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GENERIC ELEMENTS OF FREE COLOR LIE SUPERALGEBRAS

Dilek Kahyalar

Department of Mathematics
Faculty of Arts and Sciences
Çukurova University
Balcali 01330, Adana, TURKEY
e-mail: dilekah@cu.edu.tr

Abstract: Let L be a finitely generated free color Lie superalgebra. We construct a series of generic elements of L.

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1. Introduction

In the articles [1], [2], [4], [5] generic elements of free groups and of free Lie algebras were studied. In particular, series of generic elements of free groups were constructed. In [1] it was shown that for a variety with solvable word problem there is an algorithm to recognize generic elements.

Let U be a variety of Lie algebras defined by the set of laws, L the free color Lie superalgebra and $D(L)$ be the verbal ideal. An element f of L is said to be $D-$generic if $f \in D(L)$ but $f \notin D(H)$ for every proper subalgebra H of L.

In general, generic does not imply almost primitive element, see [3]. At the same time in [3] it was proved that if U is a nontrivial variety defined by the set of laws D and $w \in D(F)$ and w is almost primitive element of F then w is $D-$generic.

In this article we consider generic elements in free color Lie superalgebras.

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2. Preliminaries

Let G be an abelian group, K a field, char$K \neq 2$, K^* the multiplicative group of K, $\varepsilon : G \times G \to K^*$ a skew symetric bilinear form for which

$$\begin{aligned} \varepsilon(g,g) &= \pm 1, \\ G_- &= \{g \in G : \varepsilon(g,g) = -1\}. \end{aligned}$$

A graded algebra $R = \bigoplus_{g\in G} R_g$ over K is color Lie superalgebra if;

$$\begin{aligned} [x,y] &= -\varepsilon(d(x),d(y))[y,x], \\ [x,[y,z]] &= [[x,y],z] + \varepsilon(d(x),d(y))[y,[x,z]], \\ [v,[v,v]] &= 0, \end{aligned}$$

with $d(v) \in G_{-1}$ for $G-$homogeneous elements $x, y, z, v \in R$, where $d(a) = g$ if $a \in R_g$.

If H is $G-$graded associative algebra over K then $[H]$ denotes the color Lie superalgebra with operation $[,]$, where $[u,v] = uv - \varepsilon(d(u),d(v))vu$ for $G-$homogeneous elements $u, v \in H$.

Let $X = \bigcup_{g\in G} X_g$ be a $G-$graded set, $d(x) = g$ for $x \in X_g$ and let $A(X)$ be the free $G-$graded associative $K-$algebra , $L(X)$ is the free color Lie superalgebra with the set X of free generators.

Let X be a finite set and $L = L(X)$, be a free color Lie superalgebra on X. For $u \in L$, by $d(u)$ we denote the usual degree of u. Consider a weight function $\mu : X \to N$, where N is the set of positive integers. Let $\Gamma(X)$ be the free groupoid of nonassosociative monomials in the alphabet X, $S(X)$ the free semigroup of associative words in X, and

$$\sim : \Gamma(X) \to S(X)$$

the bracket removing homomorphism. We set $\mu(x_1, ..., x_n) = \sum_{i=1}^{n} \mu(x_i)$ for $x_1, ..., x_n \in X$ and $\mu(u) = \mu(\widetilde{u})$ for $u \in \Gamma(X)$.

Let l be the usual length function on the free associative algebra A. For element $a \in A$, by a^0 we denote the sum of monomials of the highest degree in a with respect to the function l.

Definition 1. A set $Y \subseteq L$ is called reduced if for every $y \in Y$, the element y^0 does not belong to the subalgebra of L generated by the set $\{u^0 : u \in Y, u \neq y\}$.

Definition 2. An element u of $L(X)$ is said to be primitive if it is an element of some set of free generators of the algebra $L(X)$.

Definition 3. An almost primitive element of the free color Lie superalgebra $L = L(X)$ is an element which is not primitive in L but which is primitive in any proper subalgebra of L containing it.

Definition 4. Let U be a abelyen variety of Lie algebras (i.e. $[L, L] = 0$). An element f of L is said to be $Z-$generic if $f \in < [L, L] >$ but $f \notin < [H, H] >$ for every proper subalgebra H of L.

An element f of $[L, L]$ is $Z-$generic if and only if $u = \sum \alpha_i [f_i, g_i]$, where $\alpha_i \in K$ and the elements $\{f_i, g_i\}$ generate the algebra $L = L(X)$. Therefore if $u = \sum_{i=1}^{s} \alpha_i [u_i, v_i]$, where u_i, v_i are monomials, $d(u_i), d(v_i) > 1$, $\alpha_i \in K$ then u is not a $Z-$generic element of $L(X)$.

