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ON THE EXPECTED FISHER INFORMATION FOR THE WEIBULL DISTRIBUTION WITH TYPE II CENSORED DATA

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Abstract: We consider the expected Fisher information for the two parameter Weibull distribution with Type II censored data. This information matrix yields asymptotically valid variances and covariances of maximum likelihood estimators of the distribution parameters, and hence features widely in statistical inference on these parameters. However, published discussions generally give only the observed information, or leave the expected information in terms of further basic expectations, with little discussion on computation or interpretation. This paper provides formulae for the elements of the matrix, discusses their computation and behaviour, and summarises the agreement between theoretical results and counterparts observed in simulation experiments for data arising under this censoring regime.

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1. Introduction

We consider the expected Fisher information for the two parameter Weibull distribution with Type II censored data. This information matrix, and its

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observed counterpart, yield asymptotically valid variances and covariances of maximum likelihood estimators of the distribution parameters, and hence feature in hypothesis testing, confidence intervals and other inferential calculations. However, published discussions (Nelson, [8]; Cohen, [3]) generally give only the observed information, or leave the expected information in terms of other, more basic, expectations, with few details on computation or further interpretation. We provide formulae for the elements of the matrix, discuss the computation of these elements, and summarise the agreement between theoretical results and counterparts observed in simulation experiments for data arising under this censoring regime. The next section outlines some necessary background, Section 3 gives the elements of the expected Fisher information, and Section 4 discusses their computation and briefly summarises the results of some simulation experiments to assess agreement between asymptotic theory and behaviour observed in large but finite samples. Section 5 concludes.

2. Preliminaries

2.1. Statistical Background

Here, Y denotes a random variable following the two parameter Weibull distribution with shape parameter β and scale parameter θ. This distribution is widely used in reliability analysis; see, for example, (Nelson, [8]; Cohen, [3]; Ansell and Phillips, [1]) and the references therein. The cumulative distribution function of Y is

$$F(y;\beta,\theta) = 1 - \exp\left\{-\left(\frac{y}{\theta}\right)^{\beta}\right\} \tag{1}$$

for $y \geq 0$; the corresponding probability density function is

$$f(y;\beta,\theta) = \frac{\beta}{\theta}\left(\frac{y}{\theta}\right)^{\beta-1}\exp\left\{-\left(\frac{y}{\theta}\right)^{\beta}\right\},$$

again for $y \geq 0$. Ansell and Phillips [1] give a full discussion on important practical aspects of reliability testing, the role of censoring, and the benefits of Type II censoring. Here, we assume that the sample information comprises n independent items subject to Type II censoring, so that the experiment stops after some (pre-specified) number m of failures. The distinction between Type II censoring and complete sampling thus decreases as $m \to n$, and disappears with $m = n$. In general, the duration of the experiment is random; with the assumption that all items entered into life simultaneously, the times to failure are the first m order statistics from the Weibull distribution, and the duration of the experiment has probability density function

$$\frac{n!}{(n-m)!\,(m-1)!}\left\{1-F\left(y\right)\right\}^{m-1}\left\{F\left(y\right)\right\}^{n-m}f\left(y\right);$$

see, for instance, David [4]. Adopting standard notation for order statistics, in which $Y_{(i:n)}$ denotes the i-th order statistic in a sample of size n, the data for analysis comprises the m times to failure $Y_{(1:n)},..,Y_{(m:n)}$ and $n-m$ equal censored values $Y_{(m:n)}$; without loss of generality, the log-likelihood is

$$\begin{aligned} l &= \sum_{i=1}^{m}\ln\left\{f(Y_{(i:n)};\beta,\theta)\right\}+(n-m)\ln\left\{1-F(Y_{(m:n)};\beta,\theta)\right\} \\ &= m\ln\left(\frac{\beta}{\theta}\right)+(\beta-1)\sum_{i=1}^{m}\ln\left(\frac{Y_{(i:n)}}{\theta}\right)-\sum_{i=1}^{m}\left(\frac{Y_{(i:n)}}{\theta}\right)^{\beta} \\ &\qquad -(n-m)\left(\frac{Y_{(m:n)}}{\theta}\right)^{\beta}. \qquad (2) \end{aligned}$$

