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DISTANCE FORMULAE IN THE CHINESE CHECKER SPACE

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Abstract: In this work, we generalize Chinese checker metric to three dimensional case and use it to give distance formulas between a point and a line, a point and a plane, and distance between two lines in three dimensional Chinese checker space.

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1. Chinese Checker Metric for Three Dimensional Space

Krause [4] asked the question of how to develop a metric which would be similar to the movement made by playing Chinese checker. Later Chen [2] developed Chinese checker metric for plane as follows:

$$d_c(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \left(\sqrt{2} - 1\right) \min\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. The Chinese checker plane geometry has been studied and improved up to now (see [3], [5], [6], [7]). The above metric

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can be generalized and the Chinese checker space of dimension three can be introduced using this metric in three dimensional analytical space by

$$d_c(P_1,P_2) = d_L(P_1,P_2) + \left(\sqrt{2}-1\right) d_S(P_1,P_2),$$

where

$$d_L(P_1,P_2) = \max\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\}$$

and

$$\begin{aligned} &d_S(P_1,P_2) \\ &\quad = \min\{|x_1-x_2|+|y_1-y_2|, |x_1-x_2|+|z_1-z_2|, |y_1-y_2|+|z_1-z_2|\}, \end{aligned}$$

instead of the well known Euclidean metric

$$d_E(P_1,P_2) = \left[(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\right]^{1/2}$$

where $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. That is, three dimensional Chinese checker (CC) space $\mathbb{R}^3_c$ is constructed by simply replacing the Euclidean distance function d_E by the Chinese checker distance function d_c.

According to the definition of d_c-metric, the shortest way between the points P_1 and P_2 is the union of three line segments, one of which is parallel to a coordinate axis, direction vector of the second segment can be chosen as an element of

$$\{(p,q,r) : \text{only one of } p,q,r \text{ is zero and the others are element of } \{-1,1\}\}$$

and direction vector of the third segment can be chosen as an element of $\{(p,q,r) : p,q,r \in \{-1,1\}\}$ as in Figure 1.

In this work, we study Chinese checker analogues of some topics of the Euclidean space which include the concept of distance. These topics are distances of a point to a plane, and to a line, and distance between two lines. Taxicab analogues of all of these topics have been given in [1] which led us to this subject.

2. Distance Formulae

In the three dimensional Chinese checker (CC) space, points, lines and planes are the same with in the Euclidean case. It can be easily shown that if

$$d_c : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$$

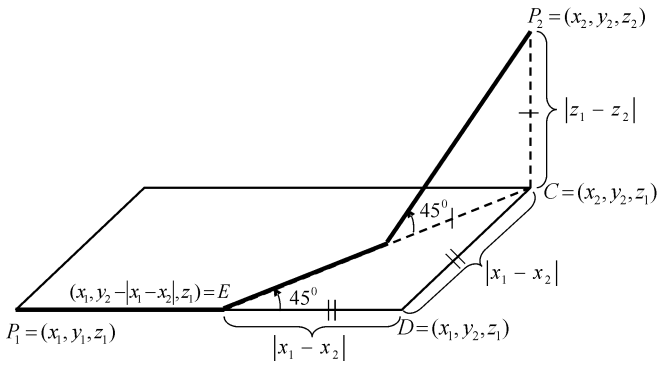

Figure 1: CC way from P_1 to P_2 in the case $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$

is defined as

$$d_c(P_1, P_2) = d_L(P_1, P_2) + \left(\sqrt{2} - 1\right) d_S(P_1, P_2),$$

then $\mathbb{R}_c^3 := \left(\mathbb{R}^3, d_c\right)$ forms a metric space.

