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ON THE EXISTENCE OF 3 AND 4-DESIGNS IN COMPLETE GRAPHS

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Abstract: It is shown how a select set of subgraphs of the complete graph on 13 vertices generate 3-designs when acted upon with a group of suitable transitivity properties. An argument for the non-existence of 4-designs generated by subgraphs of K_n (for $1 \leq n \leq 100$) is given in the last section.

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1. Introduction and Method of Construction

A pair (P, B) consisting of a set of P of v points and a set B of distinct subsets of size k of P, called blocks, is called a $t-(v, k, \lambda)$ design (t-design, for short) if any t points of P are contained in exactly λ blocks. It can be shown that a $t-(v, k, \lambda)$ design is a $s-(v, k, \lambda_s)$ design for s integral, $0 \leq s \leq t$, where

$$\lambda_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\lambda. \quad (1)$$

Designs and finite geometries arise in the work of Bose [1]. We suggest Constantine [3] for a clear exposition of the fundamental results and Farrell [9] for many other contributions to the field. Cyclic methods of constructing equineighboured incomplete block designs having blocks of size 3 which are efficient under standard intrablock analysis can be found in Jacroux [11]. Recursive methods

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of constructing large sets of t-designs are presented in Chee [2], Magliveras [13], and van Trung [14]. It was shown in Constantine [4] that by using t-designs one can recapture a probability density in up to $\frac{t}{2}$ dimensions by the use of the discrete Radon transform. Further applications to statistical design, specifically construction of optimal designs against loss of varieties, are found in Constantine [5]. Optimality of t-designs, from a statistical viewpoint, is studied through Schur-convex functions defined on the spectra of Fisher information matrices; see Constantine [6]. Methods for constructing optimal designs related to experimental situations in which v treatments are applied to experimental units arranged in b blocks of size k that have uncontrollable trends within blocks are discussed in Jacroux [12].

The automorphism group of a $t-(v, k, \lambda)$ design consists of the permutations on points which preserve the blocks. Two t-designs (defined on the same set of points) are called isomorphic if they could be made to have the same set of blocks by a permutation on points.

The simple graph on n vertices with an edge between any two vertices is called the complete graph and is written as K_n. A m-claw is a graph with $m+1$ vertices and m edges in which a vertex has degree m. The automorphism group of a graph consists of those permutations on vertices which preserve the edges of the graph. We denote the automorphism group of a graph H by $\mathrm{Aut}(H)$. $\mathrm{Aut}(K_n)$ acts (by its definition) on the edges of K_n in a natural way, i.e., $(ab)^g = a^g b^g$, where ab is an edge with endpoints a and b and $g \in \mathrm{Aut}(K_n)$. Obviously $\mathrm{Aut}(K_n)$ is the symmetric group $\mathrm{Sym}(n)$ on the n vertices. By $\mathrm{Alt}(n)$ we denote the alternating group on n vertices; $|\mathrm{Sym}(n)| = n!$, $|\mathrm{Alt}(n)| = 2^{-1}n!$.

We seek a graph with k edges whose orbit under the action of a group with suitable transitivities is a 3- or 4-design. The points of the design are the edges of K_n.

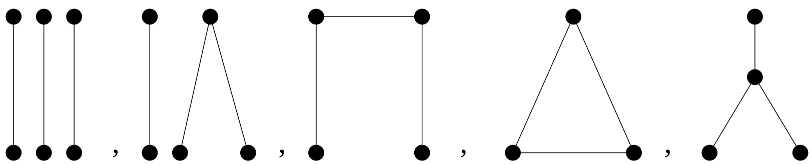
2. Generating 3-Designs in K_n

We shall first find conditions on the subgraph S of K_n such that $B = \{S^g : g \in G\}$ are the blocks of a 3-design. Here S^g is the subgraph of K_n whose edges are obtained as images of edges of S under the action of elements g of a group G. We shall require that G has certain transitivity properties.

We consider all the possible configurations of 3 edges in K_n. These are:

We refer to these subgraphs as being of types 1, 2, 3, 4, and 5, respectively.

Consider an arbitrary subgraph S of K_n. Let m_i denote the number of subgraphs of type i in K_n, and s_i denote the same in S.



