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AN ABS METHOD FOR SOLVING LINEAR INEQUALITIES SYSTEM

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Abstract: A method, called the Multi-Stage ABS algorithm, for solving a system of linear inequalities is presented. This method is characterized by giving the explicit solution of linear inequalities system in finite steps, and it can determine the compatible of the system.

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1. Introduction

Consider the following system of linear inequalities

$$Ax \leq b, \tag{1}$$

where $A = (a_1, \cdots, a_m)^T \in I\!R^{m,n}$, $x \in I\!R^n$, $b \in I\!R^m$, $m > n$, and m, n are any positive. Suppose A is regular. Various methods have been designed for solving system (1), or for solving systems of equalities and inequalities via the ABS algorithms, for instance, a method due to Esmaeili, Mahdavi-Amiri, [3].

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Esmaeili, Mahdavi-Amiri and Spedicato proposed a method, called simply as EMS method, for solving system (1) under the assumptions that A is full rank in row, i.e., the rank of A, $\mathrm{r}(A) = m$ and $m \leq n$, see [3], [4]. It can be proved, under these assumptions, that $Ax \leq d \Longleftrightarrow \{Ax = y,\, y \leq d\}$. An explicit general solution set is $x \in \{H_1^T W_m (A_m^T H_1^T W_m)^{-T} y + H_{m+1}^T H_1^{-T} x_1 + H_{m+1}^T q \,|\, y \leq d,\, q \in I\!R^n\}$, where $y \in I\!R^m$, H_1 is an arbitrary nonsingular matrix and x_1 is an arbitrary starting point. Shi [5] proposed a globally convergent ABS algorithm for generating a nonnegative solution to a linear system being of the form $\{Ax = b,\, x \geq 0, \quad A \in I\!R^{m,n}, b \in I\!R^m, x \in I\!R^n, m \leq n\}$ by carrying out a special iteration via the Huang algorithms so that iterative points asymptomatically satisfy the nonnegative conditions. A projection algorithm of the ABS form, equivalent to the simplex method via Bland's rules to a linear programming, for finding a feasible point of a general system of linear inequalities $\{Ax \geq b, A \in I\!R^{m,n}, b \in I\!R^m, x \in I\!R^n, m \leq n\}$, in a finite number of steps was constructed, and the compatibility for a system of linear inequalities with deficient rank one was investigated due to Zhang [6]. An algorithm for the least Euclidean norm solution of a linear system of inequalities via the Huang ABS algorithm and the Goldfarb-Idnani strategy was proposed by Zhang [7]. A class of direct methods and ABS algorithms for solving linear inequalities by Zhao, see [8], [9]. Much progress has been made by Li and his students on the entropy method used to solve (1).

There are many special properties of the ABS algorithm, see for instance, Abaffy and Spedicato [2]. Among them the following one is the most important for the derivation in this paper: the general solution to $Ax = b$ is formulated in the form

$$x = x_{m+1} + H_{m+1}^T q, \tag{2}$$

where $q \in I\!R^n$ is arbitrary, i. e., the linear variety containing all solutions consists of the vectors formulated by

$$H_{i+1} = H_i - H_i a_i w_i^T H_i / w_i^T H_i a_i. \tag{3}$$

For solving the system (1), we construct an equivalent system. If the equivalent system can be solved, we show that general solutions of the system can be expressed using a special solution and the matrix H_{m+1} generated by the ABS algorithm.

2. The ABS Method for Solving Over-Determined Inequalities System

Consider the linear inequalities system

$$\begin{cases} Ax + Ss = b, \\ s \geq 0, \end{cases} \tag{4}$$

where $S = \begin{pmatrix} 0 & I_{n,n} \\ I_{l,l} & 0 \end{pmatrix}$, s is the slack variable, $(x, s)^T = (x_1, x_2, \cdots, x_n, s_1, s_2, \cdots, s_m)^T$, and $l = m - n$. Denote the matrix (A, S) by $G = (g_1, g_2, \cdots, g_m)^T$, the vector (x, s) by $\widehat{x}$. Denote the matrix $(g_1, g_2, \cdots, g_i)^T$ by G_i.

