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ON RELATIONS BETWEEN CERTAIN q-POLYNOMIAL FAMILIES, GENERATED BY THE FINITE FOURIER TRANSFORM

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Abstract: Some q-extensions of Mehta's eigenvectors of the finite Fourier transform are studied. It is shown that the finite Fourier transform operator interrelates certain well-known q-polynomial families.

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1. Introduction

The finite Fourier transform, defined in terms of the operator (or the equivalent $N \times N$ unitary matrix)

$$A_{jk}^{(N)} = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N}\, jk\right), \quad i = \sqrt{-1}, \tag{1}$$

appears in various mathematical problems and has been studied in detail (see,

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for example, Berndt et al [9] and references therein). In particular, Schur [15] was the first who determined the eigenvalues of $A^{(N)}$ in order to compute the trace of $A^{(N)}$ or the quadratic Gauss sum (see also Landau [12]). Mehta [13] studied eigenvalues and eigenvectors of $A^{(N)}$ and proposed a simple argument, which enables one to recover the eigenvalues of $A^{(N)}$ from its trace. Mehta next introduced a set of eigenvectors of $A^{(N)}$,

$$F_{jk}^{(N)} := \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2} H_k \left(\sqrt{\frac{2\pi}{N}} (nN+j) \right), \tag{2}$$

associated with the eigenvalues i^k, i.e.,

$$\sum_{l=0}^{N-1} A_{jl}^{(N)} F_{lk}^{(N)} = i^k F_{jk}^{(N)}. \tag{3}$$

The $H_k(x)$ in (2) is the classical Hermite polynomial of degree k and (3) thus represents the discrete analogue of the important continuous case where the Hermite functions $H_k(x)\exp\left(-x^2/2\right)$ are their own Fourier transforms:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy-x^2/2} H_n(x)\, dx = i^n H_n(y)\, e^{-y^2/2}. \tag{4}$$

It is worth mentioning that Mehta [13] makes vital use of (4) in proving the eigenvalue problem (3). Observe also that the "lowest" (since it does contain the polynomial factor $H_0(x) \equiv 1$ of the lowest degree $k=0$) eigenvector $F_{j0}^{(N)}$ in (2) is related to the $\theta_3(z|\tau)$-function, namely, $F_{j0}^{(N)} = e^{-\frac{\pi}{N}j^2}\, \theta_3(\pi i j | iN)$.

On the other hand, recently it became clear that the Fourier integral transform turns out to be very useful in revealing close relations between various q-special functions (see Atakishiyev [6] and references therein). For instance, the continuous q-Hermite polynomials $H_n(x|q)$, studied by Rogers [14], Allaway [1], and Askey et al [5], have the following transformation property

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy-x^2/2} H_n(\sin \kappa x|q)\, dx = i^n q^{n^2/4} h_n(\sinh \kappa y|q)\, e^{-y^2/2} \tag{5}$$

with respect to the Fourier integral transform (see Atakishiyev et al [8]). Here $q := \exp\left(-2\kappa^2\right)$, $0 \le \kappa < \infty$, and $h_n(x|q) := i^{-n} H_n\left(ix|q^{-1}\right)$, as in Askey [4]. So the question naturally arises whether it is possible to find a discrete analogue of (5), associated with the finite Fourier transform operator (1). The present work is aimed at proving that certain q-polynomial families admit discrete analogues of this type. In Section 2 we derive a discrete analofue of the Fourier integral transform (5). In Section 3 we discuss another example, which reveals that the finite Fourier transform interrelates also Rogers–Szegő and Stieltjs-Wigert families of q-polynomials.

Motivation for this study comes from Mehta's work [4]: Once his technique of constructing the eigenvectors $F_{jk}^{(N)}$ of the finite Fourier transform is comprehended, q-extensions of the $F_{jk}^{(N)}$ seem to emerge quite naturally.

Throughout this paper we employ standard notations of the theory of special functions (see, for example, Andrews et al [3] and Gasper et al [10]).

2. The Continuous q-Hermite Case

We begin with the explicit representations

$$H_n\left(\sin\alpha x|q\right) := i^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q e^{i(n-2k)\alpha x},$$

$$h_n\left(\sinh\beta x|q\right) := \sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)}e^{(n-2k)\beta x}, \tag{6}$$

for the continuous q-Hermite $H_n\left(z|q\right)$ and q^{-1}-Hermite $h_n\left(z|q\right)$ polynomials, which are valid for arbitrary constants α and β. The symbol $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q$ in (6) stands for the q-binomial coefficient.