Here, and throughout, ln denotes natural logarithms. For a given set of data, we obtain maximum likelihood estimates of β,θ by ascertaining the values of these parameters which maximise (2). This maximisation is a numerical exercise, and, in practice, we benefit from known methods; see Kalbfleisch [7] and Cohen [3] for further details. Thus, for both practical and theoretical reasons, we study the derivatives of l. From (2), we have

$$l_{\theta}=\beta\theta^{-1}\left\{\sum_{i=1}^{m}\left(\frac{Y_{(i:n)}}{\theta}\right)^{\beta}+(n-m)\left(\frac{Y_{(m:n)}}{\theta}\right)^{\beta}-m\right\}, \qquad (3)$$

and

$$\begin{aligned} l_{\beta} &= m\beta^{-1}+\sum_{i=1}^{m}\ln\left(\frac{Y_{(i:n)}}{\theta}\right)-\sum_{i=1}^{m}\left(\frac{Y_{(i:n)}}{\theta}\right)^{\beta}\ln\left(\frac{Y_{(i:n)}}{\theta}\right) \\ &\qquad -(n-m)\left(\frac{Y_{(m:n)}}{\theta}\right)^{\beta}\ln\left(\frac{Y_{(m:n)}}{\theta}\right), \qquad (4) \end{aligned}$$

on an appropriate arrangement of terms. From (3) above, we have

$$l_{\theta\theta}=-\beta\theta^{-2}\left[(\beta+1)\left\{\sum_{i=1}^{m}\left(\frac{Y_{(i:n)}}{\theta}\right)^{\beta}+(n-m)\left(\frac{Y_{(m:n)}}{\theta}\right)^{\beta}\right\}-m\right],$$

and

$$l_{\theta\beta} = \theta^{-1} \left[\sum_{i=1}^{m} \left(\frac{Y_{(i:n)}}{\theta} \right)^{\beta} \left\{ 1 + \beta \ln \left(\frac{Y_{(i:n)}}{\theta} \right) \right\} + (n-m) \left(\frac{Y_{(m:n)}}{\theta} \right)^{\beta} \left\{ 1 + \beta \ln \left(\frac{Y_{(m:n)}}{\theta} \right) \right\} - m \right].$$

Finally, from (4), we have

$$l_{\beta\beta} = -\beta^{-2} \left[m + \left\{ \sum_{i=1}^{m} \left(\frac{Y_{(i:n)}}{\theta} \right)^{\beta} \left(\beta \ln \frac{Y_{(i:n)}}{\theta} \right)^{2} + (n-m) \left(\frac{Y_{(m:n)}}{\theta} \right)^{\beta} \left(\beta \ln \frac{Y_{(m:n)}}{\theta} \right)^{2} \right\} \right].$$

The expectations of these derivatives lead to asymptotically valid standard deviations for the maximum likelihood estimators of β, θ; we first consider these expectations, and show that they depend on known constants, quantities m, β, θ associated with the model and censoring regime, and two summations which, in practice, must be obtained numerically. We then consider the computation of these summations, and briefly assess the agreement between related theoretical results and finite sample practice, using simulation studies for specific values of n, β and θ, and a range of values for m.

2.2. A Useful Transformation

We consider derivatives of l using the transformation

$$Z = \left(\frac{Y}{\theta} \right)^{\beta}$$

which links the general Weibull distribution (1) to the standard negative exponential distribution, obtained from (1) on setting $\beta = \theta = 1$. Thus, we have

$$l_{\theta} = \beta\theta^{-1} \left\{ \sum_{i=1}^{m} Z_{(i:n)} + (n-m) Z_{(m:n)} - m \right\}, \tag{5}$$

and

$$l_\beta = \beta^{-1}\left[m + \sum_{i=1}^{m} \ln Z_{(i:n)} - \sum_{i=1}^{m} Z_{(i:n)} \ln Z_{(i:n)} - (n-m)\, Z_{(m:n)} \ln Z_{(m:n)}\right], \quad (6)$$

