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LOW RANK INDECOMPOSABLE VECTOR BUNDLES ON CERTAIN COMPACT COMPLEX MANIFOLDS

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a compact and connected n-dimensional manifold ($n \geq 2$) such that $TX \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L$ for some $L \in \mathrm{Pic}(X)$. Then one of the following cases occurs:

(a) for every integer $r \geq 2$ there is an indecomposable rank r holomorphic vector bundle on X;

(b) $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, TX) = n$;

(c) $h^0(X, TX) \geq n+1$, $h^1(X, \mathcal{O}_X) = 0$ and X is homogeneous.

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1. Indecomposable Vector Bundles

Here we prove the following result.

Theorem 1. *Let X be a compact and connected n-dimensional manifold ($n \geq 2$) such that $TX \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L$ for some $L \in \mathrm{Pic}(X)$. Then one of the following cases occurs:*

(a) for every integer $r \geq 2$ there is an indecomposable rank r holomorphic vector bundle on X;

(b) $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, TX) = n$;

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(c) $h^0(X,TX) \geq n+1$, $h^1(X,\mathcal{O}_X)=0$ *and* X *is homogeneous.*

For much more in the case $L \cong \mathcal{O}_X^{\oplus n}$, see [6] and [7], Theorem 7.13.1. For deeper results in the algebraic case, see [4] and [3]. The property listed in case (a) of Theorem 1 is very strong: compare with Hartshorne's conjecture on non-existence of rank two indecomposable vector bundles on $\mathbf{P}^n$.

Remark 1. Let X be a smooth and connected n-dimensional ($n \geq 2$) compact complex Kähler manifold such that $TX \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L$ for some $L \in \mathrm{Pic}(X)$. Hence $\Omega_X^1 \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L^*$. Thus $h^0(X,\Omega_X^1) \geq n-1 > 0$. Since X is Kähler, we get $h^1(X,\mathcal{O}_X) > 0$. It is easy to check (see Lemma 1 below) that for every integer $r \geq 2$ there is a rank r indecomposable vector bundle on X.

Lemma 1. *Assume* $h^1(X,\mathcal{O}_X) \neq 0$*. Then for every integer* $r \geq 2$ *there is a rank* r *indecomposable vector bundle on* X*.*

Proof. Since $h^1(X,\mathcal{O}_X) \neq 0$, there is a rank two vector bundle E on X which is a non-zero extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. The non-triviality of the extension is equivalent to $h^0(X,E)=1$. Since E^* is a non-zero extension of $\mathcal{O}_X$ by $\mathcal{O}_X$, we also get $h^0(X,E^*)=0$. Assume E decomposable, say $E \cong A \oplus B$ with $A, B \in \mathrm{Pic}(X)$, $h^0(X,A)=1$, $h^0(X,B)=0$. The inclusion $\mathcal{O}_X \to E$ with locally free cokernel and the assumption $h^0(X,B)=0$ gives $A \cong \mathcal{O}_X$. The composition of the map $B \to E$ with the map $E \to \mathcal{O}_X$ gives $h^0(X,B^*) \neq 0$. Thus $h^0(X,E^*) \geq 2$, contradiction. Taking a non-zero extension of E by $\mathcal{O}_X$ we get a rank 3 indecomposable vector bundle F with $h^0(X,F)=h^0(X,F^*)=1$. If $r \geq 4$, then we make $r-3$ further non-zero extensions with $\mathcal{O}_X$. □

Lemma 2. *Let* X *be a compact and connected* n*-dimensional complex manifold such that* $TX \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L$ *for some* $L \in \mathrm{Pic}(X)$*. Assume* $h^0(X,L)=0$*. Then* $h^1(X,\mathcal{O}_X) \neq 0$*.*

Proof. Assume $h^1(X,\mathcal{O}_X)=0$. Since $h^0(X,TX)=n-1$, we have $\dim(\mathrm{Aut}(X))=n-1$. Furthermore, since $h^0(X,\mathcal{O}_P \otimes TX)=0$ for all $P \in X$, we see that every $P \in X$ is fixed only by a discrete subset of $\mathrm{Aut}(X)$. Hence each orbit of $\mathrm{Aut}(X)$ is a hypersurface of X. Thus all such orbits are closed and disjoints. Hence the algebraic reduction of X is a curve (see [5]). Since $h^1(X,\mathcal{O}_X)=0$, the algebraic reduction of X is isomorphic to $\mathbf{P}^1$. Since all orbits of $\mathrm{Aut}(X)$ are closed and disjoint hypersurfaces of X, we get that the algebraic reduction of X is a holomorphic map $\alpha_X : X \to \mathbf{P}^1$, not just a meromorphic map from X onto $\mathbf{P}^1$. Since $T\mathbf{P}^1 \cong \mathcal{O}_{\mathbf{P}^1}(2))$ and α_X is surjective, we get an inclusion of sheaves $\alpha_X^*(\mathcal{O}_{\mathbf{P}^1}(2))) \to TX \cong \mathcal{O}_X^{n-1} \oplus L$ and hence an inclusion of sheaves $\alpha_X^*(\mathcal{O}_{\mathbf{P}^1}(2))) \to L$. Thus $h^0(X,L) \geq 3$, contradiction. □

Lemma 3. *Let X be a compact and connected n-dimensional complex manifold such that $TX \cong \mathcal{O}_X^{\oplus(n-1)} \oplus L$ for some $L \in \mathrm{Pic}(X)$. Assume $h^0(X,L) \geq 2$ and $h^1(X,\mathcal{O}_X) = 0$. Then X is homogeneous.*

Proof. Let B the set-theoretic base locus of $H^0(X,L)$. By assumption B has codimension at least two in X. Assume $B \neq \emptyset$ and fix $P \in B$. Since $h^0(X,\mathcal{I}_P \otimes TX) = h^0(X,TX) - (n-1)$, we get that B has dimension $n-1$, contradiction. Thus $B = \emptyset$. Hence X is homogeneous. □

Proof of Theorem 1. By Lemma 1 we may assume $h^1(X,\mathcal{O}_X) = 0$. If $h^0(X,L) = 0$, then we may apply Lemma 2. If $h^0(X,L) \geq 2$, then we may apply Lemma 3. □

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