3. Main Theorem

Let X be a finite set and $L = L(X)$ be a free color Lie superalgebra on X.

Proposition 5. *Let u be an almost primitive element of $L = L(X)$, then u is a $Z-$generic element of L.*

Proof. Assume that $u \in [H, H]$ for some proper subalgebra H of L. Since u is an almost primitive element of L, u is primitive element of H. $\{u = h_1, h_2, ..., h_m\}$ is a generated set of H. Since u can be written in this form $u = [f, g]$ where $f, g \in H$. We obtain $h_1 = [f, g]$ which is contradiction. Therefore, u is a $Z-$generic element of L. □

Theorem 6. *Let K be a field,* char$K \neq 2$, *$X = \{x_1, ..., x_n\}$ and let $L(X)$ be the free color Lie superalgebra over K.*

1. *If $n = 2m$ then*

$$[x_1, x_2] + ... + [x_{2m-1}, x_{2m}]$$

   is a $Z-$generic element of $L(X)$.

2. *If $n = 2m$, $k_i, l_i \geq 2$, $k_i \neq l_i$ then*

$$u_{k_1,l_1}(x_1, x_2) + ... + u_{k_m,l_m}(x_{2m-1}, x_{2m})$$

   is a $Z-$generic element of $L(X)$.

3. *If* $n = 2m+1$ *and* $l \geq 2$ *then*

$$[x_1, x_2] + (x_1)(Adx_3)^l + [x_4, x_5] + ... + [x_{2m}, x_{2m+1}]$$

is a $Z-$*generic element of* $L(X)$.

4. *If* $n = 3m$, $l_1, ..., l_m \geq 2$ *then*

$$[x_1, x_2] + (x_1)(Adx_3)^{l_1} + ... + [x_{3m-2}, x_{3m-1}] + (x_{3m-2})(Adx_{3m})^{l_m}$$

is a $Z-$*generic element of* $L(X)$.

Proof. Suppose that $u \in [H, H]$ and $u = [h_1, h_2]$. Apply a left Fox derivation u according to $x_1, x_2, ..., x_{2m-1}$, this gives

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= -x_2 = -h_2 \frac{\partial h_1}{\partial x_1} + h_1 \frac{\partial h_2}{\partial x_1}, \\ \frac{\partial u}{\partial x_2} &= x_1 = -h_2 \frac{\partial h_1}{\partial x_2} + h_1 \frac{\partial h_2}{\partial x_2}, \\ &\vdots \\ \frac{\partial u}{\partial x_{2m-1}} &= -x_{2m} = -h_2 \frac{\partial h_1}{\partial x_{2m-1}} + h_1 \frac{\partial h_2}{\partial x_{2m-1}}. \end{aligned}$$

Since $x_1, ..., x_n \in H$ we get $H = L$. Therefore u is $Z-$generic element of $L(X)$. The proof of (2), (3) and (4) is the same as that of (1). □

Lemma 7. *Let* l *be a nonzero linear combination of* x *and* y, $L = (x, y)$. *Then:*

1. *If* $[g, l] \in L^{(2)}$ *for a homogeneous element* g *of* L *of degree 4, then* $g = 0$.
2. *If* $[[[x, y], x], l] = [f, l_1]$ *for some homogeneous element* $f \in L$ *of degree 3 and a linear element* l_1 *of* L *then either* $[l, l_1] = 0$ *or* $[l_1, x] = 0$.

Proof. 1) We may assume that $l = x$. The element g can be written in the left normed form

$$g = \alpha_1[x, y, y, y] + \alpha_2[x, y, y, x] + \alpha_3[x, y, x, x]$$

Since

$$[g, l] = g = \alpha_1[x, y, y, y, x] + \alpha_2[x, y, y, x, x] + \alpha_3[x, y, x, x, x] \in L^{(2)}$$

$\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $g = 0$.

2) Suppose that $l = \alpha_1 x + \alpha_2 y$, $l_1 = \beta_1 x + \beta_2 y$, $f = \gamma_1[[x, y], x] + \gamma_2[[x, y], y]$. Then the equality $[x, y, x, l] = [f, l_1]$ is equivalent to the system equation

$$\alpha_1 = \gamma_1\beta_1,\ \alpha_2 = \gamma_1\beta_2,\ \gamma_2\beta_1 = 0,\ \ \gamma_2\ \beta_2 = 0.$$

Therefore the proof is completed. □

Theorem 8. *The element*

$$u = [x, y] + [[x, y], x] + [[[x, y], x], [x, y]]$$

is a $Z-$generic element of the free color Lie superalgebra but at the same time it is not almost primitive element of this algebra.

