$${}^*j^L({}^\circ({}^*f)) = j(f) \;\textit{for all}\; f \in \mathcal{D} \;\textit{with}\; 0 \leq f \leq 1.$$

Proof. As $0 \leq {}^*f \in {}^*\mathcal{D}$ and ${}^*j({}^*f) = j(f)$ we have ${}^*f \in {}^*\mathcal{D}^{\text{fin}}$ and hence ${}^\circ({}^*f) \in \mathcal{L}({}^*j)$ by Lemma 2 of [7]. Now for each $n \in \mathbb{N}$

$$ {}^*\mathcal{D} \ni {}^*f - {}^*f \wedge \frac{1}{n} \leq 2\ {}^\circ({}^*f) \in \mathcal{L}({}^*j). $$

Hence we obtain from Lemma 6 of [7] that ${}^\circ({}^*f) -\ {}^\circ({}^*f) \wedge \frac{1}{n} \in \mathcal{L}({}^*j)$ and

$$ {}^*j^L({}^\circ({}^*f) -\ {}^\circ({}^*f) \wedge \frac{1}{n}) =\ {}^\circ({}^*j({}^*f - {}^*f \wedge \frac{1}{n})) = j(f - f \wedge \frac{1}{n}) \leq j(f). \tag{1} $$

As f is σ-continuous we have

$$ j(f - f \wedge \frac{1}{n}) \to j(f). \tag{2} $$

According to Theorem 2 of [7] we obtain (use (1))

$$ {}^*j^L({}^\circ({}^*f) -{}^\circ ({}^*f) \wedge \frac{1}{n}) \to {}^*j^L({}^\circ({}^*f)). \tag{3} $$

From (1)–(3) we obtain $j(f) =\ {}^\circ j^L({}^\circ({}^*f))$. □

Lemma 5. *Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. If $j : \mathcal{D} \to \mathbb{R}$ is a τ-continuous linear functional, then*

$$ j(f) = \overline{{}^*j}_{st}(f) \quad \textit{for all } f \in \mathcal{D}. $$

Proof. It suffices to prove the assertion for $f \in \mathcal{D}$ with $0 \leq f \leq 1$. We have

$$ \overline{{}^*j}_{st}(f) = \overline{{}^*j}((f \circ st)1_{ns\,{}^*X}) = \overline{{}^*j}({}^\circ({}^*f)1_{ns\,{}^*X}). \tag{1} $$

Hence it suffices to prove

$$ j(f) = \overline{{}^*j}({}^\circ({}^*f)1_{ns\,{}^*X}). \tag{2} $$

Now $\overline{{}^*j}({}^\circ({}^*f)1_{ns\,{}^*X}) \leq \overline{{}^*j}({}^\circ({}^*f)) = j(f)$ according to Lemma 4.
Hence we have to prove

$$ \overline{{}^*j}({}^\circ({}^*f)1_{ns\,{}^*X}) \geq j(f). \tag{3} $$

Let $0 \leq h \in {}^*\mathcal{D}^{\text{fin}}$ with

$$ {}^\circ({}^*f)1_{ns\,{}^*X} \leq h \leq 1 \tag{4} $$

be given. We show that for each $n \in \mathbb{N}$, there exist internal sets $C_n \supset ns\,{}^*X$ with

$$\overline{{}^*j}({}^\circ({}^*f)1_{C_n}) \leq (1+\frac{1}{n})\,{}^\circ({}^*j(h)) + \frac{1}{n}. \tag{5}$$

Put for fixed $n \in \mathbb{N}$ and all $k \in \mathbb{N}$

$$D_k := \{{}^*f \leq (1+\frac{1}{n})h\} \cup \{{}^*f \leq \frac{1}{k}\}.$$

Then D_k are internal sets with $D_k \supset ns\,{}^*X$ (use (4)), $D_k \downarrow D = \{{}^*f \leq (1+\frac{1}{n})h\} \cup \{{}^*f \approx 0\}$. We have

$$^\circ({}^*f)\cdot 1_D \leq (1+\frac{1}{n})\,{}^\circ h. \tag{6}$$

By (6) and Lemma 2 of [7] we obtain

$$\overline{{}^*j}({}^\circ({}^*f)1_D) \leq (1+\frac{1}{n})\,\overline{{}^*j}({}^\circ h) \leq (1+\frac{1}{n})\,{}^\circ({}^*j(h)). \tag{7}$$