Theorem 1. *The system (4) is equivalent to system (1).*

When $s_i \geq 0$, $i = 1, 2, \cdots, m$, if x is the solution of system (4), it must be the solution of system (1). On the contrary, we have the same conclusion.

Theorem 2. *A is the coefficient matrix of system (1), G is the matrix (A, S). If A is regular, then G is also regular.*

Proof. Let $G^{(i)}$, $A^{(i)}$ denote the i-th principal submatrices of G and A. $A^{(i)} = G^{(i)}$, $i \in [1, n]$. When $i \in (n, m]$,

$$G^{(i)} = \begin{pmatrix} A^{(n)} & 0 \\ \widetilde{A} & I_{i-n} \end{pmatrix},$$

where $\widetilde{A} \in I\!R^{i-n,n}$, $\widetilde{A} = (a_{n+1}, a_{n+2}, \cdots, a_i)^T$. We have the conclusion that $\det(G^{(i)}) = \det(A^{(n)}) \neq 0$, $i \in (n, m]$. $\square$

Hence we divide the proceed of searching the solution of system (4) into two parts. In Section 1, we use the implicit LU algorithm to solve equations system $G\widehat{x} = b$. In Section 2, we find the slack variable s, which satisfies $s \geq 0$.

Our aim is to find the solution of the equation $Ax + Ss = b$ in (4). Since the matrix A is regular, then the matrix G is regular through the Theorem 2. Use the implicit LU algorithm to get the general solution of the equations system $G\widehat{x} = b$. The solution is

$$\widehat{x} = \widehat{x}^{*T} + H_{m+1}^T q, \qquad q \in I\!R^{m+n},$$

where $H_{m+1} \in I\!R^{m+n,m+n}$,

$$H_{m+1} = \begin{pmatrix} 0 & 0 \\ K_m & I_n \end{pmatrix}, \tag{5}$$

with $K_m = (K_1, K_2) \in I\!R^{n,m}$, $K_1 \in I\!R^{n,n}$, $K_2 \in I\!R^{n,m-n}$.

Lemma 1. *If the inverse matrix of C_1 exists, the inverse matrix of $A = \begin{pmatrix} C_1 & C_2 \\ 0 & I \end{pmatrix}$ is*

$$A^{-1} = \begin{pmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I \end{pmatrix}.$$

Theorem 3. *If the coefficient matrix G of the system (4) is regular, then the matrix K_m in (5) has the following structure*

$$K_m = [A_1^{-1}, A_1^{-1}A_2],$$

where the matres A_1 and A_2 are as follows

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{n+11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{n+1n} & \cdots & a_{mn} \end{pmatrix}.$$

Proof. The matrix has the formulation as follows

$$(K_1, K_2) = \quad [I_n, 0] \begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2 \\ 0 & I \end{pmatrix}.$$

Since $[A_1^{-1}, -A_1^{-1}A_2] \in I\!R^{n,m+n}$, we have that $K_m = [A_1^{-1}, A_1^{-1}A_2]$. □

We have proved that x is the solution of inequalities system (1) iff $s \geq 0$, then we search the parameter q that satisfies the inequalities system $s \geq 0$. Since the structure of K_2 is very particular, we can get the range of q easily.

