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HOLOMORPHIC VECTOR BUNDLES ON A FORMAL NEIGHBORHOOD OF A $\mathbf{P}^2$ INSIDE A SMOOTH 3-FOLD WITH NEGATIVE NORMAL BUNDLE

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Abstract: Let $\widehat{S}$ be a formal neighborhood of $\mathbf{P}^2$ in a smooth 3-fold with negative normal bundle and E a vector bundle on $\widehat{S}$ such that $E|\mathbf{P}^2$ is simple. Then the local deformation space of E is finite dimensional and smooth and a general element of it is "as balanced as possible".

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Set $S := \mathbf{P}^2$ and let W be a smooth complex 3-dimensional manifold containg S. We will only W as a germ around S. Let $\mathcal{I}_S$ denote the ideal sheaf of S in W. The conormal bundle $\mathcal{I}_S/(\mathcal{I}_S)^2$ of S in W is a holomorphic line bundle on S. Set $x := \deg(\mathcal{I}_S/(\mathcal{I}_S)^2)$. We assume $x > 0$, i.e. we assume that the conormal bundle of S in W is ample. For all integers $n \geq 0$, let $S^{(n)}$ denote the order n infinitesimal neighborhhod of S in W, i.e. the complex subspace of W with $(\mathcal{I}_S)^{n+1}$ as its ideal sheaf. Thus $S_0 = S$ and $(S_n)_{red} = S$ for all n. Let $\widehat{S}$ denote the formal neighborhood of S in W. Thus $\widehat{S}$ is a formal complex space with S as its reduction. Let $\mathcal{I}$ denote the ideal sheaf of S in $\widehat{S}$. For any vector bundle E on $\widehat{S}$ and any integer $n \geq 0$, set $E_n := E|S^{(n)}$. In this paper

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we extend [1], Theorem A, to the case $S = \mathbf{P}^2$ and prove the following result.

Theorem 1. *Take $S := \mathbf{P}^2 \subset W$ with W smooth 3-fold with negative normal bundle, $P \in S$, and $E = \{E_n\}$ a vector bundle on $\widehat{S}$ such that $E_0 := E|S$ is simple. Set $r := \mathrm{rank}(E)$ and $d := c_1(E_0) \in \mathbb{Z}$. Then the local moduli space of E is finite dimensional and smooth. Moreover, E is a flat limit of a flat family of vector bundles on $\widehat{S}$ such that the general member $G = \{G_n\}$ of it satisfies the following conditions:*

(i) if $d = ar - x$ with $a, x \in \mathbb{Z}$ and $0 < x < r$, then for every line $D \subset S$ such that $P \in D$, the vector bundle $G_0|$ has splitting type $a \geq \cdots \geq a - 1$, i.e. it is rigid;

(ii) if $d = ar$ with $a \in \mathbb{Z}$, then for a general line $D \subset S$ such $P \in D$ the vector bundle $G_0|D$ is isomorphic to $\mathcal{O}_D(a)^{\oplus r}$, but there are finitely many lines $R \subset S$ such that $P \in S$ such that $G_0|S \ncong \mathcal{O}_D(a)^{\oplus r}$; for all these jumping lines R the vector bundle $G_0|R$ has splitting type $(a+1, a, \ldots, a, a-1)$; furthermore, the number of such jumping lines depends only from the integers r and $c_2(G_0)$.

We lift from [1] a few definitions and extend to the case $S = \mathbf{P}^2$ a few lemmas proved in [1] when S is a Hirzebruch surface. For more details concerning the definitions, see [1], Section 2. Let E be a formal vector bundle on $\widehat{S}$, i.e. a compatible sequence of vector bundles $\{E_n\}$, $n \geq 0$, with E_n a vector bundle on $S^{(n)}$. The local deformation of $E := \{E_n\}$ is smooth if $H^2(\widehat{S}, \mathrm{End}(E)) = 0$ ([4], Remark 2.4, or [1], Remark 2.9).