To prove (5) it suffices to show, according to (7), that

$$\overline{{}^*j}({}^\circ({}^*f)1_{D_k}) \downarrow {}^*j({}^\circ({}^*f)1_D), \tag{8}$$

as $D_k \supset ns\,{}^*X$ are internal. Hence it suffices to show

$$\lim_{k\to\infty} \overline{{}^*j}({}^\circ({}^*f)1_{D_k}) \leq \overline{{}^*j}({}^\circ({}^*f)1_{\{{}^*f\leq(1+\frac{1}{n})h\}}). \tag{8}$$

Now

$$\overline{{}^*j}({}^\circ({}^*f)1_{D_k}) \leq \overline{{}^*j}({}^\circ({}^*f)1_{\{{}^*f\leq(1+\frac{1}{n})h\}}) + \overline{{}^*j}({}^\circ({}^*f)1_{\{{}^*f\leq 1/k\}}). \tag{9}$$

As

$$\overline{{}^*j}({}^\circ({}^*f)1_{\{{}^*f\leq 1/k\}}) \leq \overline{{}^*j}({}^\circ({}^*f)1_{\{{}^\circ({}^*f)\leq 1/k\}}) \underset{k\to\infty}{\rightarrow} 0, \tag{10}$$

(10) implies (9). To prove (3) choose $x \in X$. As $ns\,{}^*X \subset C_n$ and C_n is internal, there exist $O_x \in \mathcal{T}_x$ with

$${}^*O_x \subset C_n. \tag{11}$$

As $\mathcal{D}$ is a completely regular Stonian lattice, to each $x \in X$ there exists $g_x \in \mathcal{D}$ with

$$0 \leq g_x \leq 1,\ g_x(x) = 1 \text{ and } g_x(y) = 0 \text{ for } y \notin O_x. \tag{12}$$

Hence

$$0 \le {}^*g_x \le 1,\ {}^*g_x(x) = 1,\ {}^*g_x(y) = 0 \text{ for } y \notin {}^*O_x. \tag{13}$$

According to (11) we obtain by (13)

$$0 \le {}^*g_x \le 1_{C_n}. \tag{14}$$

Put $G = \{g_{x_1} \vee \ldots \vee g_{x_k} : x_1, \ldots, x_k \in X, k \in \mathbb{N}\}$. Then $G \uparrow 1$ by (12). As $0 \le {}^\circ({}^*g) \le 1_{C_n}$ for $g \in G$ (use (14)) the τ-continuity of j and Lemma 4 imply

$$\begin{array}{rcccl} \overline{{}^*j}({}^\circ({}^*f)1_{C_n}) & \ge & \sup\limits_{g\in G} \overline{{}^*j}({}^\circ({}^*f) \wedge {}^\circ({}^*g)) & = & \sup\limits_{g\in G} \overline{{}^*j}({}^\circ({}^*(f \wedge g))) \\ & = & \sup\limits_{g\in G} j(f \wedge g) & = & j(f). \end{array} \tag{15}$$

Hence by (15) and (5) we obtain for all $n \in \mathbb{N}$,

$$(1 + \frac{1}{n})\ {}^\circ({}^*j(h)) + \frac{1}{n} \ge j(f).$$

Therefore $n \to \infty$ implies ${}^\circ({}^*j(h)) \ge j(f)$. As $0 \le h \in {}^*\mathcal{D}^{\text{fin}}$ with ${}^\circ({}^*f)1_{ns\,{}^*X} \le h \le 1$ was arbitrary we obtain the assertion (3). □

Proof of Theorem 6. Apply Theorem 4 to $i := {}^*j$. Then $\overline{{}^*j}_{st}$ is a finitely τ-continuous outer regular measure with

$$\overline{{}^*j}_{st}(f_0) = \int f_0 \, d\overline{{}^*j}_{st} \text{ for all } 0 \le f_0 \in \mathcal{D}.$$

As, by Lemma 5, $\overline{{}^*j}_{st}(f_0) = j(f_0)$ we obtain the assertion. □

Lemma 6. *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. Let $O \in \mathcal{T}$ and $f_0 \in \mathcal{D}$ with $0 \le f_0 \le 1$ be given. Let $K \in \mathcal{K}$ with $K \subset O$ be fixed. Then there exists $f_K \in \mathcal{D}$ with $0 \le f_K \le f_0 1_O$ and $f_K = f_0$ on K.*

Proof. We construct below $g_K \in \mathcal{D}$ with $0 \le g_K \le 1_O$ and $g_K|K \equiv 1$. Then we have $f_K := f_0 \wedge g_K \in \mathcal{D}$ with $0 \le f_K \le f_0 1_O$ and $f_K = f_0$ on K.

