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TAXICAB VERSIONS OF SOME EUCLIDEAN THEOREMS

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Abstract: In this paper, we give the taxicab versions of Pythagorean Theorem, Stewart's Theorem and a median property.

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1. Introduction

The taxicab plane geometry is introduced by Menger [7] and is developed by Krause [6]. Now, there are about fifty articles published on the subject. The taxicab plane $\mathbb{R}^2_T$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. Taxicab distance between the points P and Q is the length of a shortest path from P to Q composed of the line segments parallel to the coordinate axes. That is, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then the taxicab distance from P to Q is $d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|$.

The taxicab plane geometry is non-Euclidean since it fails to satisfy the side-

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angle-side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry. Since the taxicab plane geometry has a different distance function it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors [1, 2, 3, 4, 5, 8, 9, 10, 11, 13]. The group of isometries that preserve taxicab distance is determined in [12].

Taxicab analogouses of Ceva's Theorem, Menelaus' Theorem and Thales' Theorems are proven in [9]. Here in this study, we give taxicab versions of Stewart's Theorem, median property and Pythagorean Theorem.

2. A Taxicab Version of the Stewart's Theorem

It is known that for any triangle ABC in the Euclidean plane, if $X \in [BC]$ and $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, $p = d(B,X)$, $q = d(C,X)$, $x = d(A,X)$ then

$$x^2 = \frac{b^2 p + c^2 q}{p+q} - pq$$

which is known as *Stewart's Theorem*. We use the following definitions given in [8] to give a taxicab version of this theorem.

Let ABC be any triangle in the taxicab plane. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a *base line* of ABC if and only if:

1. l passes through a vertex,
2. l is parallel to a coordinate axis,
3. l intersects the opposite side (as a line segment) to the vertex in condition 1.

Clearly, at least one of the vertices of the triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex*. A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

The next theorem gives a taxicab version of the Stewart's Theorem.

Theorem 1. *Let the sides of a triangle ABC in the taxicab plane have lengths $\mathbf{a} = d_T(B,C)$, $\mathbf{b} = d_T(A,C)$ and $\mathbf{c} = d_T(A,B)$. If $X \in [BC]$ and $\mathbf{p} = d_T(B,X)$, $\mathbf{q} = d_T(C,X)$ and $\mathbf{x} = d_T(A,X)$, then*

$$\mathbf{x} = \begin{cases} \frac{\mathbf{bp}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has no base line through the vertex } A, \\ \frac{(\mathbf{b}-2\alpha)\mathbf{p}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has only one base line through the vertex } A, \\ & \textit{and } D \textit{ is between } X \textit{ and } C, \\ \frac{\mathbf{bp}+(\mathbf{c}-2\alpha)\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has only one base line through the vertex } A, \\ & \textit{and } D \textit{ is between } X \textit{ and } B, \\ \frac{(\mathbf{b}-2\alpha)\mathbf{p}+(\mathbf{c}-2\beta)\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has two base lines through the vertex } A, \textit{ and} \\ & X \textit{ is between the intersection points of the base lines} \\ & \textit{and the opposite side, and } D \textit{ is between } X \textit{ and } C, \\ \frac{(\mathbf{b}-2\beta)\mathbf{p}+(\mathbf{c}-2\alpha)\mathbf{q}}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has two base lines through the vertex } A \\ & \textit{and } X \textit{ is between the intersection points of the base} \\ & \textit{lines and the opposite side, and } D \textit{ is between} \\ & X \textit{ and } B, \\ \frac{|\mathbf{bp}-\mathbf{cq}|}{\mathbf{p}+\mathbf{q}} & \textit{If } ABC \textit{ has two base lines through the vertex } A, \textit{ and} \\ & X \textit{ is not between the intersection points of the base} \\ & \textit{lines and the opposite side,} \end{cases}$$

where $\alpha = d_T(\text{base line}, E)$, $\beta = d_T(A, E')$ *and*

$D =$ *Intersection point of a base line and the opposite side,*
$E =$ *One of the vertices B and C such that D lies between X and E,*
$E' =$ *The point of orthogonal projection of the vertex distinct from A and E on the same base line.*

Proof. Let

$B' =$ Orthogonal projection of B to the line through A and parallel to y-axis,
$C' =$ Orthogonal projection of C to the line through A and parallel to x-axis,
$X' =$ Orthogonal projection of X to the line through A and parallel to x-axis,
$T =$ Orthogonal projection of X to the line CC',
$T' =$ Orthogonal projection of X to the line BB',

and $d(A, C') = b_1$, $d(C, C') = b_2$, $d(A, B') = c_2$, $d(B, B') = c_1$. Thus $\mathbf{b} = b_1 + b_2$, $\mathbf{c} = c_1 + c_2$ and $\mathbf{x} = d(A, X') + d(X, X')$.

