

ON REGULAR HARMONICS OF ONE  
QUATERNIONIC VARIABLE

A. Perotti

Department of Mathematics

University of Trento

Via Sommarive, 14, Povo-Trento, I-38050, ITALY

e-mail: perotti@science.unitn.it

**Abstract:** We prove some results about the Fueter-regular homogeneous polynomials, which appear as components in the power series of any quaternionic regular function. We obtain a differential condition that characterizes the homogeneous polynomials whose trace on the unit sphere extends as a regular polynomial. We apply this result to define an injective linear operator from the space of complex spherical harmonics to the module of regular homogeneous polynomials of a fixed degree  $k$ .

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1. Introduction

Let  $\mathbf{H}$  be the algebra of quaternions. Let  $B$  denote the unit ball in  $\mathbf{C}^2 \simeq \mathbf{H}$  and  $S = \partial B$  the group of unit quaternions. In Section 3.1 we obtain a differential condition that characterizes the homogeneous polynomials whose restriction to  $S$  coincides with the restriction of a regular polynomial. This result generalizes a similar characterization for holomorphic extensions of polynomials proved by Kytmanov (cf. [2] and [3]).

In Section 3.2 we show how to define an injective linear operator  $R : \mathcal{H}_k(S) \rightarrow U_k^\psi$  from the space  $\mathcal{H}_k(S)$  of complex-valued spherical harmonics of degree  $k$  to the  $\mathbf{H}$ -module  $U_k^\psi$  of  $\psi$ -regular homogeneous polynomials of the same degree (cf. Section 2 and Section 3.2 for precise definitions). As an appli-

cation, we show how to construct bases of the module of regular homogeneous polynomials of a fixed degree starting from any choice of  $\mathbf{C}$ -bases of the spaces of complex harmonic homogeneous polynomials.

## 2. Notations and Definitions

**2.1.** Let  $\Omega = \{z \in \mathbf{C}^2 : \rho(z) < 0\}$  be a bounded domain in  $\mathbf{C}^2$  with smooth boundary. Let  $\nu$  denote the outer unit normal to  $\partial\Omega$  and  $\tau = i\nu$ . For every  $F \in C^1(\overline{\Omega})$ , let  $\overline{\partial}_n F = \frac{1}{2} \left( \frac{\partial F}{\partial \nu} + i \frac{\partial F}{\partial \tau} \right)$  be the normal component of  $\overline{\partial} F$  (see for example Kytmanov [2] Section 3.3 and Section 14.2). It can be expressed by means of the Hodge  $*$ -operator and the Lebesgue surface measure as  $\overline{\partial}_n f d\sigma = * \overline{\partial} f|_{\partial\Omega}$ . In a neighbourhood of  $\partial\Omega$  we have the decomposition of  $\overline{\partial} F$  in the tangential and the normal parts:  $\overline{\partial} F = \overline{\partial}_b F + \overline{\partial}_n F \frac{\overline{\partial} \rho}{|\overline{\partial} \rho|}$ . We denote by  $L$  the tangential Cauchy-Riemann operator  $L = \frac{1}{|\overline{\partial} \rho|} \left( \frac{\partial \rho}{\partial \bar{z}_2} \frac{\partial}{\partial z_1} - \frac{\partial \rho}{\partial \bar{z}_1} \frac{\partial}{\partial z_2} \right)$ .

Let  $\mathbf{H}$  be the algebra of quaternions  $q = x_0 + ix_1 + jx_2 + kx_3$ , where  $x_0, x_1, x_2, x_3$  are real numbers and  $i, j, k$  denote the basic quaternions. We identify the space  $\mathbf{C}^2$  with  $\mathbf{H}$  by means of the mapping that associates the quaternion  $q = z_1 + z_2 j$  with the element  $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ . We refer to Sudbery [8] for the basic facts of quaternionic analysis. We will denote by  $\mathcal{D}$  the left Cauchy-Riemann-Fueter operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3}.$$