To construct g_K choose for each $x \in K$ an $g_x \in \mathcal{D}$ with $g_x(x) = 1$, $g_x \equiv 0$ on $\overline{O}$ and $0 \le g_x \le 1$. Then there exists $O_x \in \mathcal{T}$ with $O_x \subset O$ and $g_x|O_x \ge 1/2$. Hence, as K is compact, there exists $h_K \in \mathcal{D}$ with $h_K \ge 1/2$ on K, $0 \le h_K \le 1$ and $h_K = 0$ on $\overline{O}$. Now $g_K = (2h_K \wedge 1) \in \mathcal{D}$, $0 \le g_K \le 1$, $g_K = 0$ on $\overline{O}$ and $g_K = 1$ on K. □

Lemma 7. *Let $i : \mathcal{E} \to \mathbb{R}$ be a positive ${}^*\mathbb{R}$-linear internal functional. Let $0 \le g \le 1$ internal with ${}^\circ g \in \mathcal{L}(i)$. Let $\mathcal{C}$ be a downwards directed system of internal sets with $|\mathcal{C}| \le |\widehat{S}|$. Then*

$$\underline{i}({}^\circ g 1_{\bigcap_{C\in\mathcal{C}} C}) = \inf_{C\in\mathcal{C}} \underline{i}\,{}^\circ(g 1_C).$$

Proof. We show the assertion at first for functions g for which there exists $\delta \in \mathbb{R}_+$ with

$$g(y) = 0 \text{ or } g(y) \geq \delta \text{ for each } y \in Y. \tag{1}$$

As $^\circ g \in \mathcal{L}(i)$ we have $\alpha := \inf\limits_{C \in \mathcal{C}} \underline{i}(^\circ g 1_C) < \infty$. Choose $\varepsilon \in \mathbb{R}_+$. It suffices to show

$$\underline{i}(^\circ g 1_{\bigcap\limits_{C \in \mathcal{C}} C}) \geq \alpha - \varepsilon. \tag{2}$$

Define for $n \in \mathbb{N}$

$$\mathcal{F}_{C,\varepsilon,n} = \{0 \leq h \in \mathcal{E} : h \leq (1 + \frac{1}{n}) g 1_C,\ i(h) \geq \alpha - \varepsilon/2\}.$$

Then $\mathcal{S} := \{\mathcal{F}_{C,\varepsilon,n} : C \in \mathcal{C}, \varepsilon \in \mathbb{R}_+, n \in \mathbb{N}\}$ is a familiy of internal sets. We show that this family has the finite intersection property.

Let $\mathcal{F}_{C_i,\varepsilon_i,n_i}$ for $i = 1, \dots, k$ be given. Put

$$\varepsilon := \min \varepsilon_i,\ n := \max n_i \tag{3}$$

and choose a set C_0 with

$$C_0 \in \mathcal{C},\ C_0 \subset \bigcap_{i=1}^{k} C_i. \tag{4}$$

It suffices — using the definitions (3) and (4) — to find $h \in \mathcal{F}_{C_0,\varepsilon,n}$. According to the definition of α there exists

$$0 \leq h \in \mathcal{E}^{\text{fin}},\ h \leq {}^\circ g 1_{C_0},\ {}^\circ i(h) \geq \alpha - \varepsilon/4. \tag{5}$$

Hence $i(h) \geq \alpha - \varepsilon/2$ and $h \leq (1 + \frac{1}{n}) g 1_{C_0}$, whence $h \in \mathcal{F}_{C_0,\varepsilon,n}$. As $\mathcal{C}$ is a system of internal sets with the finite intersection property and $|\mathcal{S}| \leq |\widehat{S}|$ (use $|\mathcal{C}| \leq |\widehat{S}|$), there exists

$$\widetilde{h} \in \cap \{\mathcal{F}_{C,\varepsilon,n} : C \in \mathcal{C}, \varepsilon \in \mathbb{R}_+, n \in \mathbb{N}\}. \tag{6}$$

Then $0 \leq \widetilde{h} \in \mathcal{E}$ and $i(\widetilde{h}) \geq \alpha - \varepsilon/2$.