Figure 1:

Figure 2:

Case I. If ABC is a triangle which has no base line through the vertex A as in Figure 1, then one can easily obtain

$$\mathbf{q}\cdot(b_1 - c_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T)\,, \quad \mathbf{p}\cdot(c_2 - b_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')$$

by [9], Theorem 5. Thus

$$d(X,T) = \mathbf{q}\cdot(b_1 - c_1)\diagup(\mathbf{p}+\mathbf{q})\,, \quad d(X,T') = \mathbf{p}\cdot(c_2 - b_2)\diagup(\mathbf{p}+\mathbf{q})\,.$$

Since $d(A,X') = b_1 - d(X,T)$, $d(X,X') = c_2 - d(X,T')$ and $\mathbf{x} = d(A,X') + d(X,X')$, we get

$$\begin{aligned}\mathbf{x} &= b_1 - \frac{\mathbf{q}(b_1 - c_1)}{\mathbf{p}+\mathbf{q}} + c_2 - \frac{\mathbf{p}(c_2 - b_2)}{\mathbf{p}+\mathbf{q}} = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}}\,.\end{aligned}$$

Case II. Let ABC be a triangle which has only one base line through the vertex A. If D is between X and C as in Figure 2, then one can easily obtain

$$\mathbf{q}\cdot(c_1 - b_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T)\,, \quad \mathbf{p}\cdot(b_2 + c_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')\,,$$

by [9], Theorem 5. Thus

$$d(X,T) = \mathbf{q}\cdot(c_1 - b_1)\diagup(\mathbf{p}+\mathbf{q})\,, \quad d(X,T') = \mathbf{p}\cdot(b_2 + c_2)\diagup(\mathbf{p}+\mathbf{q})\,.$$

Since $d(A,X') = b_1 + d(X,T)$, $d(X,X') = c_2 - d(X,T')$ and $\mathbf{x} = d(A,X') + d(X,X')$ we get

$$\begin{aligned}\mathbf{x} &= b_1 + \frac{\mathbf{q}(c_1 - b_1)}{\mathbf{p}+\mathbf{q}} + c_2 - \frac{\mathbf{p}(b_2 + c_2)}{\mathbf{p}+\mathbf{q}} = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2b_2\mathbf{p}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{(\mathbf{b}-2\alpha)\mathbf{p}+\mathbf{cq}}{\mathbf{p}+\mathbf{q}}\,.\end{aligned}$$

Case III. Let ABC be a triangle which has only one base line through the vertex A. If D is between X and B as in Figure 3, then one can easily obtain

$$\mathbf{q}\cdot(c_1 - b_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T)\,, \quad \mathbf{p}\cdot(b_2 + c_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')\,,$$

Figure 3:

Figure 4:

by [9, Theorem 5]. Thus

$$d(X,T) = \mathbf{q}\cdot(c_1 - b_1)/(\mathbf{p}+\mathbf{q})\,, \quad d(X,T') = \mathbf{p}\cdot(b_2 + c_2)/(\mathbf{p}+\mathbf{q})\,.$$

Since $d(A,X') = b_1 + d(X,T)$, $d(X,X') = d(X,T') - c_2$ and $\mathbf{x} = d(A,X') + d(X,X')$ we get

$$\begin{aligned} \mathbf{x} &= b_1 + \frac{\mathbf{q}(c_1 - b_1)}{\mathbf{p}+\mathbf{q}} + \frac{\mathbf{p}(b_2 + c_2)}{\mathbf{p}+\mathbf{q}} - c_2 = \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{b}\mathbf{p} + (\mathbf{c} - 2\alpha)\mathbf{q}}{\mathbf{p}+\mathbf{q}}\,. \end{aligned}$$

Case IV. Let ABC be a triangle which has two base lines through the vertex A, and X be between the intersection points of the base lines and the opposite side. If D is between X and C as in Figure 4, then one can easily obtain

$$\mathbf{q}\cdot(b_1 + c_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T)\,, \quad \mathbf{p}\cdot(b_2 + c_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')$$

by [9, Theorem 5]. Thus

$$d(X,T) = \mathbf{q}\cdot(b_1 + c_1)/(\mathbf{p}+\mathbf{q})\,, \quad d(X,T') = \mathbf{p}\cdot(b_2 + c_2)/(\mathbf{p}+\mathbf{q})\,.$$