We define next a q-extension of the Mehta eigenvectors $F_{jk}^{(N)}$ of the form

$$F_{jk}^{(N)}\left(q\right) := \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2} H_k\left(\sin\sqrt{\frac{2\pi}{N}}\kappa\left(nN+j\right)\middle|\, q\right), \tag{7}$$

where $0 < q := \exp\left(-2\kappa^2\right) < 1$. Then from (7) it follows that

$$F_{jk}^{(N)}\left(q^{-1}\right) := i^k\sum_{m=-\infty}^{\infty} e^{-\frac{\pi}{N}(mN+j)^2}\times\, h_k\left(\sinh\sqrt{\frac{2\pi}{N}}\kappa\left(mN+j\right)\middle|\, q\right) \tag{8}$$

extends $F_{jk}^{(N)}\left(q\right)$ to the values of the parameter q in the interval $(1,\infty)$.

From the definitions (7) and (8) it is clear that both $F_{jk}^{(N)}\left(q\right)$ and $F_{jk}^{(N)}\left(q^{-1}\right)$ are periodic functions of j with the period N. So we can write the Fourier expansion

$$F_{jk}^{(N)}\left(q\right) = \sum_{l=-\infty}^{\infty} a_{lk}^{(N)}\left(q\right)\exp\left(\frac{2\pi i}{N}\,lj\right), \tag{9}$$

with the expansion coefficients

$$a_{lk}^{(N)}(q) = \frac{1}{N}\int_0^N e^{-\frac{2\pi i}{N}lx} F_{xk}^{(N)}(q)\,dx. \tag{10}$$

An explicit form of the coefficients $a_{lk}^{(N)}(q)$ is evaluated exactly in the same way as that for the a_{lk} in Mehta [13], except that the Hermite polynomials $H_k\left(\sqrt{\frac{2\pi}{N}}\,(nN+j)\right)$ there should be replaced by the q-Hermite polynomials $H_k\left(\sin\sqrt{\frac{2\pi}{N}}\,\kappa\,(nN+j)\Big|\,q\right)$. Needless to say that in the case under discussion one should employ the integral transform (5) rather than (4). This yields

$$a_{lk}^{(N)}(q) = \frac{(-i)^k}{\sqrt{N}}\, q^{k^2/4}\, h_k\left(\sinh\sqrt{\frac{2\pi}{N}}\kappa l\,\Big|\, q\right) e^{-\frac{\pi}{N}l^2}. \tag{11}$$

Substituting now (11) into (9), one obtains that

$$\begin{aligned} F_{jk}^{(N)}(q) &= \frac{(-i)^k}{\sqrt{N}}\, q^{k^2/4} \sum_{l=-\infty}^{\infty} e^{\frac{2\pi i}{N}lj-\frac{\pi}{N}l^2} h_k\left(\sinh\sqrt{\frac{2\pi}{N}}\kappa l\,\Big|\, q\right) \\ &= \frac{(-i)^k}{\sqrt{N}}\, q^{k^2/4} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}mj} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+m)^2} \\ \times\, h_k &\left(\sinh\sqrt{\frac{2\pi}{N}}\kappa\,(nN+m)\,\Big|\, q\right) = (-1)^k\, q^{\frac{k^2}{4}} \sum_{m=0}^{N-1} A_{jm}^{(N)} F_{mk}^{(N)}\left(q^{-1}\right). \end{aligned} \tag{12}$$

Similarly, when $1 < q < \infty$ one may write the Fourier expansion

$$F_{jk}^{(N)}\left(q^{-1}\right) = \sum_{l=-\infty}^{\infty} b_{lk}^{(N)}(q) \exp\left(\frac{2\pi i}{N}\,lj\right), \tag{13}$$

with the expansion coefficients

$$\begin{aligned} b_{lk}^{(N)}(q) &= \frac{1}{N}\int_0^N e^{-\frac{2\pi i}{N}lx} F_{xk}^{(N)}\left(q^{-1}\right)dx \\ &= \frac{1}{\sqrt{N}}\, q^{-k^2/4}\, H_k\left(\sin\sqrt{\frac{2\pi}{N}}\kappa l\,\Big|\, q\right) e^{-\frac{\pi}{N}l^2}. \end{aligned} \tag{14}$$

Consequently, putting (13), (14) and (2) together gives

$$F_{jk}^{(N)}\left(q^{-1}\right) = q^{-k^2/4} \sum_{m=0}^{N-1} A_{jm}^{(N)}\, F_{mk}^{(N)}(q). \tag{15}$$

Thus, relations (12) and (15), written as

$$\sum_{m=0}^{N-1} A_{jm}^{(N)} F_{mk}^{(N)} \left(q^{-1}\right) = (-1)^k \, q^{-k^2/4} \, F_{jk}^{(N)} \left(q\right),$$

$$\sum_{m=0}^{N-1} A_{jm}^{(N)} F_{mk}^{(N)} \left(q\right) = q^{k^2/4} \, F_{jk}^{(N)} \left(q^{-1}\right), \tag{16}$$

respectively, are q-extensions of the eigenvalue problem (3). They describe the simple transformation property of the q-Hermite polynomials $H_n\left(x|q\right)$ of Rogers with respect to the finite Fourier transform operator (1).