for the first order derivatives. The second order derivatives are

$$l_{\theta\theta} = -\beta\theta^{-2}\left[(\beta+1)\left\{\sum_{i=1}^{m} Z_{(i:n)} + (n-m)\, Z_{(m:n)}\right\} - m\right],$$

$$l_{\theta\beta} = \theta^{-1}\left[\sum_{i=1}^{m} Z_{(i:n)}\left(1+\ln Z_{(i:n)}\right) + (n-m)\, Z_{(m:n)}\left(1+\ln Z_{(m:n)}\right) - m\right],$$

and

$$l_{\beta\beta} = -\beta^{-2}\left[m + \left\{\sum_{i=1}^{m} Z_{(i:n)}\left(\ln Z_{(i:n)}\right)^2 + (n-m)\, Z_{(m:n)}\left(\ln Z_{(m:n)}\right)^2\right\}\right].$$

The form of these derivatives indicates that we will require results on the expectation of various functions $h\left(Z_{(i:n)}\right)$, on the sum of these expectations, and, in particular, on expressions of the form

$$\left[\sum_{i=1}^{m} h\left(Z_{(i:n)}\right)\right] + (n-m)\, h\left(Z_{(m:n)}\right).$$

Note that, with $m = n$, these derivatives and their expectations reduced to those for complete case, covered in Watkins [9].

2.3. Expectations as Summations

Here, and in other reliability settings, it is convenient to express these expectations in terms of the expectations for the first order statistic in different sample sizes; this approach allows us to exploit the connection between the distribution of $Z_{(1:i)}$ and the underlying distribution. By repeated use of the recursive result

$$i \times E\left[h\left(Z_{i+1:n+1}\right)\right] = (n+1) \times E\left[h\left(Z_{i:n}\right)\right] - (n-i+1) \times E\left[h\left(Z_{i:n+1}\right)\right],$$

which links expectations of order statistics with neighbouring sample sizes (David, [4]; Balakrishnan and Rao [2]), we obtain

$$E\left[h\left(Z_{i:n}\right)\right] = \sum_{j=1}^{i} (-1)^{i-j} \binom{n}{j-1}\binom{n-j}{i-j} h_{n+1-j},$$

where, with a slight simplification of notation, we write

$$h_i = E\left[h\left(Z_{1:i}\right)\right]$$

for $i = 1, \ldots, n$. This result can be established by mathematical induction, and is independent of both the function $h(.)$ and the underlying distribution. We also have $\sum_{i=1}^{m} E\left[h\left(Z_{i:n}\right)\right]$ as

$$\sum_{i=1}^{m}\sum_{j=1}^{i}(-1)^{i-j}\binom{n}{j-1}\binom{n-j}{i-j}h_{n+1-j}$$

for $m = 1, \ldots, n$. Manipulating the indices, this double summation can be written as

$$\sum_{i=1}^{m}\binom{n}{i-1}\left\{\sum_{j=0}^{m-i}(-1)^{j}\binom{n-i}{j}\right\}h_{n+1-i},$$

in which

$$\sum_{j=0}^{m-i}(-1)^{j}\binom{n-i}{j} = (-1)^{m-i}\binom{n-i-1}{m-i}$$

(see, for instance, Gradshteyn and Ryzhik [5]), so that

$$\sum_{i=1}^{m} E\left[h\left(Z_{i:n}\right)\right] = \sum_{i=1}^{m}(-1)^{m-i}\binom{n}{i-1}\binom{n-i-1}{m-i}h_{n+1-i}. \tag{7}$$

Finally, by combining the above results and simplifying, we may write

$$\sum_{i=1}^{m} E\left[h\left(Z_{i:n}\right)\right] + (n-m)\,E\left[h\left(Z_{(m:n)}\right)\right]$$

as

$$n\sum_{i=1}^{m}(-1)^{m-i}\binom{n-1}{i-1}\binom{n-i-1}{m-i}h_{n+1-i}. \tag{8}$$

The coefficients in (7) and (8) can be expressed in various alternative forms, and there is also considerable structure to the terms in these summations. In particular, we note that:

1. The formulae remain valid for $m = n$ under the conventions that

$$\binom{i}{0} = 1$$

for all $i < 0$, and

$$\binom{i-1}{i} = 0$$

for $i < 0$. These conventions are consistent with those used in *Mathematica*; (Wolfram, [11]).