Let l be a line through the points P_1 and P_2. If l has direction vector (p, q, r), then it can be show that

$$\frac{d_E(P_1, P_2)}{(p^2+q^2+r^2)^{1/2}} = \frac{d_c(P_1, P_2)}{\left(\max\{|p|, |q|, |r|\} + \left(\sqrt{2}\text{-}1\right)\min\{|p|+|q|, |p|+|r|, |q|+|r|\}\right)}.$$

It is easy to show that $d_E(P_1, P_2) \leq d_c(P_1, P_2)$ for every P_1 and P_2 by the above result. Furthermore, $d_c(P_1, P_2)$ is minimum if and only if direction vector of the line passing through P_1 and P_2 can be taken as an element of $\Delta = \{(p, q, r) \mid p, q, r \in \{-1, 0, 1\} \text{ and } (p, q, r) \neq (0, 0, 0)\}$.

Theorem 1. *The CC-distance is invariant under all translations in $\mathbb{R}_c^3$. In other words,*

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \ni T(x, y, z) = (x + a, y + b, z + c) \quad , \quad a, b, c \in \mathbb{R}$$

does not change the distance between any two points in $\mathbb{R}_c^3$.

Proof is trivial.

Theorem 2. *CC-distance of a point $P=(x_0,y_0,z_0)$ to a plane $\mathcal{P}: Ax+By+Cz+D=0$ is*

$$d_c(P,\mathcal{P}) = \min\left\{\begin{array}{l} \dfrac{(2\sqrt{2}-1)\ |Ax_0+By_0+Cz_0+D|}{\max\{|A+B+C|,|-A+B+C|,|A-B+C|,|A+B-C|\}}, \\ \dfrac{\sqrt{2}\ |Ax_0+By_0+Cz_0+D|}{\max\{|A+B|,|A-B|,|A+C|,|A-C|,|B+C|,|B-C|\}}, \\ \dfrac{|Ax_0+By_0+Cz_0+D|}{\max\{|A|,|B|,|C|\}}, \end{array}\right.$$

where it is obvious that each of the expressions in the denominators is not zero.

Proof. In the CC-space $\mathbb{R}^3_c$, the distance from a point P to a plane $\mathcal{P}$ is defined as

$$d_c(P,\mathcal{P}) = \min\{d_c(P,X) \mid X\in\mathcal{P}\}\ .$$

If $d_c(P,X)$ is minimum, then direction vector of the line passing through P and X is an element of Δ. So consider lines l_i $(i=1,2,\ldots,13)$ passing through P and each of them has an element of Δ as direction vector. Let $P_i = l_i\cap\mathcal{P}$, $i=1,2,\ldots,13$. Clearly, there exist at least three such points P_i since $\mathcal{P}$ cannot be parallel to all of the coordinate axes. Thus, $d_c(P,\mathcal{P}) = \min\{d_c(P,P_i) : i=1,2,\ldots,13\}$. According to values of p,q,r the following three main cases are possible:

Case I. $p\neq 0$, $q\neq 0$, $r\neq 0$. In this case, if (p,q,r) is $(1,1,1)$, then $P_1=(k_1+x_0,k_1+y_0,k_1+z_0)$ such that $k_1=\frac{-Ax_0-By_0-Cz_0-D}{A+B+C}$. Therefore one can obtain

$$d_c(P,P_1) = \frac{(2\sqrt{2}-1)\,|Ax_0+By_0+Cz_0+D|}{|A+B+C|}\ .$$

Similarly, if l_i (i=2,3,4) has a direction vector in $\{(-1,1,1),(1,-1,1),(1,1,-1)\}$, then $d_c(P,P_i)=\frac{(2\sqrt{2}-1)|Ax_0+By_0+Cz_0+D|}{t_i}$ such that

$$t_2=|-A+B+C|,\quad t_3=|A-B+C|,\quad t_4=|A+B-C|\,.$$

Case II. Only one of p, q, r is zero. Say $p=0$, $q\neq 0\neq r$. In this case, if (p,q,r) is $(0,1,1)$, then $P_5=(x_0,k_2+y_0,k_2+z_0)$ such that

$$k_2=\frac{-Ax_0-By_0-Cz_0-D}{B+C} \text{ and } d_c(P,P_5)=\frac{\sqrt{2}\ |Ax_0+By_0+Cz_0+D|}{|B+C|}.$$