To prove (2) it suffices to show

$$\widetilde{h} \leq (1 + \frac{1}{n})\, {}^\circ g 1_{\bigcap\limits_{C \in \mathcal{C}}}. \tag{7}$$

By (6) we have

$$\widetilde{h} \leq (1 + \frac{1}{2n}) g 1_{\bigcap\limits_{C \in \mathcal{C}} C} \text{ for all } n \in \mathbb{N}. \tag{8}$$

Now (7) follows from (1). This proves the assertion for g fulfilling (1).

Let now g be an arbitrary internal function with $0 \le g \le 1$ and $^\circ g \in \mathcal{L}(i)$. Then $g_\delta := g1_{\{g \ge \delta\}}$ is an internal function fulfilling (1) with $0 \le g_\delta \le 1$ and $^\circ g_\delta \in \mathcal{L}(i)$ for all $\delta \in \mathbb{R}_+$ for which $\{g \ge \delta\} \in \mathcal{M}(\underline{i})$ (use Theorem 3 (v) of [7]).

Now $\{g \ge \delta\} \notin \mathcal{M}(\underline{i})$ for at most countably many $\delta \in \mathbb{R}_+$, because $^\circ g \in \mathcal{L}(i)$ and hence by Theorem 4(v) of [7] $\underline{i}(^\circ g = \delta) > 0$ for at most countably many $\delta \in \mathbb{R}_+$. Therefore, for δ with $\underline{i}(^\circ g = \delta) = 0$, we have $\{g \ge \delta\} \in \mathcal{M}(\underline{i})$ as

$$\{^\circ g > \delta\} \subset \{g \ge \delta\} \subset \{^\circ g \ge \delta\}$$

and $\underline{i}|\mathcal{M}(\underline{i})$ is a complete measure. Choose for $n \in \mathbb{N}$ an $\delta_n \le \frac{1}{n}$ such that $^\circ g_{\delta_n} \in \mathcal{L}(i)$. Hence, as g_{δ_n} fulfills (1), we have

$$\underline{i}(^\circ g 1_{\bigcap\limits_{C\in\mathcal{C}} C}) \;\ge\; \underline{i}(^\circ g_{\delta_n} 1_{\bigcap\limits_{C\in\mathcal{C}} C}) \underset{(1)}{=} \inf_{C\in\mathcal{C}} \underline{i}(^\circ g_{\delta_n} 1_C).$$

To prove the assertion we have to show

$$\lim_{n\in\mathbb{N}} \inf_{C\in\mathcal{C}} \underline{i}(^\circ g_{\delta_n} 1_C) = \inf_{C\in\mathcal{C}} \underline{i}(^\circ g 1_C). \tag{10}$$

Now by Theorem 7 (ii) of [7]

$$\begin{array}{rcccl} 0 & \le & \inf\limits_{C\in\mathcal{C}} \underline{i}(^\circ g 1_C) - \inf\limits_{C\in\mathcal{C}} \underline{i}(^\circ g_{\delta_n} 1_C) & \le & \sup\limits_{C\in\mathcal{C}} [\underline{i}(^\circ g 1_C) - \underline{i}(^\circ g_{\delta_n} 1_C)] \\ & = & \sup\limits_{C\in\mathcal{C}} \underline{i}[(^\circ g - {}^\circ g_{\delta_n}) 1_C] & \le & \underline{i}(^\circ g - {}^\circ g_{\delta_n}) \\ & = & \underline{i}(^\circ g 1_{\{g<\delta_n\}}) & \le & \underline{i}(^\circ g 1_{\{^\circ g \le \delta_n\}}) \underset{n\to\infty}{\rightarrow} 0, \end{array}$$

as $^\circ g \; 1_{\{^\circ g \le \delta_n\}} \to 0$ and $^\circ g \in \mathcal{L}(i)$. This proves (10). □

Lemma 8. *Let $(X, \mathcal{T})$ be a Hausdorff space. Let $\mathcal{D} \subset \mathcal{C}(X)$ be a completely regular Stonian lattice. Let furthermore $j : \mathcal{D} \to \mathbb{R}$ be a positive Radon functional. Then*

$$^*\underline{j}_{st}(f) = j(f) \text{ for all } f \in \mathcal{D}.$$

Proof. As by definition $^*\underline{j}_{st}(f) = {}^*\underline{j}_{st}(f^+) - {}^*\underline{j}_{st}(f^-)$ for $f \in \mathcal{D}$, we may assume $0 \le f \in \mathcal{D}$.