A quaternionic  $C^1$  function  $f = f_1 + f_2 j$ , is (*left-regular*) on a domain  $\Omega \subseteq \mathbf{H}$  if  $\mathcal{D}f = 0$  on  $\Omega$ . We prefer to work with another class of regular functions, which is more explicitly connected with the hyperkähler structure of  $\mathbf{H}$ . It is defined by the Cauchy-Riemann-Fueter operator associated with the structural vector  $\psi = \{1, i, j, -k\}$ :

$$\mathcal{D}' = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3} = 2 \left( \frac{\partial}{\partial \bar{z}_1} + j \frac{\partial}{\partial \bar{z}_2} \right).$$

A quaternionic  $C^1$  function  $f = f_1 + f_2 j$ , is called (*left- $\psi$ -regular*) on a domain  $\Omega$ , if  $\mathcal{D}'f = 0$  on  $\Omega$ . This condition is equivalent to the following system of complex differential equations:

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \overline{f_2}}{\partial z_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \overline{f_2}}{\partial z_1}.$$

The identity mapping is  $\psi$ -regular, and any holomorphic mapping  $(f_1, f_2)$  on  $\Omega$  defines a  $\psi$ -regular function  $f = f_1 + f_2j$ . This is no more true if we replace  $\psi$ -regularity with regularity. Moreover, the complex components of a  $\psi$ -regular function are either both holomorphic or both non-holomorphic (cf. Vasilevski et al [9], Mitelman et al [4] and Perotti [5]). Let  $\gamma$  be the transformation of  $\mathbf{C}^2$  defined by  $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$ . Then a  $C^1$  function  $f$  is regular on the domain  $\Omega$  if, and only if,  $f \circ \gamma$  is  $\psi$ -regular on  $\gamma^{-1}(\Omega)$ .

**2.2.** The two-dimensional Bochner-Martinelli form  $U(\zeta, z)$  is the first complex component of the Cauchy-Fueter kernel  $G'(p - q)$  associated with  $\psi$ -regular functions (cf. Fueter [1], Vasilevski et al [9], Mitelman et al [4]). Let  $q = z_1 + z_2j$ ,  $p = \zeta_1 + \zeta_2j$ ,  $\sigma(q) = dx[0] - idx[1] + jdx[2] + kdx[3]$ , where  $dx[k]$  denotes the product of  $dx_0, dx_1, dx_2, dx_3$  with  $dx_k$  deleted. Then  $G'(p - q)\sigma(p) = U(\zeta, z) + \omega(\zeta, z)j$ , where  $\omega(\zeta, z)$  is the complex  $(1, 2)$ -form

$$\omega(\zeta, z) = -\frac{1}{4\pi^2}|\zeta - z|^{-4}((\bar{\zeta}_1 - \bar{z}_1)d\zeta_1 + (\bar{\zeta}_2 - \bar{z}_2)d\zeta_2) \wedge \overline{d\zeta}.$$

Here  $\overline{d\zeta} = \overline{d\zeta_1} \wedge \overline{d\zeta_2}$  and we choose the orientation of  $\mathbf{C}^2$  given by the volume form  $\frac{1}{4}dz_1 \wedge dz_2 \wedge \overline{dz_1} \wedge \overline{dz_2}$ . Given  $g(\zeta, z) = \frac{1}{4\pi^2}|\zeta - z|^{-2}$ , we can also write  $U(\zeta, z) = -2 * \partial_{\zeta}g(\zeta, z)$  and  $\omega(\zeta, z) = -\partial_{\zeta}(g(\zeta, z)\overline{d\zeta})$ .

### 3. Regular Polynomials

**3.1.** In this section we will obtain a differential condition that characterizes the homogeneous polynomials whose restrictions to the unit sphere extend regularly or  $\psi$ -regularly. We will use a computation made by Kytmanov in [3] (cf. also [2] Corollary 23.4), where the analogous result for holomorphic extensions is proved.

Let  $\Omega$  be the unit ball  $B$  in  $\mathbf{C}^2$ ,  $S = \partial B$  the unit sphere. In this case the operators  $\bar{\partial}_n$  and  $L$  have the following forms:

$$\bar{\partial}_n = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}, \quad L = z_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial \bar{z}_2}$$

and they preserve harmonicity. Let  $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}$  be the Laplacian in  $\mathbf{C}^2$  and  $D_k$  the differential operator

$$D_k = \sum_{0 \leq l \leq k/2-1} \frac{(k - 2l - 1)!(2l - 1)!!}{k!(l + 1)!} 2^l \Delta^{l+1}.$$