Similarly, if l_i $(i=6,7,\ldots,10)$ has a direction vector in $\{(0,1,-1),(1,0,1),(1,0,-1),(1,1,0),(1,-1,0)\}$, then $d_c(P,P_i)=\frac{\sqrt{2}|Ax_0+By_0+Cz_0+D|}{t_i}$ such that $t_6=|B-C|$, $t_7=|A+C|$, $t_8=|A-C|$, $t_9=|A+B|$, $t_{10}=|A-B|$.

Case III. Exactly two of p, q, r are zero. Say $p \neq 0$, $q = 0 = r$. In this case, direction vector is $(1, 0, 0)$. Therefore $P_{11} = (k_3 + x_0, y_0, z_0)$ such that $k_3 = \frac{-Ax_0 - By_0 - Cz_0 - D}{A}$ and $d_c(P, P_{11}) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{|A|}$, respectively.

Similarly, if direction vector is $(0, 1, 0)$ or $(0, 0, 1)$, then $d_c(P, P_{12}) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{|B|}$ or $d_c(P, P_{13}) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{|C|}$.

Notice that the results obtained in the above cases give the required formula.

□

The following corollary immediately follows from Theorem 2.

Corollary 3. *Distance of point $P = (x_0, y_0)$ to a line $l : Ax + By + D = 0$ in the Chinese checker plane is*

$$d_c(P, l) = \min \begin{cases} \dfrac{\sqrt{2}\ |Ax_0 + By_0 + D|}{\max\{|A + B|, |A - B|\}}, \\ \dfrac{|Ax_0 + By_0 + D|}{\max\{|A|, |B|\}}. \end{cases}$$

Theorem 4. *CC - distance of a point $P = (x_0, y_0, z_0)$ to a line l given by*

$$\frac{x - a}{p} = \frac{y - b}{q} = \frac{z - c}{r}$$

is

$$d_c(P, l) = \min \begin{cases} E_1, & p \neq 0, \\ E_2, & q \neq 0, \\ E_3, & r \neq 0, \\ E_4, & p \neq q, \\ E_5, & p \neq -q, \\ E_6, & p \neq r, \\ E_7, & p \neq -r, \\ E_8, & q \neq r, \\ E_9, & q \neq -r, \\ E_{10}, & p + q + r \neq 0, \\ E_{11}, & p \neq q + r, \\ E_{12}, & q \neq p + r, \\ E_{13}, & r \neq p + q, \end{cases}$$

where

$$E_1 = \left|p^{-1}\right| [\max\{|pB - qA|, |pC - rA|\} + (\sqrt{2} - 1) \min\{|pB - qA|, |pC - rA|\}],$$

$$E_2 = \left|q^{-1}\right| [\max\{|qA - pB|\,, |qC - rB|\} + (\sqrt{2}-1)\min\{|qA - pB|\,, |qC - rB|\}],$$

$$E_3 = \left|r^{-1}\right| [\max\{|rA - pC|\,, |rB - qC|\} + (\sqrt{2}-1)\min\{|rA - pC|\,, |rB - qC|\}],$$

$$E_4 = \left|(p-q)^{-1}\right| [\max\{|pB - qA|\,, |r(B-A) + (p-q)C|\} + (\sqrt{2}-1)\min\{|pB - qA| + |r(B-A) + (p-q)C|\,, 2\,|pB - qA|\}],$$

$$E_5 = \left|(p+q)^{-1}\right| [\max\{|pB - qA|\,, |r(-A-B) + (p+q)C|\} + (\sqrt{2}-1)\min\{|pB - qA| + |r(-A-B) + (p+q)C|\,, 2\,|pB - qA|\}],$$