As $^*\underline{j}_{st}(f \wedge n) = {}^*\underline{j}([^\circ(^*f) \wedge n] 1_{ns\,^*X})$, we obtain by Theorem 7 (ii) and Lemma 15 (ii) of [7]

$$^*\underline{j}([^\circ(^*f) \wedge n] 1_{ns\,^*X}) \uparrow {}^*\underline{j}(^\circ(^*f) 1_{ns\,^*X}) = {}^*\underline{j}_{st}(f).$$

As furthermore j is σ-continuous we may assume $0 \le f \le 1$.

By Lemma 4 we have ${}^*\underline{j}_{st}(f) = {}^*\underline{j}({}^\circ({}^*f)1_{ns\,{}^*X}) \leq {}^*\underline{j}({}^\circ({}^*f)) = j(f)$. Therefore it suffices to prove

$$f_0 \in \mathcal{D},\ 0 \leq f_0 \leq 1 \Rightarrow {}^*\underline{j}_{st}(f_0) \geq j(f_0). \tag{1}$$

By Lemma 7 we have for each $K \in \mathcal{K}$

$$\begin{aligned} {}^*\underline{j}_{st}(f_0) &\geq {}^*\underline{j}_{st}(f_0 1_K) = {}^*\underline{j}(((f_0 \cdot 1_K) \circ st)\ 1_{ns\,{}^*X}) \\ &= {}^*\underline{j}({}^\circ({}^*f_0)1_{st^{-1}K}) = {}^*\underline{j}({}^\circ({}^*f_0)1_{\bigcap_{\mathcal{T}\ni O\supset K} {}^*O}) \\ &= \inf_{\mathcal{T}\ni O\supset K} {}^*\underline{j}({}^\circ({}^*f_0)1_{{}^*O}). \end{aligned} \tag{2}$$

Now let $\varepsilon \in \mathbb{R}_+$ and $K \in \mathcal{K}$ be given. According to (2) there exists $O \in \mathcal{T}$ with $K \subset O$ and

$${}^*\underline{j}_{st}(f_0) \geq {}^*\underline{j}({}^\circ({}^*f_0)1_O) - \varepsilon/2 = {}^*\underline{j}({}^\circ({}^*(f_0 \cdot 1_O)) - \varepsilon/2. \tag{3}$$

As j is a Radon functional there exists $K_\varepsilon \in \mathcal{K}$ such that

$$f \in \mathcal{D},\ 0 \leq f \leq f_0,\ f = f_0 \text{ on } K_\varepsilon \Rightarrow j(f) \geq j(f_0) - \varepsilon/2. \tag{4}$$

By (3), applied to K_ε, there exists $O_\varepsilon \in \mathcal{T}$ with $K_\varepsilon \subset 0_\varepsilon$ and

$${}^*\underline{j}_{st}(f_0) \geq {}^*\underline{j}({}^\circ({}^*(f_0)1_{O_\varepsilon})) - \varepsilon/2. \tag{5}$$

Now, according to Lemma 6, there exists $f_{K_\varepsilon} \in \mathcal{D}$ with

$$0 \leq f_{K_\varepsilon} \leq f_0 1_{O_\varepsilon},\ f_{K_\varepsilon} = f_0 \text{ on } K_\varepsilon.$$

Hence by Lemma 4

$$\begin{aligned} {}^*\underline{j}_{st}(f_0) &\underset{(5)}{\geq} {}^*\underline{j}({}^\circ({}^*(f_0)1_{O_\varepsilon}))) - \varepsilon/2 &\geq& {}^*\underline{j}({}^\circ({}^*(f_{K_\varepsilon}))) - \varepsilon/2 \\ &= j(f_{K_\varepsilon}) - \varepsilon/2 &\underset{(4)}{\geq}& j(f_0) - \varepsilon. \end{aligned}$$

□

Proof of Theorem 7. Applying Theorem 5 to $i = {}^*j$ we obtain that

$${}^*\underline{j}_{st} \text{ is a compact approximable measure} \tag{1}$$

with

$${}^*\underline{j}_{st}(f_0) = \int f_0\ d{}^*\underline{j}_{st} \ \text{ for all } 0 \leq f_0 \in \mathcal{D}. \tag{2}$$