2. The coefficient of h_{n+1-i} in (8) is $n+1-i$ times its counterpart in (7).

3. The sum of coefficients is m in (7) and n in (8); one way to establish the first result is to use the previous property to write

$$\sum_{i=1}^{m} (-1)^{m-i} \binom{n}{i-1} \binom{n-i-1}{m-i}$$
$$= n \sum_{i=1}^{m} (-1)^{m-i} \binom{n-1}{i-1} \binom{n-i-1}{m-i} (n+1-i)^{-1} . \quad (9)$$

Then, by writing binomial coefficients as factorials, we see that the sum of coefficients in (7) is

$$\frac{n!}{(m-1)!\,(n-m-1)!} \sum_{i=1}^{m} (-1)^{m-i} \binom{m-1}{i-1} \left[\frac{1}{(n-i)\,(n+1-i)} \right] .$$

We now use partial fractions and the identities in John, Johnson and Watkins [6] to reduce this to

$$\frac{n!\,[B\,(n-m,m) - B\,(n+1-m,m)]}{(m-1)!\,(n-m-1)!},$$

where $B\,(.,.)$ is the usual beta function. Further simplification then gives

$$\sum_{i=1}^{m} (-1)^{m-i} \binom{n}{i-1} \binom{n-i-1}{m-i} = m. \quad (10)$$

The second result can be established by adapting the above argument.

2.4. Expectations as Functions of ϕ_1, ϕ_2

We now use (7) and (8) for appropriate choices of $h\,(.)$ to obtain the required expectations in terms of known constants, quantities m, β, θ associated with the model and censoring regime, and two summations ϕ_1 and ϕ_2. From the formulae for the derivatives, we see that we will need to use (7) with $h\left(Z_{(i:n)}\right) =$

$\ln Z_{(i:n)}$, and (8) with $h\left(Z_{(i:n)}\right) = Z_{(i:n)}, Z_{(i:n)} \ln Z_{(i:n)}$, and $Z_{(i:n)} \left\{\ln Z_{(i:n)}\right\}^2$. We also exploit the fact that $iZ_{(1:i)}$ follows the standard negative exponential distribution, so that, in all four cases, the expression for h_i below is based directly on formulae in Watkins [9].

1. First, we take $h\left(Z_{(1:i)}\right) = Z_{(1:i)}$, so that writing $h_i = i^{-1}$ in (8), we obtain

$$\sum_{i=1}^{m} E\left[Z_{(i:n)}\right] + (n-m) E\left[Z_{(m:n)}\right]$$

as

$$n \sum_{i=1}^{m} (-1)^{m-i} \binom{n-1}{i-1} \binom{n-i-1}{m-i} (n+1-i)^{-1} = m,$$

from (9) and (10). This result is also available via the lack-of-memory property of the negative exponential distribution.

2. With $h\left(Z_{(1:i)}\right) = \ln Z_{(1:i)}$, we have $h_i = -(\gamma + \ln i)$, where $\gamma = 0.57721\cdots$ is Euler's number. Thus, from (7), we have

$$\begin{aligned} \sum_{i=1}^{m} E\left[\ln Z_{(i:n)}\right] &= -\sum_{i=1}^{m} (-1)^{m-i} \binom{n}{i-1} \binom{n-i-1}{m-i} \{\gamma + \ln(n+1-i)\} \\ &= -m(\gamma + \phi_1) , \end{aligned}$$

where we define

$$\phi_j = m^{-1} \sum_{i=1}^{m} (-1)^{m-i} \binom{n}{i-1} \binom{n-i-1}{m-i} \{\ln(n+1-i)\}^j$$

for $j = 1, 2$, with the convention $0^0 \equiv 1$. Although not required here, this notation extends; for instance, (10) implies $\phi_0 = 1$. We discuss the computation of the summations ϕ_1 and ϕ_2 below.