Since $d(A,X') = d(X,T) - b_1$, $d(X,X') = d(X,T') - c_2$ and $\mathbf{x} = d(A,X') + d(X,X')$ we get

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{q}(b_1 + c_1)}{\mathbf{p}+\mathbf{q}} - b_1 + \frac{\mathbf{p}(b_2 + c_2)}{\mathbf{p}+\mathbf{q}} - c_2 \\ &= \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2b_1\mathbf{p} - 2c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} = \frac{(\mathbf{b} - 2\alpha)\mathbf{p} + (\mathbf{c} - 2\beta)\mathbf{q}}{\mathbf{p}+\mathbf{q}}\,. \end{aligned}$$

Case V. Let ABC be a triangle which has two base lines through the vertex A, and X be between the intersection points of the base lines and the opposite side. If D is between X and B as in Figure 5, then one can easily obtain

$$\mathbf{q}\cdot(b_1 + c_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T)\,, \quad \mathbf{p}\cdot(b_2 + c_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')$$

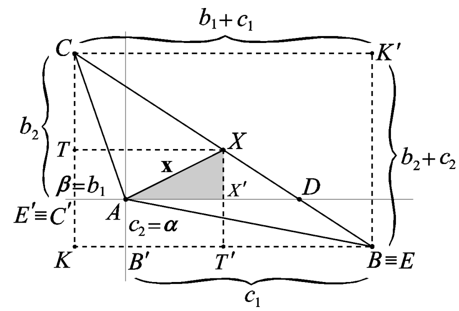

Figure 5:

by [9], Theorem 5. Thus

$$d(X,T) = \mathbf{q}\cdot(b_1+c_1)\diagup(\mathbf{p}+\mathbf{q}), \quad d(X,T') = \mathbf{p}\cdot(b_2+c_2)\diagup(\mathbf{p}+\mathbf{q}).$$

Since $d(A,X') = d(X,T) - b_1$, $d(X,X') = d(X,T') - c_2$ and $\mathbf{x} = d(A,X') + d(X,X')$ we get

$$\mathbf{x} = \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}} - b_1 + \frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}} - c_2$$
$$= \frac{b_1\mathbf{p} + b_2\mathbf{p} + c_1\mathbf{q} + c_2\mathbf{q} - 2b_1\mathbf{p} - 2c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} = \frac{(\mathbf{b}-2\beta)\mathbf{p} + (\mathbf{c}-2\alpha)\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

Case VI. Let ABC be a triangle which has two base lines passing through the vertex A. If X is not between the intersection points of the base lines and the opposite side as in Figure 6 and Figure 7, then one can easily obtain

$$\mathbf{q}\cdot(b_1+c_1) = (\mathbf{p}+\mathbf{q})\cdot d(X,T), \quad \mathbf{p}\cdot(b_2+c_2) = (\mathbf{p}+\mathbf{q})\cdot d(X,T')$$

by [9], Theorem 5. Thus

$$d(X,T) = \mathbf{q}\cdot(b_1+c_1)\diagup(\mathbf{p}+\mathbf{q}), \quad d(X,T') = \mathbf{p}\cdot(b_2+c_2)\diagup(\mathbf{p}+\mathbf{q}).$$

Now, two subcases are possible. If D is between X and C as in Figure 6, then $d(A,X') = d(X,T) - b_1$, $d(X,X') = c_2 - d(X,T')$ and we get

$$\mathbf{x} = d(A,X') + d(X,X') = \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}} - b_1 + c_2 - \frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}}$$
$$= \frac{c_1\mathbf{q} + c_2\mathbf{q} - b_1\mathbf{p} - b_2\mathbf{p}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{cq}-\mathbf{bp}}{\mathbf{p}+\mathbf{q}}.$$

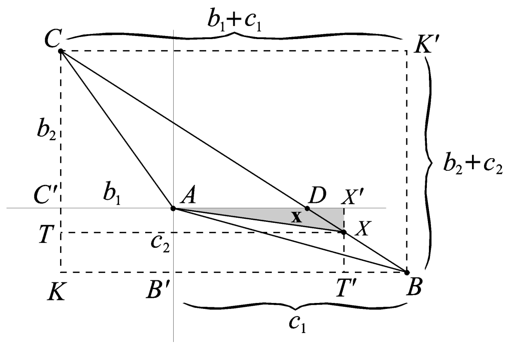

Figure 6:

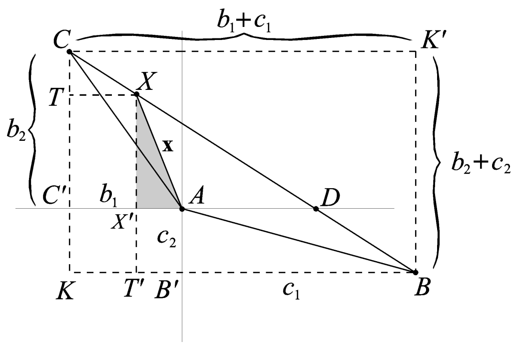

Figure 7:

If D is between X and B as in Figure 7, then $d(A,X') = b_1 - d(X,T)$, $d(X,X') = d(X,T') - c_2$ and we get

$$\begin{aligned}\mathbf{x} = \;& d(A,X') + d(X,X') = b_1 - \frac{\mathbf{q}(b_1+c_1)}{\mathbf{p}+\mathbf{q}} + \frac{\mathbf{p}(b_2+c_2)}{\mathbf{p}+\mathbf{q}} - c_2 \\ = \;& \frac{b_1\mathbf{p} + b_2\mathbf{p} - c_1\mathbf{q} - c_2\mathbf{q}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{bp} - \mathbf{cq}}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

Consequently, $\mathbf{x} = \dfrac{|\mathbf{bp} - \mathbf{cq}|}{\mathbf{p}+\mathbf{q}}$ which completes the proof. □

If X is the midpoint of $[BC]$ of any triangle ABC in the Euclidean plane with $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$ and $V_a = d(A,X)$ then

$$2V_a^2 = b^2 + c^2 - a^2/2$$

which is known as *median property*. The following corollary gives a taxicab version of this property, for $\mathbf{p} = \mathbf{q}$ in Theorem 1.

Corollary 2. *Let the sides of a triangle ABC in the taxicab plane have lengths $\mathbf{a} = d_T(B,C)$, $\mathbf{b} = d_T(A,C)$ and $\mathbf{c} = d_T(A,B)$. If X is the midpoint of $[BC]$ and $V_{\mathbf{a}} = d_T(A,X)$, then*

$$2V_{\mathbf{a}} = \begin{cases} \mathbf{b}+\mathbf{c} & \text{If } ABC \text{ has no base line through the vertex } A, \\ \mathbf{b}+\mathbf{c}-2\alpha & \text{If } ABC \text{ has only one base line through the vertex } A, \\ \mathbf{b}+\mathbf{c}-2(\alpha+\beta) & \text{If } ABC \text{ has two base lines through the vertex } A \\ & \text{and } X \text{ is between the intersection points of the base} \\ & \text{lines and the opposite side,} \\ |\mathbf{b}-\mathbf{c}| & \text{If } ABC \text{ has two base lines through the vertex } A \\ & \text{and } X \text{ is not between the intersection points of the} \\ & \text{base lines and the opposite side,} \end{cases}$$

where $\alpha = d_T(\text{base line}, E)$, $\beta = d_T(A, E')$ and

$D =$ *Intersection point of a base line and the opposite side,*
$E =$ *One of the vertices B and C such that D lies between X and E,*
$E' =$ *The point of orthogonal projection of the vertex distinct from A and E on the same base line.*

3. A Taxicab Version of the Pythagorean Theorem

It is well known that for any right triangle ABC in the Euclidean plane, if $[BC]$ is its hypotenuse and $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$ then

$$a^2 = b^2 + c^2 ,$$

which is known as *Pythagorean Theorem.* A taxicab version of this theorem can be stated as follows.

Theorem 3. *Let* $\mathbf{a}$ *denote the length of the hypotenuse,* $\mathbf{b}$ *and* $\mathbf{c}$ *denote the lengths of the legs of a triangle ABC with right angle A in the taxicab plane. Then*

$$\mathbf{a} = \begin{cases} \mathbf{b}+\mathbf{c}-2\gamma & \text{If there exists only one base line through the vertex } A, \\ \mathbf{b}+\mathbf{c} & \text{If there exist two base lines through the vertex } A, \end{cases}$$

where $\gamma = d_T(A,H)$ and $H=$ The point of orthogonal projection of B or C to the base segment through A.

Figure 8:

Figure 9:

Proof. A is a basic vertex since ABC is a triangle with right angle A. That is, this triangle always has one or two base lines passing through A.