**Theorem 1.** *Let  $f = f_1 + f_2j$  be a  $\mathbf{H}$ -valued, homogeneous polynomial of degree  $k$ . Then its restriction to  $S$  extends as a  $\psi$ -regular function into  $B$  if, and only if,*

$$(\bar{\partial}_n - D_k)f_1 + \overline{L(f_2)} = 0 \quad \text{on } S.$$

*Proof.* In the first part we can proceed as in [3]. The harmonic extension  $\tilde{f}_1$  of  $f_1|_S$  into  $B$  is given by Gauss's formula:  $\tilde{f}_1 = \sum_{s \geq 0} g_{k-2s}$ , where  $g_{k-2s}$  is the homogeneous harmonic polynomial of degree  $k - 2s$  defined by

$$g_{k-2s} = \frac{k-2s+1}{s!(k-s+1)!} \sum_{j \geq 0} \frac{(-1)^j (k-j-2s)!}{j!} |z|^{2j} \Delta^{j+s} f_1. \quad (*)$$

Then  $\bar{\partial}_n \tilde{f}_1 = \bar{\partial}_n f_1 - D_k f_1$  on  $S$  (cf. [2] Section 23). Let  $\tilde{f}_2$  be the harmonic extension of  $f_2$  into  $B$  and  $\tilde{f} = \tilde{f}_1 + \tilde{f}_2j$ . Then  $(\bar{\partial}_n - D_k)f_1 + \overline{L(f_2)} = 0$  on  $S$  is equivalent to  $\bar{\partial}_n \tilde{f}_1 + \overline{L(f_2)} = 0$  on  $S$ . We now show that this implies the  $\psi$ -regularity of  $\tilde{f}$ . Let  $F^+$  and  $F^-$  be the  $\psi$ -regular functions defined respectively on  $B$  and on  $\mathbf{C}^2 \setminus \overline{B}$  by the Cauchy-Fueter integral of  $\tilde{f}$ :

$$F^\pm(z) = \int_S U(\zeta, z) \tilde{f}(\zeta) + \int_S \omega(\zeta, z) j \tilde{f}(\zeta).$$

From the equalities  $U(\zeta, z) = -2 * \partial_\zeta g(\zeta, z)$ ,  $\omega(\zeta, z) = -\partial_\zeta (g(\zeta, z) d\bar{\zeta})$ , we get that

$$F^-(z) = -2 \int_S (\tilde{f}_1(\zeta) + f_2(\zeta)j) * \partial_\zeta g(\zeta, z) - \int_S \partial_\zeta (g(\zeta, z) d\bar{\zeta}) (\tilde{f}_1j - \tilde{f}_2)$$

for every  $z \notin \overline{B}$ . From the complex Green formula and Stokes' Theorem and from the equality  $\bar{\partial} \tilde{f}_2 \wedge d\zeta|_S = 2L(f_2)d\sigma$  on  $S$ , we get that the first complex component of  $F^-(z)$  is

$$\begin{aligned} -2 \int_S \tilde{f}_1 \partial_n g d\sigma + \int_S \tilde{f}_2 \partial_\zeta g \wedge d\bar{\zeta} &= -2 \int_S g \bar{\partial}_n \tilde{f}_1 d\sigma - \int_S g \partial_\zeta \tilde{f}_2 \wedge d\bar{\zeta} \\ &= -2 \int_S g (\bar{\partial}_n \tilde{f}_1 + \overline{L(f_2)}) d\sigma \end{aligned}$$

and then it vanishes on  $\mathbf{C}^2 \setminus \overline{B}$ . Therefore,  $F^- = F_2j$ , with  $F_2$  a holomorphic function that can be holomorphically continued to the whole space. Let  $\tilde{F}^- = \tilde{F}_2j$  be such extension. Then  $F = F^+ - \tilde{F}_2j$  is a  $\psi$ -regular function on  $B$  (indeed a polynomial of the same degree  $k$ ), continuous on  $\overline{B}$ , such that  $F|_S = f|_S$ . The converse is immediate from the equations of  $\psi$ -regularity.  $\square$

Let  $N$  and  $T$  be the differential operators

$$N = \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_2 \frac{\partial}{\partial z_2}, \quad T = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_2}.$$

$T$  is a tangential operator w.r.t.  $S$ , while  $N$  is non-tangential, such that  $N(\rho) = |\bar{\partial}\rho|^2$ ,  $\text{Re}(N) = |\bar{\partial}\rho| \text{Re}(\bar{\partial}_n)$ , where  $\rho = |z_1|^2 + |z_2|^2 - 1$ . Let  $\gamma$  be the reflection introduced at the end of Section 2.1. The operator  $D_k$  is  $\gamma$ -invariant, i.e.  $D_k(f \circ \gamma) = D_k(f) \circ \gamma$ , since  $\Delta$  is invariant. It follows a criterion for regularity of homogeneous polynomials.