$$E_6 = \left|(p-r)^{-1}\right| [\max\{|pC - rA|\,, |q(C-A) + (p-r)B|\} + (\sqrt{2}-1)\min\{|pC - rA| + |q(C-A) + (p-r)B|\,, 2\,|pC - rA|\}],$$

$$E_7 = \left|(p+r)^{-1}\right| [\max\{|pC - rA|\,, |q(-A-C) + (p+r)B|\} + (\sqrt{2}-1)\min\{|pC - rA| + |q(-A-C) + (p+r)B|\,, 2\,|pC - rA|\}],$$

$$E_8 = \left|(q-r)^{-1}\right| [\max\{|qC - rB|\,, |p(C-B) + (q-r)A|\} + (\sqrt{2}-1)\min\{|qC - rB| + |p(C-B) + (q-r)A|\,, 2\,|qC - rB|\}],$$

$$E_9 = \left|(q+r)^{-1}\right| [\max\{|qC - rB|\,, |p(-B-C) + (q+r)A|\} + (\sqrt{2}-1)\min\{|qC - rB| + |p(-B-C) + (q+r)A|\,, 2\,|qC - rB|\}],$$

$$E_{10} = \left|(p+q+r)^{-1}\right| \times [\max\{|(q+r)A - p(B+C)|\,, |(p+r)B - q(A+C)|\,, |(p+q)C - r(A+B)|\} + (\sqrt{2}-1)\min\{|(q+r)A - p(B+C)| + |(p+r)B - q(A+C)|\,, |(q+r)A - p(B+C)| + |(p+q)C - r(A+B)|\,, |(p+r)B - q(A+C)| + |(p+q)C - r(A+B)|\}],$$

$$E_{11} = \left|(q{-}p{+}r)^{-1}\right| \times [\max\{|(q{+}r)A{-}p(B{+}C)|\,,|(r{-}p)B{+}q(A{-}C)|\,,|(q{-}p)C{+}r(A{-}B)|\} +(\sqrt{2}{-}1)\min\{|(q{+}r)A{-}p(B{+}C)|+|(r{-}p)B{+}q(A{-}C)|\,,|(q{+}r)A{-}p(B{+}C)| +|(q{-}p)C{+}r(A{-}B)|\,,|(r{-}p)B{+}q(A{-}C)|+|(q{-}p)C{+}r(A{-}B)|\}],$$

$$E_{12} = \left|(p{-}q{+}r)^{-1}\right| \times [\max\{|(r{-}q)A{+}p(B{-}C)|\,,|(p{+}r)B{-}q(A{+}C)|\,,|(p{-}q)C{+}r(B{-}A)|\} +(\sqrt{2}{-}1)\min\{|(r{-}q)A{+}p(B{-}C)|+|(p{+}r)B{-}q(A{+}C)|\,,|(r{-}q)A{+}p(B{-}C)| +|(p{-}q)C{+}r(B{-}A)|\,,|(p{+}r)B{-}q(A{+}C)|+|(p{-}q)C{+}r(B{-}A)|\}],$$

$$E_{13} = \left|(p{+}q{-}r)^{-1}\right| \times [\max\{|(q{-}r)A{+}p(C{-}B)|\,,|(p{-}r)B{+}q(C{-}A)|\,,|(p{+}q)C{-}r(A{+}B)|\} +(\sqrt{2}{-}1)\min\{|(q{-}r)A{+}p(C{-}B)|+|(p{-}r)B{+}q(C{-}A)|\,,|(q{-}r)A{+}p(C{-}B)| +|(p{+}q)C{-}r(A{+}B)|\,,|(p{-}r)B{+}q(C{-}A)|+|(p{+}q)C{-}r(A{+}B)|\}],$$

and $A = a - x_0$, $B = b - y_0$, $C = c - z_0$.