It is well known that the q-Hermite polynomials $H_n\left(x|q\right)$ coincide with the Hermite polynomials $H_n\left(x\right)$ in the limit as the parameter q tends to 1 (or the parameter κ vanishes), namely,

$$\lim_{q\to 1} \kappa^{-n} H_n\left(\sin\kappa y|q\right) = \lim_{q\to 1} \kappa^{-n} h_n\left(\sinh\kappa y|q\right) = H_n\left(y\right).$$

So it is readily verified that in the limit as $q \to 1$

$$\lim_{q\to 1} \kappa^{-k} F_{jk}^{(N)}\left(q\right) = F_{jk}^{(N)}, \quad \lim_{q\to 1} \kappa^{-k} F_{jk}^{(N)}\left(q^{-1}\right) = i^k F_{jk}^{(N)},$$

where $F_{jk}^{(N)}$ are Mehta's eigenvectors (2). Consequently, both relations in (16) coincide with (3) when $q \to 1$.

Observe that the parameters q and N are independent here and the continuous limit (i.e., the limit as $N \to \infty$) of relations (16) are the Fourier integral transform (5) and its inverse.

Also, as is evident from (16), both of $F_{jk}^{(N)}\left(q\right)$ and $F_{jk}^{(N)}\left(q^{-1}\right)$ are eigenvectors of the operator $\left\{A^{(N)}\right\}^2$ (which represents the reflection operator), associated with the eigenvalues $(-1)^k$.

3. The Rogers–Szegő and Stieltjes–Wigert Polynomials

In this section we consider another pair of q-polynomial families, interrelated by the transformation $q \to q^{-1}$. The Rogers–Szegő polynomials $H_n\left(x;q\right)$ are defined in Szegő [16] and Al-Salam et al [2] as

$$H_n\left(x;q\right) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \tag{17}$$

From the inversion formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

for the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ it is evident that

$$H_n\left(x; q^{-1}\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} x^k. \tag{18}$$

It proves convenient to define the Stieltjes–Wigert polynomials $\widetilde{S}_n(x;q)$ in terms of the $H_n\left(x;q^{-1}\right)$ as (cf. Koekoek et al [11], p. 116)

$$\widetilde{S}_n(x;q) := (q;q)_n \, S_n(-x;q) := H_n\left(q^n x; q^{-1}\right). \tag{19}$$

As in the case of continuous q-Hermite polynomials (5) with $0 < q < 1$ and $1 < q < \infty$, the Rogers–Szegő $H_n(x;q)$ and the Stieltjes–Wigert $\widetilde{S}_n(x;q)$ polynomials are linked by the Fourier integral transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n\left(\alpha e^{2i\kappa x}; q\right) e^{ixy - x^2/2} dx = \widetilde{S}_n\left(\alpha e^{-2\kappa y}; q\right) e^{-y^2/2}, \tag{20}$$

where α is an arbitrary complex number, Atakishiyev et al [7].

Now we are in a position to define another q-extension of the Mehta eigenvectors $F_{jk}^{(N)}$ of the form

$$F_{jk}^{(N)}(\alpha;q) := \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2} H_k\left(\alpha e^{2i\sqrt{\frac{2\pi}{N}}\kappa(nN+j)}; q\right), \tag{21}$$

where $0 < q = \exp\left(-2\kappa^2\right) < 1$ as before and $H_k(x;q)$ is the Rogers–Szegő polynomial (17) of degree k. Then from (21) it follows that

$$F_{jk}^{(N)}\left(\alpha; q^{-1}\right) := \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+j)^2} \widetilde{S}_k\left(\alpha q^{-k} e^{-2\sqrt{\frac{2\pi}{N}}\kappa(nN+j)}; q\right) \tag{22}$$

extends $F_{jk}^{(N)}(\alpha;q)$ to the values of the parameter q in the interval $(1,\infty)$.