**Corollary 2.** *Let  $f = f_1 + f_2j$  be a  $\mathbf{H}$ -valued, homogeneous polynomial of degree  $k$ . Then its restriction to  $S$  extends as a regular function into  $B$  if, and only if,*

$$(N - D_k)f_1 + \overline{T(f_2)} = 0 \quad \text{on } S.$$

Let  $g = \sum_k g^k$  be the homogeneous decomposition of a polynomial  $g$ . After replacing  $D_k g$  by  $\sum_k D_k g^k$ , we can extend the preceding results also to non-homogeneous polynomials.

**3.2.** Let  $\mathcal{P}_k$  denote the space of homogeneous complex-valued polynomials of degree  $k$  on  $\mathbf{C}^2$ , and  $\mathcal{H}_k$  the space of harmonic polynomials in  $\mathcal{P}_k$ . The space  $\mathcal{H}_k$  is the sum of the pairwise  $L^2(S)$ -orthogonal spaces  $\mathcal{H}_{p,q}$  ( $p + q = k$ ), whose elements are the harmonic homogeneous polynomials of degree  $p$  in  $z_1, z_2$  and  $q$  in  $\bar{z}_1, \bar{z}_2$  (cf. for example Rudin [7] Section 12.2). The spaces  $\mathcal{H}_k$  and  $\mathcal{H}_{p,q}$  can be identified with the spaces of the restrictions of their elements to  $S$  (*spherical harmonics*). These spaces will be denoted by  $\mathcal{H}_k(S)$  and  $\mathcal{H}_{p,q}(S)$  respectively.

Let  $U_k^\psi$  be the right  $\mathbf{H}$ -module of (left)  $\psi$ -regular homogeneous polynomials of degree  $k$ . The elements of the modules  $U_k^\psi$  can be identified with their restrictions to  $S$ , which we will call *regular harmonics*.

**Theorem 3.** *For every  $f_1 \in \mathcal{P}_k$ , there exists  $f_2 \in \mathcal{P}_k$  such that the trace of  $f = f_1 + f_2j$  on  $S$  extends as a  $\psi$ -regular polynomial of degree at most  $k$  on  $\mathbf{H}$ . If  $f_1 \in \mathcal{H}_k$ , then  $f_2 \in \mathcal{H}_k$  and  $f = f_1 + f_2j \in U_k^\psi$ .*

*Proof.* We can suppose that  $f_1$  has degree  $p$  in  $z$  and  $q$  in  $\bar{z}$ ,  $p + q = k$ , and then extend by linearity. Let  $\tilde{f}_1 = \sum_{s \geq 0} g_{p-s, q-s}$  be the harmonic extension of  $f_1$  into  $B$ , where  $g_{p-s, q-s} \in \mathcal{H}_{p-s, q-s}$  is given by formula (\*). Then  $\overline{\partial_n L(g_{p-s, q-s})} = (p - s + 1)L(g_{p-s, q-s})$ . We set

$$\tilde{f}_2 = \sum_{s \geq 0} \frac{1}{p - s + 1} \overline{L(g_{p-s, q-s})} \in \bigoplus_{s \geq 0} \mathcal{H}_{k-2s}.$$

Then  $\bar{\partial}_n \tilde{f}_2 = \overline{L(f_1)}$  on  $S$  and we can conclude as in the proof of Theorem 1 that  $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 j$  is a  $\psi$ -regular polynomial of degree at most  $k$ . Now it suffices to define

$$f_2 = \sum_{s \geq 0} \frac{|z|^{2s}}{p-s+1} \overline{L(g_{p-s,q-s})} \in \mathcal{P}_k$$

to get a homogeneous polynomial  $f = f_1 + f_2 j$ , of degree  $k$ , that has the same restriction to  $S$  as  $\tilde{f}$ . If  $f_1 \in \mathcal{H}_k$ , then  $\tilde{f}_1 = f_1$ ,  $\tilde{f}_2 = f_2$  and therefore  $f \in U_k^\psi$ .  $\square$