Lemma 1. *Let $E = \{E_n\}$ be a vector bundle on $\widehat{S}$ such that $E_0 := E|S$ is simple. Then $H^2(\widehat{S}, \mathrm{End}(E)) = 0$.*

Proof. It is sufficient to prove that $h^2(S^{(n)}, \mathrm{End}(E_n)) = 0$ for all $n \geq 0$. First assume $n = 0$. Since E_0 is simple, we have $h^0(S, \mathrm{End}(E)(-3)) = 0$. Since $\omega_S \cong \mathcal{O}_S(-3)$, we have $h^0(S, \mathrm{End}(E_0)(-3)) = h^2(S, \mathrm{End}(E_0))$ (Serre duality), concluding the case $n = 0$. Now assume $n > 0$ and that the result is true for the integer $n' := n - 1$. Notice that $(\mathcal{I}_S)^n/(\mathcal{I}_S)^{n+1} \cong \mathcal{O}_S(nx)$. Consider the exact sequence:

$$0 \to (\mathcal{I}_S)^n/(\mathcal{I}_S)^{n+1} \to \mathcal{O}_{S^{(n)}} \to \mathcal{O}_{S^{(n-1)}} \to 0\,. \tag{1}$$

Since $S^{(n)}$ is a two-dimensional locally Cohen Macaulay projective scheme, we have $h^2(S^{(n)}, \mathrm{End}(E_n)) = h^0(S^{(n)}, \mathrm{End}(E_n) \otimes \omega_{S^{(n)}})$. Furthermore, there is an exact sequence relating $\omega_{S^{(n)}}$ with $\omega_{S^{(n)}}$. We get $h^0(S^{(n)}, \mathrm{End}(E_n) \otimes \omega_{S^{(n)}}) = 0$ tensoring (1) with $\mathrm{End}(E_n) \otimes \omega_{S^{(n)}}$ and using the inductive assumption. $\square$

By tensoring (1) with $\mathrm{End}(E_n)$) and using induction on n we also get the following results (see [1], Lemma 3.3, Lemma 3.4 and Proposition 3.5) in which we use the assumption $x > 0$.

Lemma 2. *Let $E = \{E_n\}$ be a vector bundle on $\widehat{S}$ such that E_0 is simple. Then for all integers $n \geq 0$ the restriction map $H^1(S^{(n)}, \mathrm{End}(E_n)) \to H^1(S^{(n-1)}, \mathrm{End}(E_{n-1}))$ is surjective.*

Lemma 3. *Let $E = \{E_n\}$ be a vector bundle on $\widehat{S}$ such that E_0 is simple. There is an integer τ (depending only from E_0) such that for all integers $n \geq \tau$ the restriction map $H^1(S^{(n)}, \mathrm{End}(E_n)) \to H^1(S^{(n-1)}, \mathrm{End}(E_{n-1}))$ is bijective.*

Proposition 1. *Let $E = \{E_n\}$ be a vector bundle on $\widehat{S}$ such that E_0 is simple. There is an integer τ (depending only from E_0) such that for all integers $n \geq \tau$ every local deformation of E_{n-1} extends uniquely to a local deformation of E.*

Proof of Theorem 1. The smoothness and finite-dimensionality of the local moduli space of E is Lemma 1 and 3 and Proposition 1. To prove the statements concerning the vector bundle G it is sufficient to make a few quotations. For case (i) and $r \geq 3$, apply [2]. For case (i) and $r = 2$, apply [3]. For case (ii) use the proof of [2]. □

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References

[1] E. Ballico, E. Gasparim, Vector bundles on a three-dimensional neighborhood of a ruled surface, *J. Pure Appl. Algebra*, **195**, No. 1 (2005), 7-19.

[2] J. Brun, A. Hirschowitz, Droites des saut des fibrés stables de rank élévé sur $\mathbb{P}_2$, *Math. Z.*, **181**, No. 2 (1982), 171-178.

[3] K. Hulek, Stable rank-2 vector bundles on $\mathbb{P}^2$ with c_1 odd, *Math. Ann.*, **242**, No. 3 (1979), 241-266.

[4] C.S. Seshadri, Theory of moduli, *Symp. Pure Math.*, **29**, AMS, Providence, (1975) 263-304.