Proof. Let α_i denote a plane passing through P and having a normal vector (A, B, C) such that $A, B, C \in \{-1, 0, 1\}$ and $(A, B, C) \neq (0, 0, 0)$. Let $\alpha_i \cap l := P_i$ $(i = 1, 2, \ldots, 13)$. If $(A, B, C) = (1, 0, 0)$ then α_1 is $x - x_0 = 0$ and intersection point of α_1 and line l is $P_1 = (x_0, \frac{q(x_0 - a) + pb}{p}, \frac{r(x_0 - a) + pc}{p})$. Thus

$$d_c(P, P_1) = \max\left\{\frac{|pB{-}qA|}{|p|}, \frac{|pC{-}rA|}{|p|}\right\} +(\sqrt{2}\text{-}1)\min\left\{\frac{|pB{-}qA|}{|p|}, \frac{|pC{-}rA|}{|p|}\right\} = E_1.$$

Similar calculations can be easily given for every choice of (A, B, C) to complete the proof. □

Theorem 5. *CC - distance between any two lines given by*

$$l \ldots \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} = \lambda\,,$$
$$l' \ldots \frac{x-a'}{p'} = \frac{y-b'}{q'} = \frac{z-c'}{r'} = \mu\,,$$

in $\mathbb{R}_c^3$ can be expressed as follows:

If l is parallel to l', then $d_c(l,l') = d_c(A,l')$, where $A = (a,b,c)$.
If l is not parallel to l', then

$$d_c(l,l') = \min \begin{cases} \dfrac{(2\sqrt{2}\text{-}1)\,|(qr'\text{-}q'r)(a'\text{-}a)+(rp'\text{-}r'p)(b'\text{-}b)+(pq'\text{-}p'q)(c'\text{-}c)|}{\max K}, & K \neq 0, \\ \dfrac{\sqrt{2}\,|(qr'\text{-}q'r)(a'\text{-}a)+(rp'\text{-}r'p)(b'\text{-}b)+(pq'\text{-}p'q)(c'\text{-}c)|}{\max M}, & M \neq 0, \\ \dfrac{|(qr'\text{-}q'r)(a'\text{-}a)+(rp'\text{-}r'p)(b'\text{-}b)+(pq'\text{-}p'q)(c'\text{-}c)|}{\max N}, & N \neq 0, \end{cases}$$

where

$$K = \{|(q-r)(q'-p') - (q'-r')(q-p)|\,, |(p+q)(r'-q') - (p'+q')(r-q)|,$$
$$|(q+r)(p'+q') - (q'+r')(p+q)|\,, |(q+r)(q'-p') - (q'+r')(q-p)|\},$$
$$M = \{|r(p'-q') - r'(p-q)|\,, |q(p'-r') - q'(p-r)|\,, |p(r'-q') - p'(r-q)|,$$
$$|r(p'+q') - r'(p+q)|\,, |q(p'+r') - q'(p+r)|\,, |p(q'+r') - p'(q+r)|\},$$
$$N = \{|qr'-q'r|\,, |pr'-p'r|\,, |pq'-p'q|\}\,.$$

Proof. $d_c(l,l') = \min\{(X,X') : X \in l\ ,\ X' \in l'\}$. If l is parallel to l', then one can take $(p,q,r) = (p',q',r')$ and $P = (a'+\mu p, b'+\mu q, c'+\mu r)$ for any point P on l', without loss of generality. Then it can be easily computed using the formula given by Theorem 4 that $d_c(P,l') = d_c(A,l')$.