Let  $C : U_k^\psi \rightarrow \mathcal{H}_k(S)$  be the complex-linear operator that associates to  $f = f_1 + f_2 j$  the restriction to  $S$  of its first complex component  $f_1$ . The function  $\tilde{f}$  in the preceding proof gives a right inverse  $R : \mathcal{H}_k(S) \rightarrow U_k^\psi$  of the operator  $C$ . The function  $R(f_1)$  is uniquely determined by the orthogonality condition with respect to the functions holomorphic on a neighbourhood of  $\bar{B}$ :

$$\int_S (R(f_1) - f_1) \bar{h} d\sigma = 0 \quad \forall h \in \mathcal{O}(\bar{B}).$$

**Corollary 4.** (i) *The restriction operator  $C$  defined on  $U_k^\psi$  induces isomorphisms of real vector spaces*

$$\frac{U_k^\psi}{\mathcal{H}_{k,0} j} \simeq \mathcal{H}_k(S), \quad \frac{U_k^\psi}{\mathcal{H}_{k,0} + \mathcal{H}_{k,0} j} \simeq \frac{\mathcal{H}_k(S)}{\mathcal{H}_{k,0}(S)}.$$

(ii)  $U_k^\psi$  has dimension  $\frac{1}{2}(k+1)(k+2)$  over  $\mathbf{H}$ .

*Proof.* The first part follows from  $\ker C = \{f = f_1 + f_2 j \in U_k^\psi : f_1 = 0 \text{ on } S\} = \mathcal{H}_{k,0} j$ . Part (ii) can be obtained from any of the above isomorphisms, since  $\mathcal{H}_{k,0}$  (as every space  $\mathcal{H}_{p,q}$ ,  $p+q=k$ ) and  $\mathcal{H}_k(S)$  have real dimensions respectively  $2(k+1)$  and  $2(k+1)^2$ .  $\square$

As an application of Corollary 2, we have another proof of the known result (cf. Sudbery [8] Theorem 7) that the right  $\mathbf{H}$ -module  $U_k$  of left-regular homogeneous polynomials of degree  $k$  has dimension  $\frac{1}{2}(k+1)(k+2)$  over  $\mathbf{H}$ .

The operator  $R : \mathcal{H}_k(S) = \bigoplus_{p+q=k} \mathcal{H}_{p,q}(S) \rightarrow U_k^\psi$  can also be used to obtain  $\mathbf{H}$ -bases for  $U_k^\psi$  starting from bases of the complex spaces  $\mathcal{H}_{p,q}(S)$ . On  $\mathcal{H}_{p,q}(S)$ ,  $R$  acts in the following way:

$$R(h) = h + M(h)j, \quad \text{where } M(h) = \frac{1}{p+1} \overline{L(h)} \in \mathcal{H}_{q-1,p+1} \quad (h \in \mathcal{H}_{p,q}).$$

Note that  $M \equiv 0$  on  $\mathcal{H}_{k,0}(S)$ . If  $q > 0$ ,  $M^2 = -Id$  on  $\mathcal{H}_{p,q}(S)$ , since  $qh = \overline{\partial_n h} = -\overline{L(M(h))}$  on  $S$ , and therefore

$$h = -\frac{1}{q} \overline{L(M(h))} = -\frac{1}{q(p+1)} \overline{LL(h)} = -M^2(h).$$

If  $k = 2m+1$  is odd, then  $M$  is a complex conjugate isomorphism of  $\mathcal{H}_{m,m+1}(S)$ . Then  $M$  induces a quaternionic structure on this space, which has real dimension  $4(m+1)$ . We can find complex bases of  $\mathcal{H}_{m,m+1}(S)$  of the form

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}.$$

**Theorem 5.** *Let  $\mathcal{B}_{p,q}$  denote a complex base of the space  $\mathcal{H}_{p,q}(S)$  ( $p+q = k$ ). Then:*