3. Next, we take $h\left(Z_{(1:i)}\right) = Z_{(1:i)} \ln Z_{(1:i)}$, so that $h_i = i^{-1}(1 - \gamma - \ln i)$, and (8) gives

$$\sum_{i=1}^{m} E\left[Z_{(i:n)} \ln Z_{(i:n)}\right] + (n-m) E\left[Z_{(m:n)} \ln Z_{(m:n)}\right]$$

as

$$n \sum_{i=1}^{m} (-1)^{m-i} \binom{n-1}{i-1} \binom{n-i-1}{m-i} (n+1-i)^{-1} \{1 - \gamma - \ln(n+1-i)\}$$

which reduces to

$$m\left(1-\gamma-\phi_1\right)$$

on using (9) and (10).

4. Finally, we take $h\left(Z_{(1:i)}\right)=Z_{(1:i)}\left\{\ln Z_{(1:i)}\right\}^2$, so that

$$\begin{aligned} h_i &= i^{-1}\left\{\frac{\pi^2}{6}-1+\left(1-\gamma-\ln i\right)^2\right\} \\ &= i^{-1}\left\{\frac{\pi^2}{6}+\gamma^2-2\gamma-2\left(1-\gamma\right)\ln i+\left(\ln i\right)^2\right\}. \end{aligned}$$

Thus, by adapting the previous argument, (8) gives

$$\sum_{i=1}^{m} E\left[Z_{(i:n)}\left\{\ln Z_{(i:n)}\right\}^2\right]+\left(n-m\right)E\left[Z_{(m:n)}\left\{\ln Z_{(m:n)}\right\}^2\right]$$

as

$$m\left\{\frac{\pi^2}{6}+\gamma^2-2\gamma-2\left(1-\gamma\right)\phi_1+\phi_2\right\}.$$

3. Expectations of Derivatives of l

We have expressed the four expectations in the required terms, and now consider the expectation of the derivatives. From (5), we have

$$E\left[l_\theta\right]=\beta\theta^{-1}\left\{E\left[\sum_{i=1}^{m} Z_{(i:n)}+\left(n-m\right)Z_{(m:n)}\right]-m\right\}=0,$$

and, by a similar argument,

$$E\left[l_{\theta\theta}\right]=-m\beta^2\theta^{-2}.$$

From (6), we also have

$$\begin{aligned} E\left[l_\beta\right]=\beta^{-1}\Bigg\{ & m+E\left[\sum_{i=1}^{m}\ln Z_{(i:n)}\right] \\ & -E\left[\sum_{i=1}^{m} Z_{(i:n)}\ln Z_{(i:n)}-\left(n-m\right)Z_{(m:n)}\ln Z_{(m:n)}\right]\Bigg\} \end{aligned}$$

$$= \beta^{-1} \left\{ m - m\left(\gamma + \phi_1\right) - m\left(1 - \gamma - \phi_1\right) \right\} = 0$$

on cancelling, while

$$\begin{aligned} E\left[l_{\theta\beta}\right] = \theta^{-1} \Bigg\{ & E\left[\sum_{i=1}^{m} Z_{(i:n)} + (n-m) Z_{(m:n)}\right] \\ & + E\left[\sum_{i=1}^{m} Z_{(i:n)} \ln Z_{(i:n)} + (n-m) Z_{(m:n)} \ln Z_{(m:n)}\right] - m \Bigg\} \\ & = \theta^{-1}\left[m + m\left(1 - \gamma - \phi_1\right) - m\right] = m\theta^{-1}\left(1 - \gamma - \phi_1\right). \end{aligned}$$

Finally, we see that

$$\begin{aligned} & E\left[l_{\beta\beta}\right] \\ & = -\beta^{-2}\left\{ m + E\left[\sum_{i=1}^{m} Z_{(i:n)} \left(\ln Z_{(i:n)}\right)^2 + (n-m) Z_{(m:n)} \left(\ln Z_{(m:n)}\right)^2\right]\right\} \\ & = -\beta^{-2}\left(m + m\left(\frac{\pi^2}{6} + \gamma^2 - 2\gamma - 2\left(1-\gamma\right)\phi_1 + \phi_2\right)\right) \\ & = -m\beta^{-2}\left\{\frac{\pi^2}{6} + \left(1-\gamma\right)^2 - 2\left(1-\gamma\right)\phi_1 + \phi_2\right\}. \end{aligned}$$