Find the parameter q that satisfies the condition $s \geq 0$, i.e., the condition

$$s^T = s^{*T} + \overline{K}q \geq 0 = (\overline{s}_1^*, \overline{s}_2^*)^T + \overline{K}q\,, \tag{6}$$

where $\overline{s}_1^* = (s_1^*, s_2^*, \cdots, s_{m-n}^*)$, $\overline{s}_2^* = (s_{m-n+1}^*, \cdots, s_m^*)$, $\overline{K} = \begin{pmatrix} 0 & K_2^T \\ 0 & I_n \end{pmatrix}$. and $K_2 = [A_1^{-1}, -A_1^{-1}A_2]$ by Theorem 3. Therefore, (6) can be written as follows

$$\begin{cases} \overline{s}_1^* - A_2^T A_1^{-T}\overline{q} \geq 0\,, \\ \overline{s}_2^* + \overline{q} \geq 0\,, \end{cases} \tag{7}$$

where $\overline{q} = (q_{m+1}, q_{m+2}, \cdots, q_{m+n})$. System (7) is equivalent to

$$\begin{cases} (I_{l,l} - A_2^T A_1^{-T}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -A_2^T A_1^{-T}\overline{s}_2^* - \overline{s}_1^*\,, \\ (y_1, y_2) \geq 0\,. \end{cases} \tag{8}$$

Denote the matrix $I_{l,l}-A_2^T A_1^{-T}$ by M, vector (y_1, y_2) by y, and vector $-A_2^T A_1^{-T}\overline{s}_2^*-\overline{s}_1^*$ by $\overline{b}$, then system (8) can be written as

$$\left\{ \begin{array}{l} My = \overline{b}\,, \\ y \geq 0\,, \end{array} \right. \tag{9}$$

where $M \in I\!R^{m-n,m}$ and $\text{rank}(M) = m-n$. System (9) can be transformed to the nonlinear program

$$\left\{ \begin{array}{l} \min F(y) = \frac{1}{2}\sum_{j=1}^{m}(y_j^2 - y_j|y_j|), \\ My = \overline{b}. \end{array} \right. \tag{10}$$

Lemma 2. *To (10), one has the following conclusions.*

- *The function $F(x)$ is continuous differentiable convex and $F(x) \geq 0$ ($\forall x \in I\!R^{m-n}$).*
- *$F(x+\alpha d) \leq F(x) + \alpha \nabla f(x)^T d + \alpha^2 d^T d$, where $\alpha \in R$ and $d \in R^n$.*

Lemma 3. *There exists the optimal solution for the programme (10). The programme is compatible if and only if $F(y^*) = 0$. If $F(y^*) = 0$, then y^* is the solution of (9).*

The LW method is given as follows:

Step 1. Let $y^0 = e - M^T(MM^T)^{-1}[Me - \overline{b}]$, where $e = [1, \cdots, 1]^T$.

Step 2. Compute

$$d_k = -[I - D_k^{-1}M^T(MD_k^{-1}M^T)^{-1}M]D_k^{-1}\nabla F(y^*)\,,$$

where $D_k = \text{diag}(d_1^k, d_2^k, \cdots, d_n^k)$,

$$d_j^k = \left\{ \begin{array}{ll} \mu, & y_j^k \geq 0\,, \\ 1, & y_j^k < 0\,. \end{array} \right.$$

Step 3. If $d_k = 0$, stop; Otherwise, go to Step 4.

Step 4. Find $\alpha_k > 0$ satisfying

$$F(y^k + \alpha_k d_k) = \min_{\alpha \geq 0} F(y^k + \alpha d_k).$$

Let $y^{k+1} = y^k + \alpha_k d_k$, $k = k+1$ and go to Step 2.

Lemma 4. *If there exists k_0 satisfying $d_{k_0} = 0$, then x^{k_0} is the optimal solution of (10).*

There are the following conclusions for the LW algorithm:

(1) If the algorithm generated a finite set d_k, then the last point must satisfy $d_k = 0$.

(2) If the algorithm generated a infinite set, then one has

$$\lim_{k\to+\infty} d_k = 0.$$

Thus, one can get the solution of system (1).

3. Conclusion

We have proposed an approach for finding solutions of linear inequalities system. This approach is based on the implicit LU algorithm for solving system of linear equations and the special properties of it. By the LW method we can determine whether the solution exists or not. Moreover, if system (1) has a solution, we can obtain the form of the solutions.

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