(i) *if  $k = 2m$  is even, a basis of  $U_k^\psi$  over  $\mathbf{H}$  is given by the set*

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \leq q \leq p \leq k\}.$$

(ii) *if  $k = 2m+1$  is odd, a basis of  $U_k^\psi$  over  $\mathbf{H}$  is given by*

$$\mathcal{B}_k = \{R(h) : h \in \mathcal{B}_{p,q}, p+q = k, 0 \leq q < p \leq k\} \cup \{R(h_1), \dots, R(h_{m+1})\},$$

where  $h_1, \dots, h_{m+1}$  are chosen such that the set

$$\{h_1, M(h_1), \dots, h_{m+1}, M(h_{m+1})\}$$

forms a complex basis of  $\mathcal{H}_{m,m+1}(S)$ .

If the bases  $\mathcal{B}_{p,q}$  are orthogonal in  $L^2(S)$  and  $h_1, \dots, h_{m+1} \in \mathcal{H}_{m,m+1}(S)$  are mutually orthogonal, then  $\mathcal{B}_k$  is orthogonal, with norms

$$\|R(h)\|_{L^2(S, \mathbf{H})} = \left( \frac{p+q+1}{p+1} \right)^{1/2} \|h\|_{L^2(S)} \quad (h \in \mathcal{B}_{p,q})$$

w.r.t. the scalar product of  $L^2(S, \mathbf{H})$ .

*Proof.* From dimension count, it suffices to prove that the sets  $\mathcal{B}_k$  are linearly independent. When  $q \leq p$ ,  $q' \leq p'$ ,  $p+q = p'+q' = k$ , the spaces  $\mathcal{H}_{p,q}$  and  $\mathcal{H}_{q'-1, p'+1}$  are distinct. Since  $R(h) = h + M(h)j \in \mathcal{H}_{p,q} \oplus \mathcal{H}_{q-1, p+1}j$ , this implies the independence over  $\mathbf{H}$  of the images  $\{R(h) : h \in \mathcal{B}_{p,q}\}$ . It remains to consider the case when  $k = 2m+1$  is odd. If  $h \in \mathcal{H}_{m,m+1}(S)$ , the complex components  $h$  and  $M(h)$  of  $R(h)$  belong to the same space. The independence of  $\{R(h_1), \dots, R(h_{m+1})\}$  over  $\mathbf{H}$  follows from the particular form of the complex basis chosen in  $\mathcal{H}_{m,m+1}(S)$ .

The scalar product of  $L(h)$  and  $L(h')$  in  $\mathcal{H}_{p,q}(S)$  is

$$(L(h), L(h')) = (h, L^*L(h')) = -(h, \overline{L}L(h')) = q(p+1)(h, h'),$$

since the adjoint  $L^*$  is equal to  $-\overline{L}$  (cf. [7] Section 18.2.2) and  $\overline{L}L = q(p+1)M^2 = -q(p+1)Id$ . Therefore, if  $h, h'$  are orthogonal,  $M(h)$  and  $M(h')$  are orthogonal in  $\mathcal{H}_{q-1,p+1}$  and then also  $R(h)$  and  $R(h')$ . Finally, the norm of  $R(h)$ ,  $h \in \mathcal{H}_{p,q}(S)$ , is

$$\|R(h)\|^2 = \|h\|^2 + \|M(h)\|^2 = \|h\|^2 + \frac{1}{(p+1)^2}\|L(h)\|^2 = \frac{p+q+1}{p+1}\|h\|^2$$

and this concludes the proof.  $\square$

From Theorem 5 it is immediate to obtain also bases of the right  $\mathbf{H}$ -module  $U_k$  of left-regular homogeneous polynomials of degree  $k$ .

**Examples.** (i) The case  $k = 2$ . Starting from the orthogonal bases  $\mathcal{B}_{2,0} = \{z_1^2, 2z_1z_2, z_2^2\}$  of  $\mathcal{H}_{2,0}$  and  $\mathcal{B}_{1,1} = \{z_1\bar{z}_2, |z_1|^2 - |z_2|^2, z_2\bar{z}_1\}$  of  $\mathcal{H}_{1,1}$  we get the orthogonal basis of regular harmonics

$$\mathcal{B}_2 = \{z_1^2, 2z_1z_2, z_2^2, z_1\bar{z}_2 - \frac{1}{2}\bar{z}_1^2j, |z_1|^2 - |z_2|^2 + \bar{z}_1\bar{z}_2j, z_2\bar{z}_1 + \frac{1}{2}\bar{z}_2^2j\}$$

of the six-dimensional right  $\mathbf{H}$ -module  $U_2^\psi$ .