Proof. The element u belongs to the proper subalgebra of L generated by $[x, y]$ and $[[x, y], x]$. It is clear that u is not a primitive element of this subalgebra. Therefore u is not an almost primitive element of L. Suppose that u is not $Z-$generic element of L. Then there is a proper subalgebra H of L such that $u \in [H, H]$. By H^0 we denote the subalgebra of leading terms of elements of H. Since H is generated by some reduced set if $f \in [H, H]$, then $f^0 \in [H^0, H^0]$ and $u^0 = [[[x, y], x], [x, y]] \in [H^0, H^0]$. Hence u is a linear combination of homogeneous elements of degree 5 of the form $[f, g]$, where $f, g \in H^0$. We may assume that $d(f) > d(g)$. If $d(g) = 1$, then g is a linear element of L. If H^0 contain two linearly independent linear elements then $H = L$. Therefore we may suppose that $u^0 = [f, [x, y]] + [g, l]$, where l is a linear element. Then $[g, l] \in L^{(2)}$. By Lemma 3, $[g, l] = 0$. Hence $f = [[x, y], x]$ and $[[x, y], x], [x, y] \in H^0$ and H contains elements of the form $a = [[x, y], x] + l_1$, $b = [x, y] + l_2$, where l_1 and l_2 are linear elements. Consider the element

$$w = u - [a, b] = [x, y] + [[x, y], x + l_1] - [x, y, x, l_2] \in [H, H]\,.$$

Suppose that $l_2 = 0$. If $x + l_1 \neq 0$, then $w^0 = [[x, y], x + l_1] \in [H^0, H^0]$. It follows that $x + l_1 \in H$. Therefore $w - w^0 0[x, y] \in [H, H]$ and $H = L$. If $x + l_1 = 0$ then $[x, y] \in [H, H]$ and again $H = L$. If $l_2 \neq 0$ then

$$w^0 = -[x, y, x, l_2] \in [H^0, H^0].$$

This is possible only if $[x, y, x, l_2] = [g, l]$ where l is a nonzero linear element of H and g is a homogeneous element of H^0 of degree 3. By Lemma 3, either $l = \alpha l_2$ or $l = \alpha x$. If $l_2 \in H$ then $[x, y] = b - l_2 \in H$. As in the case $l_2 = 0$, we get $H = L$. Let $l = x \in H$. Then we assume that $a = [[x,y],x] + \beta y$, $b = [x,y] + \gamma y \in H$ and $0 \neq \gamma \in K$. Therefore

$$a - [b,x] - \gamma b = (\beta - \gamma^2)y \in H.$$

Since $H \neq L$, $\beta = \gamma^2$ and

$$w = u - [a,b] = [x,y] + [[x,y], x + \gamma^2 y] - \gamma[x,y,x,y] \in [H,H].$$

It follows that $w^0 = -\gamma[x,y,x,y] \in [H^0, H^0]$. The element $[x,y,x,y]$ has the unique such presentation $[x,y,x,y] = [x,y,y,x]$. Therefore $[[x,y],y] \in H^0$ and $c = [[x,y],y] + \delta y \in H$. Consider the element

$$w_1 = w + \gamma[c,x] = (1 - \gamma\delta)[x,y] + [[x,y], x + \gamma^2 y] \in [H,H].$$

We have $w_1^0 = [[x,y], x + \gamma^2 y] \in [H^0, H^0]$. Therefore $x + \gamma^2 y \in H$. Since $\gamma \neq 0$, $H = L$. □

Theorem 9. *If $n > n + k$, $n > 1$ and $k \geq 1$ then*

$$u = [[x,y], x^n, y^k] + [x^m, [x,y], x]$$

is an almost primitive and $Z-$generic element of $L(X)$.

Proof. Let H be a proper subalgebra of L and $\{h_1, ..., h_m\}$ be a reduced set of free generators of the subalgebra H. If $u \in H$ then we get $u^0 = \alpha h_i^0 + f\{h_j^0 : i \neq j\}$. If u^0 contains a linear term then u is an almost primitive element of L. Otherwise we assume that $u^0 = f\{h_j : i \neq j\}$. This means for $v_1, v_2 \in H$ assume that $u^0 = [x^m, [[x,y],x]] = [v_1, v_2]$. Thus $x \in H$. Now we consider the weight function μ given by

$$\mu(x) = 1 \text{ and } \mu(y) = N > m.$$

Therefore we have $\hat{u} = [[x,y]], x^n], y^k]$, where $\hat{u}$ is a polynomial in $\hat{h}_1, ..., \hat{h}_m$. If u is not primitive element of H then $\hat{u} = [v_1', v_2']$ for $v_1', v_2' \in H$. Thus $y \in H$ and $H = L(x,y)$. Hence u is almost primitive element of H. □

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