By Lemma 8 we have ${}^*\underline{j}_{st}(f_0) = j(f_0)$, we thus obtain from (2)

$$j(f) = \int f \, d{}^*\underline{j}_{st} \quad \text{for all } f \in \mathcal{D}.$$

As ${}^*j({}^*f) = j(f) \in \mathcal{D}$ for all $f \in \mathcal{D}$, we obtain that ${}^*\underline{j}_{st}$ is locally finite by Lemma 3 (iv) and hence a Radon measure by (1). □

Proof of Corollary 8. As X is completely regular, it follows directly that $\mathcal{C}_c(X)$ is a completely regular Stonian lattice. Let $f_0 \in \mathcal{D}_+$ and $f_\alpha \leq f_0$ be a net which converges to zero uniformly on all compact sets. Let $\varepsilon \in \mathbb{R}_+$ be given. Then, as $f_0 \in C_c(X)$, we have $\sqrt{f_\alpha} \leq \varepsilon$ for all sufficiently large α. Hence $j(f_\alpha) \leq j(\sqrt{f_\alpha}\sqrt{f_\alpha}) \leq j(\varepsilon\sqrt{f_0}) = \varepsilon j(\sqrt{f_0}))$, whence j is a Radon functional. Hence Theorem 7 implies the assertion. □

Proof of Corollary 9. It is well known that j is τ-continuous. Moreover $\mathcal{D} = \mathcal{C}_c(X)$ is a completely regular Stonian lattice. According to Theorem 6 it suffices to show

$$\mathcal{T} \supset \mathcal{T}_0 \uparrow O_1 \text{ and } \overline{{}^*j}_{st}(O_1) = \infty \Rightarrow \sup_{O \in \mathcal{T}_0} \overline{{}^*j}_{st}(O) = \infty. \tag{1}$$

Since $\mathcal{T}_0 \uparrow O_1$ it suffices to prove

$$\sup_{O_1 \supset K \in \mathcal{K}} \overline{{}^*j}_{st}(K) = \infty. \tag{2}$$

According to Lemma 3 (ii) it is sufficient to show

$${}^*\underline{j}_{st}(O_1) = \overline{{}^*j}_{st}(O_1). \tag{3}$$

As X is locally compact and 1_{O_1} is lower semicontinuous we have

$$1_{O_1} = \sup\{f \in \mathcal{D}_+(X) : f \leq 1_{O_1}\} \tag{4}$$

(see Pfeffer [11], Proposition 3.5).

As f is continuous with compact support we have

$${}^\circ({}^*f) = (f \circ st) 1_{ns\, {}^*X}. \tag{5}$$

By (4) and (5)

$$\begin{aligned} 1_{st^{-1}O_1} &= \sup\{(f \circ st)1_{ns\, {}^*X} : f \leq 1_{O_1}, f \in \mathcal{D}_+\} \\ &= \sup\{{}^\circ({}^*f) : f \leq 1_{O_1}, f \in \mathcal{D}_+\}. \end{aligned}$$

Hence by Theorem 6 (ii) of [7] and Lemma 4 we have

$$\overline{{}^*j}_{st}(O_1) = \overline{{}^*j}(\sup\{{}^\circ({}^*f) : f \leq 1_{O_1}, f \in \mathcal{D}_+\})$$
$$= \sup\{\overline{{}^*j}({}^\circ({}^*f)) : f \leq 1_{O_1}, f \in \mathcal{D}_+\} = \sup\{j(f) : \mathcal{D}_+ \ni f \leq 1_{O_1}\}.$$

Similarly (by Theorem 6 (iii) of [7] and Lemma 4) we have $\underline{{}^*j}_{st}(O_1)$ $= \sup\{j(f) : \mathcal{D}_+ \ni f \leq 1_{O_1}\}$. Hence we obtain (3). □

Proof of Corollary 10. $\mathcal{D} := C_b(X)$ is a completely regular Stonian lattice. By Theorem 6, $\overline{{}^*j}_{st} : \mathcal{B} \to [0, \infty]$ is an outer regular measure with $j(f) = \int f d\overline{{}^*j}_{st}$ for all $f \in C_b(X)$. Hence $\infty > j(1) = \overline{{}^*j}_{st}(X)$, whence $\overline{{}^*j}_{st}$ is finite and τ-continuous according to Theorem 6.

Proof of Corollary 11. This follows from Theorem 7, as $C_b(X)$ is a completely regular Stonian lattice. □

Corollaries 10 and 11 hold true even for $C(X)$ instead of $C_b(X)$.

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