(2)

Let G be a permutation group on the n vertices transitive within each of the types listed in (2). Denote by t_i the number of images in the set B defined above that contain a (specific) subgraph of type i. Note that t_i is well-defined as a function of type. Such groups G exist, $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ being immediate examples. Interesting choices should involve groups of small cardinality.

Let us count in two ways the cardinality of the set $C = \{(\alpha, \sigma) : \sigma \subset \alpha\}$, where $\alpha \in B$ and σ is a subgraph of type i. First: fix σ. There are then t_i α's containing σ; and there are m_i choices for σ; hence $|C| = t_i m_i$. Second: fix $\alpha \in B$. There are s_i σ's in α; in all, there are $|B|$ choices for α; so $|C| = |B| s_i$. These arguments give

$$t_i = \frac{s_i}{m_i}|B| \ ; \ 1 \le i \le 5. \tag{3}$$

We want $B = \{S^g : g \in G\}$ to be the set of blocks of a 3-design. Then the values t_i should be independent of type, i.e., we should have $t_1 = t_2 = t_3 = t_4 = t_5 (= \lambda)$. Making this change in (3) we obtain:

$$s_i = m_i \lambda \lambda_0^{-1} \ ; \ 1 \le i \le 5, \tag{4}$$

where $\lambda_0 = |B|$. But the m_i's can be explicitly computed. Using counting techniques as described in Farrell ([7] and [9]) we obtain $m_1 = \frac{6!}{48}\binom{n}{6}$, $m_2 = \frac{5!}{4}\binom{n}{5}$, $m_3 = \frac{4!}{2}\binom{n}{4}$, $m_4 = \frac{3!}{6}\binom{n}{3}$, and $m_5 = \frac{4!}{6}\binom{n}{4}$. The number of points in our prospective 3-design is $v = \binom{n}{2}$ and we let $|S| = k$ be the block size. Using the relation between λ and λ_0 mentioned in (1) and with the m_i's as written above, (4) becomes:

$$s_i = \frac{\binom{k}{3}}{\binom{\binom{n}{2}}{3}} m_i \ ; \ 1 \le i \le 5.$$

We summarize below.

Theorem 1. *If there exists a subgraph S of k edges in K_n containing exactly*

$$s_i = \frac{\binom{k}{3}}{\binom{\binom{n}{2}}{3}} m_i \tag{5}$$

subgraphs of type i, $1 \leq i \leq 5$, then $\{S^g : g \in G\}$ is the set of blocks of a 3-design whose points are the edges of K_n. The graph G and the m_i's are as defined previously.

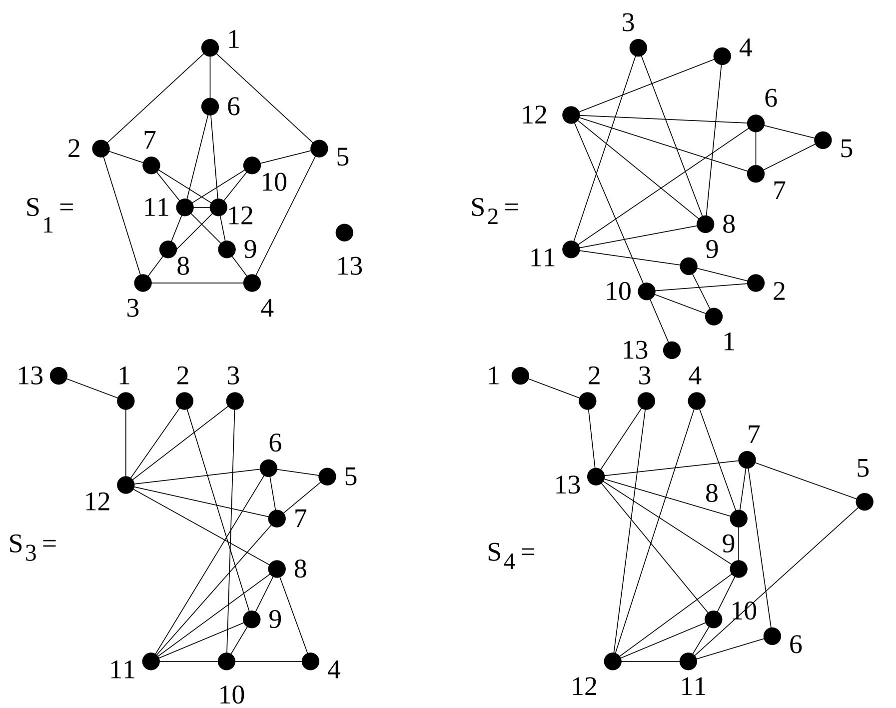
The difficulty is in pinning down a subgraph S with these properties. A computer search and some trade-off techniques proved useful in doing this. First, a diophantine system of equations, counting 1, 2, and 3-claws, gives information about the degrees of vertices in S. Once the feasible vectors of degrees are known, one can solve another diophantine system of equations having as unknowns N_{ij}, the number of edges between vertices of degree i and those of degree j (counting subgraphs of type 3 gives a crucial relation among N_{ij}'s). For $n = 13$ and $k = 21$, knowing the possible values for the N_{ij}'s and the degrees, one can construct by hand graphs S having exactly $s_4 = 5$ triangles (i.e., subgraphs of type 4). A computer count then shows that for certain choices of S all the s_i's are as desired. Once found, we can check directly that our selected examples satisfy all the necessary requirements.

