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ON INEQUALITY FOR POSITIVE DEFINITE MATRICES

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Abstract: This study is conducted to prove the inequality among determinants of positive definite matrices through maximum likelihood ratio test. Stefanski [3] already showed the inequality between arithmetic, geometric and harmonic means. However, the results in this article are derived from p-variate normal distribution.

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1. Introduction

The relation between determinants and trace of positive matrices are useful in topics of statistics. Often, by mathematical methods, mathematicians propose a function, find extreme value of function and prove some inequalities. Eigenvalues show if they are involved, inequality are stated in terms of function and making a hypothesis, and presented a new method of proving. Now, in this article exponential density is changed to Wishart distribution, and there generates some inequalities.

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2. Results

Let $\mathbf{X}_{ij}$ be independently normally distributed as $N_p(\mu_i, \Sigma_i)$, $j = 1, 2, ..., N_i$, $i = 1, 2, ..., k$. The mean sample and covariance matrix sample for i-th sample are

$$\overline{\mathbf{X}}_i = N_i^{-1} \sum_{j=1}^{N_i} \mathbf{X}_{ij}\,, \quad \mathbf{S}_i = N_i^{-1} \sum_{j=1}^{N_i} (\mathbf{X}_{ij} - \overline{\mathbf{X}}_i)(\mathbf{X}_{ij} - \overline{\mathbf{X}}_i)\prime.$$

Two estimators are sufficient and maximum likelihood estimators for mean vector and covariance matrix of population, respectively. Also, the coefficient of sample covariance is a positive matrix, with Wishart distribution,

$$\mathbf{V}_i = N_i \mathbf{S}_i = \sum_{j=1}^{N_i} (\mathbf{X}_{ij} - \overline{\mathbf{X}}_i)(\mathbf{X}_{ij} - \overline{\mathbf{X}}_i)\prime \quad i = 1, 2, ..., k, \tag{1}$$

$$\mathbf{V}_i \sim \mathbf{W}_p(\Sigma_i, \mathbf{n}_i) \quad i = 1, 2, ..., k\,, \tag{2}$$

where $N_i = n_i + 1$ (Johnson [1]). The density of $\mathbf{V}_i$ equals

$$f(v_i) = \frac{1}{2^{\frac{pn_i}{2}} \Gamma_p(\frac{n_i}{2})} |v_i|^{\frac{n_i-p-1}{2}} |\Sigma_i|^{-\frac{n_i}{2}} \exp(-\frac{1}{2} \mathrm{tr}\,(\Sigma_i^{-1} v_i))$$

$$\propto |v_i|^{\frac{n_i-p-1}{2}} |\Sigma_i|^{-\frac{n_i}{2}} \exp(-\frac{1}{2} \mathrm{tr}\,(\Sigma_i^{-1} v_i))\,, \tag{3}$$

where $\Gamma_p(.)$, multivariate Gamma function, is defined in Muirhead [2]. Then the likelihood function is

$$L(\Sigma_1, \Sigma_2, ..., \Sigma_k) = c \prod_{i=1}^{k} |v_i|^{\frac{n_i-p-1}{2}} |\Sigma_i|^{-\frac{n_i}{2}} \exp(-\frac{1}{2} \mathrm{tr}\,(\Sigma_i^{-1} v_i))\,, \tag{4}$$

where $c = \prod_{i=1}^{k} \frac{1}{2^{\frac{pn_i}{2}} \Gamma_p(\frac{n_i}{2})}$. Without any assumption on covariance matrices of populations, the maximum value of likelihood function is

$$L(n_1^{-1} v_1, n_2^{-1} v_2, ..., n_k^{-1} v_k) = \max_{\Sigma_1, \Sigma_2, ..., \Sigma_k} L(\Sigma_1, \Sigma_2, ..., \Sigma_k) \tag{5}$$

$$= \prod_{i=1}^{k} |v_i|^{\frac{n_i-p-1}{2}} |n_i^{-1} v_i|^{-\frac{n_i}{2}} \exp(-\frac{1}{2} \mathrm{tr}\,(n_i v_i^{-1} v_i)) \tag{6}$$

$$= c\ \exp(-\frac{1}{2} p \sum_{j=1}^{k} n_j) \prod_{i=1}^{k} |v_i|^{\frac{-p-1}{2}} n_i^{\frac{n_i p}{2}}\,. \tag{7}$$