The results for the first derivatives are as expected from regularity considerations. From the results on the second derivatives, we obtain

$$\operatorname{Var}\left(\frac{\sqrt{m}\beta}{\theta} \times \hat{\theta}\right) \simeq \frac{\frac{\pi^2}{6} + \left(1-\gamma\right)^2 - 2\left(1-\gamma\right)\phi_1 + \phi_2}{\frac{\pi^2}{6} + \phi_2 - \phi_1^2}, \tag{11}$$

$$\operatorname{Var}\left(\frac{\sqrt{m}}{\beta} \times \hat{\beta}\right) \simeq \frac{1}{\frac{\pi^2}{6} + \phi_2 - \phi_1^2} \tag{12}$$

and

$$\operatorname{Corr}\left(\hat{\theta}, \hat{\beta}\right) \simeq \frac{1 - \gamma - \phi_1}{\sqrt{\frac{\pi^2}{6} + \left(1-\gamma\right)^2 - 2\left(1-\gamma\right)\phi_1 + \phi_2}}. \tag{13}$$

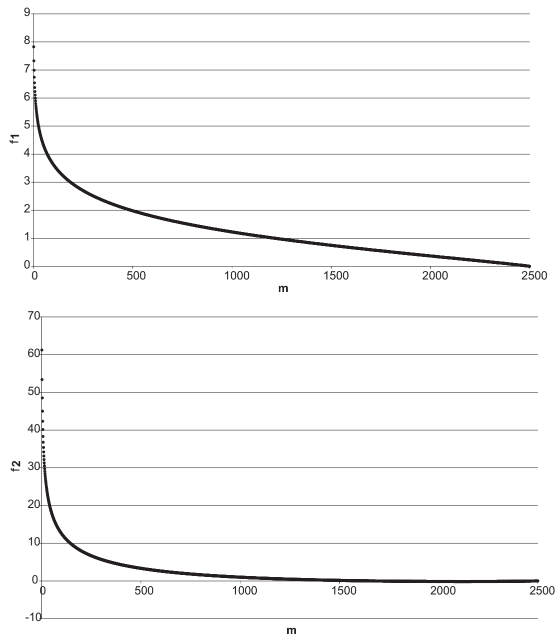

Figure 1: The functions ϕ_1 and ϕ_2 for $m = 1, \ldots, 2500$ with $n = 2500$: (a) ϕ_1, (b) ϕ_2

4. Computational Issues

The computation of ϕ_1, ϕ_2 causes few problems for small to moderate n; for instance, using *Mathematica*, we obtain accurate results for $m \leq n \leq 2500$; this covers most sample sizes and ranges of censoring encountered in practice. Figure 1 shows ϕ_1 and ϕ_2 as functions of m for $n = 2500$, and indicates that these summations have considerable structure, which may be further exploited by approximation techniques; this aspect will be discussed elsewhere. Figure 2 shows expectations of the second derivatives, suitably standardised by multiplication by $-n^{-1}$, again as functions of m for $n = 2500$, and we see that these expectations approach the complete counterparts as $m \to n$.

Figure 3 shows the standardised asymptotic standard deviations of $\hat{\theta}, \hat{\beta}$, at (11) and (12), respectively, and the asymptotic correlation between these maximum likelihood estimators, at (13), again for $n = 2500$. Figure 3 also shows sample counterparts to these quantities, based on simulation experiments with 10000 replications and $\beta = \theta = 1$, $n = 2500$, for various m corresponding to a wide range of levels of censoring. We see excellent agreement between theory