If l and l' are not parallel, then at least one of $pq'-p'q$, $qr'-q'r$, $rp'-r'p$ is not zero. Consider $P = (p\lambda + a, q\lambda + b, r\lambda + c)$ on l and $P' = (p'\mu + a', q'\mu + b', r'\mu + c')$ on l'. If $d_c(P,P')$ is minimum, then direction vector of the line l'' passing P and P' is an element of Δ. According to values of p,q,r the following three main cases are possible:

Case I. $p \neq 0$, $q \neq 0$, $r \neq 0$. In this case, if l_1'' has direction vector $(1,1,1)$, then one get

$$d_c(P,P') = \frac{(2\sqrt{2}-1)\,|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{|(q-r)(q'-p') - (q'-r')(q-p)|}.$$

Similarly, if l_i'' $(i = 2,3,4)$ has a direction vector in $\{(-1,1,1),(1,-1,1),(1,1,-1)\}$, then

$$d_c(P,P') = \frac{(2\sqrt{2}-1)\,|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{t_i},$$

such that

$$t_2 = \left|(p+q)(r'-q')-(p'+q')(r-q)\right| ,$$

$$t_3 = \left|(q+r)(p'+q')-(q'+r')(p+q)\right| ,$$

$$t_4 = \left|(q+r)(q'-p')-(q'+r')(q-p)\right| .$$

Case II. Only one of p, q, r is zero. Say $p = 0$, $q \neq 0 \neq r$. In this case, if (p, q, r) is $(0, 1, 1)$, then it is easily seen that

$$d_c(P,P') = \frac{\sqrt{2}\,|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{|p(r'-q')-p'(r-q)|}.$$

Similarly, if l''_i $(i = 6, 7, \ldots, 10)$ has a direction vector in $\{(0, 1, -1), (1, 0, 1), (1, 0, -1), (1, 1, 0), (1, -1, 0)\}$, then

$$d_c(P,P') = \frac{\sqrt{2}\,|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{t_i}$$

such that

$$t_6 = \left|p(q'+r')-p'(q+r)\right|, \quad t_7 = \left|q(p'-r')-q'(p-r)\right|,$$

$$t_8 = \left|q(p'+r')-q'(p+r)\right|, \quad t_9 = \left|r(p'-q')-r'(p-q)\right|,$$

$$t_{10} = \left|r(p'+q')-r'(p+q)\right|.$$

Case III. Exactly two of p, q, r are zero. Say $p \neq 0$, $q = 0 = r$. In this case, direction vector of l'' is $(1, 0, 0)$. Therefore CC$-$distance between P and P' is

$$d_c(P,P') = \frac{|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{|qr'-q'r|}.$$

Similarly, if l''_i $(i = 12, 13)$ has a direction vector in $\{(0, 1, 0), (0, 0, 1)\}$, then the CC$-$distance is

$$d_c(P,P') = \frac{|(qr'-q'r)(a'-a)+(rp'-r'p)(b'-b)+(pq'-p'q)(c'-c)|}{t_i}$$

such that $t_{12} = |pr' - p'r|$, $t_{13} = |pq' - p'q|$.

Notice that the results obtained in the above cases give the required formula.

References

[1] Z. Akça, R. Kaya, On the distance formulae in three dimensional taxicab space, *Hadronic Journal*, **27**, No. 5 (2004), 521-532.

[2] G. Chen, *Lines and Circles in Taxicab Geometry*, Master Thesis, Department of Mathematics and Computer Sciences, Central Missouri State University (1992).

[3] R. Kaya, Ö. Gelişgen, S. Ekmekçi, A. Bayar, Group of isometries of CC-plane, *Missouri Journal of Mathematical Sciences*, To Appear.

[4] E.F. Krause, *Taxicab Geometry*, Addision-Wesley, Menlo Park, California (1975).

[5] S. Tian, Alpha distance – A generalization of chinese checker distance and taxicab distance, *Missouri Journal of Mathematical Sciences*, **17**, No. 1 (2005), 35-40.

[6] M. Turan, M. Özcan, Two-foci CC-ellipses, *IJPAM*, **16**, No. 1 (2004), 119-127.

[7] M. Turan, M. Özcan, Two-foci CC-hyperbolas, *IJPAM*, **16**, No. 4 (2005), 509-520.