While searching for such S, the values of the s_i's given in (5), along with the corresponding s_i's for the complement of S, ought to be integral. This, along with the help of a computer, leads one to conclude that for $4 \leq k \leq n-2$, and $6 \leq n \leq 100$, the only feasible values that might admit 3-designs are $n = 13$, $k = 21$ and 57; $n = 21$, $k = 66$ and 144; $n = 34$, $k = 260$ and 301.

We investigated the smallest case $n = 13$, $k = 21$ and found four graphs that comply to our requirements.

Theorem 2. *Each of the four graphs displayed above generates a 3-design when acted upon with a group transitive within each of the five types of subgraphs displayed in (2).*

We conclude this section by observing that no two of the 3-designs generated by the subgraphs mentioned in Theorem 2 are isomorphic. Indeed, the automorphism groups are as follows: $\mathrm{Aut}(S_1) = D_{10} \oplus Z_2$ where D_{10} is the dihedral group with 10 elements and Z_2 is the cyclic group of order 2, $\mathrm{Aut}(S_2) =< (12),(67) >= Z_2 \oplus Z_2$, $\mathrm{Aut}(S_3) =< ((67) >= Z_2$, $\mathrm{Aut}(S_4) =< ((56) >= Z_2$. The number of blocks in the design generated by S_i is $|G|\,|\mathrm{Aut}(S_i)|^{-1}$. In view of this it remains to be shown that the designs generated by S_3 and S_4 are not isomorphic. It is clear from the construction that the automorphism group of the design generated by S_i is transitive on blocks. The stabilizer of S_3 is $< (5\ 6\ ,\ 5\ 7)(6\ 12\ ,\ 7\ 12)(6\ 11,\ 7\ 11) >$ (recall that edges of K_{13} are points), while the stabilizer of S_4 is $< (5\ 7\ ,\ 6\ 7)(5\ 11\ ,\ 6\ 11) >$. These two stabilizers (whilst obviously isomorphic as groups) cannot be obtained from one another by a mere permutation on points (i.e., edges of K_{13}) because they have different cycle structure. But the automorphism groups of two isomorphic de-



signs can be obtained from one another by a permutation on points - and with the existing transitivity on blocks the same is true of the one block stabilizers. We can now conclude that S_3 and S_4 do not generate isomorphic designs.

3. Nonexistence of 4-Designs in K_n for $n \leq 100$

Apart from S_1, S_2, S_3 and S_4 there are many other subgraphs of K_{13} on 21 edges, nonisomorphic to any of these, that generate 3-designs. It then seems appropriate to ask whether some of these subgraphs generate, in fact, 4-designs. As it turns out none of them do and we shall prove it in this section.

There are no 4-designs in K_{21} or K_{34} because, by counting as in (3), a set S that generates them has to have $\binom{66}{4} \cdot \binom{210}{4}^{-1} \cdot 21 \cdot \binom{20}{4}$ or $\binom{260}{4} \cdot \binom{561}{4}^{-1} \cdot 34 \cdot \binom{33}{4}$ 4-claws, respectively. But both these numbers fail to be integers. Restricting the discussion to K_n, $6 \leq n \leq 100$, it hence follows that 4-designs can only exist in K_{13} with block size $k = 21$ (and $k = 57$, which would be the complementary designs).