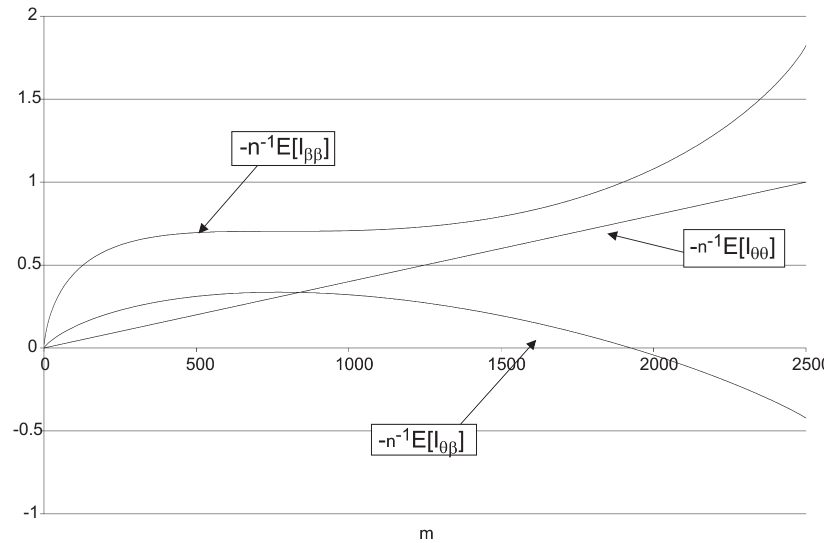

Figure 2: The expectations of $-n^{-1}l_{\theta\theta}, -n^{-1}l_{\theta\beta}$ and $-n^{-1}l_{\beta\beta}$ the case $\beta = \theta = 1$ for $m = 1, \ldots, 2500$ with $n = 2500$

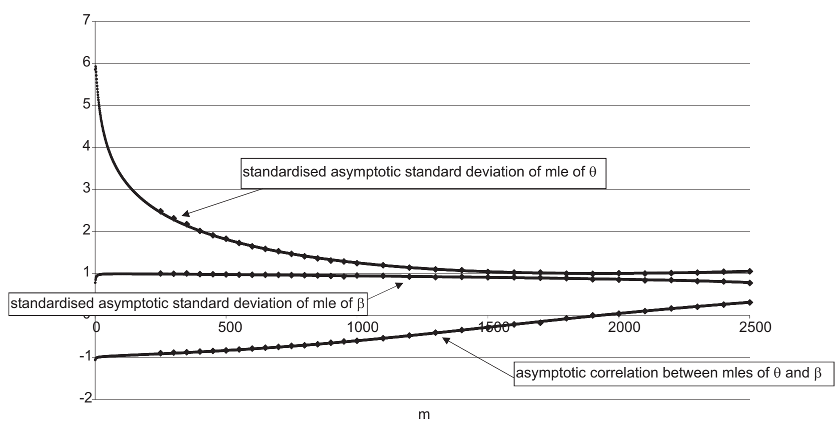

Figure 3: Standardised second moments of $\hat{\theta}, \hat{\beta}$ for the case $\beta = \theta = 1, n = 2500$ (continuous lines), with observed counterparts based on simulation experiments for selected m with $250 \leq m \leq 2500$

and practice, providing confirmation of our results. More extensive simulation experiments, considering the extent of this agreement for varying n, β and m will be presented elsewhere.

5. Conclusions

We have obtained expressions for elements in the expected Fisher information for the two parameter Weibull distribution with Type II censored data, and used this information matrix to obtain asymptotically valid standardised standard deviations and correlations of maximum likelihood estimators of θ, β. Our formulae are relatively straightforward and computationally tractable; we have also shown that they agree with the behaviour observed in simulation experiments for data arising under this censoring regime. The second derivatives in Figure 2 can, with a little further consideration, be compared with their counterparts under Type I censoring in Watkins and John [10]. In this regard, it is important to emphasise that Figure 2 is linear in the proportion of non-failures m, and therefore non-linear in terms of time to failure, whereas the reverse is the case in Watkins and John [10]; however, in both cases the derivatives approach their complete counterparts. There is also scope for further consideration of the extent of this agreement for various combinations of sample size n, values of the shape parameter β, and the censoring number m.

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