Because of the periodicity of $F_{jk}^{(N)}(\alpha;q)$ in j, one may write the Fourier expansion

$$F_{jk}^{(N)}(\alpha;q) = \sum_{l=-\infty}^{\infty} a_{lk}^{(N)}(\alpha;q) \exp\left(\frac{2\pi i}{N} lj\right), \tag{23}$$

with the expansion coefficients

$$a_{lk}^{(N)}(\alpha;q) = \frac{1}{N}\int_0^N e^{-\frac{2\pi i}{N}lx} F_{xk}^{(N)}(\alpha;q)\,dx = \frac{1}{\sqrt{N}}\widetilde{S}_k\left(\alpha e^{2\sqrt{\frac{2\pi}{N}}\kappa l};q\right) e^{-\frac{\pi}{N}l^2}. \tag{24}$$

The expression on the second line in (24) follows from (20) in exactly the same manner as (11) does from (5) in the previous section.

Substituting now (24) into (23), one readily derives that

$$\begin{aligned} F_{jk}^{(N)}(\alpha;q) &= \frac{1}{\sqrt{N}}\sum_{l=-\infty}^{\infty} e^{\frac{2\pi i}{N}jl-\frac{\pi}{N}l^2}\widetilde{S}_k\left(\alpha e^{2\sqrt{\frac{2\pi}{N}}\kappa l};q\right) \\ &= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} e^{-\frac{2\pi i}{N}mj}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{N}(nN+m)^2}\widetilde{S}_k\left(\alpha e^{-2\sqrt{\frac{2\pi}{N}}\kappa(nN+m)};q\right) \\ &= \sum_{m=0}^{N-1}\left(A^{(N)}\right)^{-1}_{jm} F_{mk}^{(N)}\left(\alpha q^k;q^{-1}\right), \end{aligned} \tag{25}$$

where $\left(A^{(N)}\right)^{-1}$ is the inverse operator with respect to the $A^{(N)}$.

Similarly, when $1 < q < \infty$ one may expand $F_{jk}^{(N)}\left(\alpha;q^{-1}\right)$ into the Fourier series

$$F_{jk}^{(N)}\left(\alpha;q^{-1}\right) = \sum_{l=-\infty}^{\infty} b_{lk}^{(N)}(\alpha;q)\exp\left(\frac{2\pi i}{N}lj\right), \tag{26}$$

with the expansion coefficients

$$\begin{aligned} b_{lk}^{(N)}(\alpha;q) &= \frac{1}{N}\int_0^N e^{-\frac{2\pi i}{N}lx} F_{xk}^{(N)}\left(\alpha;q^{-1}\right)dx \\ &= \frac{1}{\sqrt{N}}H_k\left(\alpha q^{-k}e^{2i\sqrt{\frac{2\pi}{N}}xl};q\right)e^{-\frac{\pi}{N}l^2} = a_{lk}^{(N)}\left(\alpha;q^{-1}\right). \end{aligned} \tag{27}$$

Combining now (26) and (27), yields the following relation

$$\begin{aligned} F_{jk}^{(N)}\left(\alpha;q^{-1}\right) &= \frac{1}{\sqrt{N}}\sum_{l=-\infty}^{\infty}\exp\left(\frac{2\pi i}{N}jl-\frac{\pi}{N}l^2\right) \\ &\times H_k\left(\alpha q^{-k}e^{2i\sqrt{\frac{2\pi}{N}}\kappa l};q\right) = \sum_{m=0}^{N-1}A_{jm}^{(N)}F_{mk}^{(N)}\left(\alpha q^{-k};q\right). \end{aligned} \tag{28}$$

The finite Fourier transforms (25) and (28) thus represent the desired discrete analogues of the Fourier integral transforms (20) and its inverse, which interrelate Rogers–Szegő and Stieltjes–Wigert polynomials.

As in the previous case of continuous q-Hermite polynomials, both relations (25) and (28) coincide with Mehta's eigenvalue equation (3) in the limit as $q \to 1$ (or $\kappa \to 0$).

4. Concluding Remarks

From our results in Section 2 and Section 3 it is plain that the existence of q-extensions like (5) and (20) of the integral Fourier transform (4) enables one to construct their discrete (or finite) analogues by using the mathematical technique, developed by Mehta [13]. What still remains to be understood is the following: Are there other families of q-orthogonal polynomials, which are linked by the finite Fourier transform (1)? For instance, it seems that there should be some connection between the discrete q-Hermite polynomials $h_n\,(x;q)$ of type I and $\widetilde{h}_n\,(x;q)$ of type II in terms of the finite Fourier transform. Work on answering this question is in progress.

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