The divisibility conditions given in (1) are satisfied (for $t = 4$) with $v = 78$, $k = 21$ and λ a multiple of 6. For convenience we mention that $\lambda_0 =$

$\binom{78}{m}\binom{21}{m}^{-1}\lambda_m$; $0 \leq m \leq 4$, i.e., $\frac{\lambda_0}{\lambda_4} = \frac{13\cdot 11\cdot 5}{3}$, $\frac{\lambda_0}{\lambda_3} = \frac{13\cdot 11\cdot 2}{5}$, $\frac{\lambda_0}{\lambda_2} = \frac{13\cdot 11}{5\cdot 2}$, $\frac{\lambda_0}{\lambda_1} = \frac{13\cdot 2\cdot 5}{7}$, $\frac{\lambda_0}{\lambda_0} = 1$.

We shall count in two ways the number of m-claws in S, $0 \leq m \leq 4$. Let us denote by $c_m(K_{13})$ and $c_m(S)$ the number of m-claws in K_{13} and S, respectively. As the design generated by S is supposed to be a 4-design (and thus a 3-design and a 2-design as well) a counting argument analogous to that which yielded (3) gives

$$\frac{c_4(K_{13})}{c_4(S)} = \frac{\lambda_0}{\lambda_4}, \quad \frac{c_3(K_{13})}{c_3(S)} = \frac{\lambda_0}{\lambda_3}, \quad \frac{c_2(K_{13})}{c_2(S)} = \frac{\lambda_0}{\lambda_2}.$$

Replacing $\frac{\lambda_0}{\lambda_m}$ by the values just computed in the last paragraph and the value of $c_m(K_{13})$ by $13 \cdot \binom{12}{m}$; $m = 2, 3, 4$, we obtain $c_4(S) = 27$, $c_3(S) = 50$, $c_2(S) = 60$ ($c_1(S) = 42$, which is just the number of degrees and $c_0(S) = 13$, the number of vertices). Another way of counting the number of m-claws in S, used in Farrell ([8] and [9]), is as follows. Denote by n_i the number of vertices of degree i in S (note that $n_i = 0$ for $i \geq 7$, since a single vertex of degree bigger or equal to 7 generates at least $\binom{7}{4} = 35$ 4-claws, which exceeds the value of $c_4(S) = 27$ found above). Counting the number of m-claws centered at each vertex of S we obtain $c_m(S) = \sum_{i=0}^{6} \binom{i}{m} n_i$; $m = 0, 1, 2, 3, 4$ (set $\binom{i}{m} = 0$, for $i < m$.) This two-way counting leads us to the following (diophantine) system of equations:

$$\begin{aligned}
n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + n_6 &= 13\,, \\
n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 + 6n_6 &= 42\,, \\
n_2 + 3n_3 + 6n_4 + 10n_5 + 15n_6 &= 60\,, \\
n_3 + 4n_4 + 10n_5 + 20n_6 &= 50\,, \\
n_4 + 5n_5 + 15n_6 &= 27\,.
\end{aligned}$$

We seek nonnegative integral solutions to this system and shall conclude the proof by showing that there are none.

We shall refer by eq(i) to the i-th equation in the above system. From equation (5) we conclude that $n_6 = 0$ or 1. Still from equation (5) we obtain that 5 divides $27 - n_4$; as equation (3) limits $n_4 \leq 10$, the remaining possible values for n_4 are 2 and 7. But $n_6 = 0$ and $n_4 = 7$ lead to $n_5 = 4$, and then equation (4) gives $n_3 = -18$, which is not allowed; $n_6 = 0$ and $n_4 = 2$ give $n_5 = 5$, and again n_3 becomes negative from equation (4). Let us next examine the case in which $n_6 = 1$. If also $n_4 = 7$, then $n_5 = 1$ and this forces once again a negative value for n_3 in equation (4). The remaining case is $n_6 = 1$ and $n_4 = 2$. In this case the last two equations give $n_3 = n_5 = 2$, and equation (3) gives $n_2 = 7$. The values determined so far sum up to 14 and this contradicts equation (1).

This ends our proof and we conclude that there is no subgraph in K_n $(1 \leq n \leq 100)$ that generates a 4-design.

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