Case I. Let b_1, b_2, $c_1$and c_2 denote the parameters used in the proof of Theorem 1. If there exists only one base line through the vertex A as in Figure 8, and $\mathbf{b} = d(A, C') + d(C, C') = b_1 + b_2$, $\mathbf{c} = d(A, B') + d(B, B') = c_2 + c_1$, then

$$\begin{aligned} \mathbf{a} &= b_1 - c_1 + b_2 + c_2 = \mathbf{b} + \mathbf{c} - 2c_1\,, \quad c_1 = \gamma = d(B, B') = d(A, H) \\ &= \mathbf{b} + \mathbf{c} - 2\gamma\ . \end{aligned}$$

Case II. If there exist two base lines through the vertex A, then the basic lines coincide with the perpendicular sides of ABC as in Figure 9. Thus, obviously, $\mathbf{a} = \mathbf{b} + \mathbf{c}$ which completes the proof. □

4. Another Taxicab Version of the Pythagorean Theorem

A taxicab version of the Pythagorean Theorem has been given in Section 3 using a parameter γ which is length of a part of the base segment. In this section, we use slopes of hypotenuse and a side of the right triangle to give another version of the theorem.

Theorem 4. *Let* $\mathbf{a}$ *denote the length of the hypotenuse,* $\mathbf{b}$ *and* $\mathbf{c}$ *denote the lengths of the legs of a right triangle in the taxicab plane. If the slope of the hypotenuse is* m_1 *and the slope of the anyone of the legs is* m_2*, then*

$$\mathbf{a}^2 = \rho(m_1, m_2) \cdot \left(\mathbf{b}^2 + \mathbf{c}^2\right)\ ,$$

where

$$\rho(m_1, m_2) = \begin{cases} \left(\frac{1+m_2^2}{1+m_1^2}\right)\left(\frac{1+|m_1|}{1+|m_2|}\right)^2 & , \text{if} \ \ m_1, m_2 \in \mathbb{R}\,, \\ \left(\frac{1+m_2^2}{(1+|m_2|)^2}\right) & , \text{if} \ \ m_1 \to \infty\,, \\ \left(\frac{(1+|m_1|)^2}{1+m_1^2}\right) & , \text{if} \ \ m_2 \to \infty\,. \end{cases}$$

Proof. We know from [4] that for any two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the taxicab plane, if $x_1 \neq x_2$ then

$$d(P,Q) = \left[(1+m^2)^{1/2} \big/ (1+|m|)\right] \cdot d_T(P,Q)\,,$$

where $m = (y_1 - y_2)\big/(x_1 - x_2)$ and if $x_1 = x_2$, that is, $m \to \infty$ then

$$d(P,Q) = d_T(P,Q)$$

which allows us to convert a taxicab distance to the Euclidean distance.

Let a, b and c be the corresponding Euclidean lengths of the sides of the same right triangle, m_1 denote the slope of the hypotenuse, and m_2 denote the slope of the anyone of the legs. If $m_2 \neq 0$, then the slope of the other leg is $(-1\big/m_2)$ and

$$\begin{aligned} a &= \left[(1+m_1^2)^{1/2} \big/ (1+|m_1|)\right] \cdot \mathbf{a}\,, \\ b &= \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{b}\,, \\ c &= \left[\left(1+\left(-\tfrac{1}{m_2}\right)^2\right)^{1/2} \Big/ \left(1+\left|-\tfrac{1}{m_2}\right|\right)\right] \cdot \mathbf{c} \\ &= \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{c}\,. \end{aligned}$$

If $m_2 = 0$, then the slope of the other leg is $(-1\big/m_2) \to \infty$ or if $m_2 \to \infty$, then the slope of the other leg is $(-1\big/m_2) \to 0$ and

$$a = \left[(1+m_1^2)^{1/2} \big/ (1+|m_1|)\right] \cdot \mathbf{a}\,, \quad b = \mathbf{b}\,, \quad c = \mathbf{c}\,.$$

If $m_1 \to \infty$, then $a = \mathbf{a}$,

$$b = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{b}\,, \quad c = \left[(1+m_2^2)^{1/2} \big/ (1+|m_2|)\right] \cdot \mathbf{c}\,.$$

Using these values of a, b and c in the Euclidean Pythagorean Theorem one obtains

$$\mathbf{a}^2 = \rho(m_1, m_2) \cdot \left(\mathbf{b}^2 + \mathbf{c}^2\right)\,,$$

where

$$\rho(m_1, m_2) = \begin{cases} \left(\frac{1+m_2^2}{1+m_1^2}\right)\left(\frac{1+|m_1|}{1+|m_2|}\right)^2 & \text{, if } m_1, m_2 \in \mathbb{R}\,, \\ \left(\frac{1+m_2^2}{(1+|m_2|)^2}\right) & \text{, if } m_1 \to \infty\,, \\ \left(\frac{(1+|m_1|)^2}{1+m_1^2}\right) & \text{, if } m_2 \to \infty\,. \end{cases}$$

which completes the proof. □

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