Under the null hypothesis, $\Sigma_1 = \Sigma_2 = ... = \Sigma_k = \Sigma$, the maximum likelihood estimator of Σ is $\mathbf{V} = \frac{1}{\sum_{i=1}^k n_i} \sum_{i=1}^k \mathbf{V}_i$ and the maximum value of likelihood function is

$$L(v, v, ..., v) = \max_{\Sigma_1=\Sigma_2=...=\Sigma_k} L(\Sigma_1, \Sigma_2, ..., \Sigma_k) \tag{8}$$

$$= c \prod_{i=1}^k |v_i|^{\frac{n_i-p-1}{2}} |v|^{-\frac{n_i}{2}} \exp(-\frac{1}{2}\mathrm{tr}\,(v^{-1}v_i)) \tag{9}$$

$$= c\ \exp(-\frac{1}{2}p\sum_{j=1}^k n_j) \prod_{i=1}^k |v_i|^{\frac{n_i-p-1}{2}|v|^{-\frac{n_i}{2}}}\ . \tag{10}$$

It is obvious

$$\frac{L(v, v, ..., v)}{L(n_1^{-1}v_1, n_2^{-1}v_2, ..., n_k^{-1}v_k)} = \frac{\max_{\Sigma_1=\Sigma_2=...=\Sigma_k} L(\Sigma_1, \Sigma_2, ..., \Sigma_k)}{\max_{\Sigma_1,\Sigma_2,...,\Sigma_k} L(\Sigma_1, \Sigma_2, ..., \Sigma_k)} \le 1. \tag{11}$$

Thus

$$\frac{\prod_{i=1}^k |v_i|^{\frac{n_i-p-1}{2}} |v|^{-\frac{n_i}{2}}}{\prod_{i=1}^k |v_i|^{\frac{-p-1}{2}} n_i^{-\frac{n_i p}{2}}} = |v|^{-\frac{\sum_{i=1}^k n_i}{2}} \prod_{i=1}^k n_i^{-\frac{pn_i}{2}} |v_i|^{\frac{n_i}{2}} \le 1 \tag{13}$$

and

$$|\frac{1}{\sum_{i=1}^k n_i} \sum_{i=1}^k v_i|^{\frac{\sum_{i=1}^k n_i}{2}} \ge \prod_{i=1}^k n_i^{-\frac{n_i p}{2}} |v_i|^{n_i} |\sum_{i=1}^k v_i|^{\sum_{i=1}^k n_i}$$
$$\ge (\sum_{i=1}^k n_i)^{p\sum_{i=1}^k n_i} \prod_{i=1}^k n_i^{-pn_i} |v_i|^{n_i}. \tag{15}$$

If the sample sizes from populations are equal to n, then

$$|\sum_{i=1}^k v_i|^{k\,n} \ge (kn)^{p\,k\,n} \prod_{i=1}^k n^{-n\,p} |v_i|^n.$$

Theorem. *For positive definite matrices* $V_1, V_2, ..., V_k$,

$$|\sum_{i=1}^k v_i|^k \ge (k)^{p\,k} \prod_{i=1}^k |v_i|.$$

Sinece inverse of any positive definite matrices is positive definite matrix we have Result 1.

Result 1. *For positive definite matrices,* $V_1, V_2, ..., V_k$,

$$|\sum_{i=1}^{k} v_i^{-1}|^{-k} \geq (k)^{-p\,k} \prod_{i=1}^{k} |v_i|.$$

With choice of matrices V_i =diag$(a_{i1}, a_{i2}, ..., a_{ip})$, $i = 1, 2, ..., k$, where a_{ij}, Result 2 is obtained.

Result 2. *For any set of positive values,* a_{ij}, $i = 1, 2, ..., k \quad j = 1, 2, ..., p$,

$$(\Pi_{j=1}^{p} \sum_{i=1}^{k} a_{ij})^k \geq k^{p\,k} \Pi_{i=1}^{k} \Pi_{j=1}^{p} a_{ij},$$

and for $p = 1$,

$$\sum_{i=1}^{k} a_i/k \geq (\Pi_{i=1}^{k} a_i)^{k^{-1}}.$$

Again, choosing matrices V_i =diag$(a_{i1}^{-1}, a_{i2}^{-1}, ..., a_{ip}^{-1})$, $i = 1, 2, ..., k$, and combine Result 2, in case $p = 1$, we obtain Result 3.

Result 3. *For any set of positive values,* $a_i > 0$, $i = 1, 2, ..., k$,

$$\sum_{i=1}^{k} a_i/k \geq (\Pi_{i=1}^{k} a_i)^{k^{-1}} \geq \frac{k}{\sum_{i=1}^{k} a_i^{-1}}.$$

This is the famous inequality that Stefanski proved by the exponential density function. The positive matrix has positive eigen values, so we can write the following result.

Result 4. *For any positive definite matrix,* A,

$$|A| \geq (\frac{1}{k}\text{trace}\,(A))^k.$$

If we replace the

$$s_1\Sigma_1, s_2\Sigma_2, ..., s_k\Sigma_k\,,$$

where $s_i > 0$, with null hypothesis $\Sigma_1 = \Sigma_2 = ... = \Sigma_k = \Sigma$, then the inequalities change to weighted quanties.

References

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