(ii) The case  $k = 3$ . From the orthogonal bases

$$\mathcal{B}_{3,0} = \{z_1^3, 3z_1^2z_2, 3z_1z_2^2, z_2^3\},$$

$$\mathcal{B}_{2,1} = \{z_1^2\bar{z}_2, 2z_1|z_2|^2 - z_1|z_1|^2, 2z_2|z_1|^2 - z_2|z_2|^2, z_2^2\bar{z}_1\},$$

$$\mathcal{B}_{1,2} = \{h_1 = z_1\bar{z}_2^2, M(h_1) = -z_2\bar{z}_1^2, h_2 = -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2,$$

$$M(h_2) = -2\bar{z}_1|z_2|^2 + \bar{z}_1|z_1|^2\},$$

we get the orthogonal basis of regular harmonics

$$\mathcal{B}_3 = \{z_1^3, 3z_1^2z_2, 3z_1z_2^2, z_2^3, z_1^2\bar{z}_2 - \frac{1}{3}\bar{z}_1^3j, 2z_1|z_2|^2 - z_1|z_1|^2 - \bar{z}_1^2\bar{z}_2j,$$

$$2z_2|z_1|^2 - z_2|z_2|^2 + \bar{z}_1\bar{z}_2^2j,$$

$$z_2^2\bar{z}_1 + \frac{1}{3}\bar{z}_2^3j, z_1\bar{z}_2^2 - z_2\bar{z}_1^2j, -2\bar{z}_2|z_1|^2 + \bar{z}_2|z_2|^2 + (\bar{z}_1|z_1|^2 - 2\bar{z}_1|z_2|^2 + j)\}.$$

of the ten-dimensional right  $\mathbf{H}$ -module  $U_3^\psi$ .



In general, for any  $k$ , an orthogonal basis of  $\mathcal{H}_{p,q}$  ( $p + q = k$ ) is given by the polynomials  $\{P_{q,l}^k\}_{l=0,\dots,k}$  defined by formula (6.14) in Sudbery [8]. The basis of  $U_k$  obtained from these bases by means of Theorem 5 and applying the reflection  $\gamma$  is essentially the same given in Proposition 8 of Sudbery [8].

Another spanning set of the space  $\mathcal{H}_{p,q}$  is given by the functions

$$g_\alpha^{p,q}(z_1, z_2) = (z_1 + \alpha z_2)^p (\bar{z}_2 - \alpha \bar{z}_1)^q \quad (\alpha \in \mathbf{C})$$

(cf. Rudin [7] Section 12.5.1). Since  $M(g_\alpha^{p,q}) = \frac{(-1)^q q \bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1}$  for  $\alpha \neq 0$  and  $M(g_0^{p,q}) = -\frac{q}{p+1} z_2^{q-1} \bar{z}_1^{p+1}$ , where we set  $g_\alpha^{p,q} \equiv 0$  if  $p < 0$ , from Theorem 5 we get that  $U_k^\psi$  is spanned over  $\mathbf{H}$  by the polynomials

$$R(g_\alpha^{p,q}) = \begin{cases} g_\alpha^{p,q} + \frac{(-1)^q q \bar{\alpha}^{p+q}}{p+1} g_{-1/\bar{\alpha}}^{q-1,p+1} j & \text{for } \alpha \neq 0, \\ z_1^p \bar{z}_2^q - \frac{q}{p+1} z_2^{q-1} \bar{z}_1^{p+1} j & \text{for } \alpha = 0, \end{cases} \quad (\alpha \in \mathbf{C}, p + q = k).$$

Any choice of  $k + 1$  distinct numbers  $\alpha_0, \alpha_1, \dots, \alpha_k$  gives rise to a basis of  $U_k^\psi$ .

The results obtained in this paper enabled the writing of a *Mathematica* package [6], named **RegularHarmonics**, which implements efficient computations with regular and  $\psi$ -regular functions and with harmonic and holomorphic functions of two complex variables.

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