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UNIQUE FACTORIZATION AND RING EXTENSIONS

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Abstract: In this paper, we define the notion of an essential factor extension of a Noetherian integral domain and obtain a characterization of unique factorization domains in terms of the non existence of "proper essential factor extensions". The inspiration for this comes from the rather unrelated fact that a module is injective if and only if it has no proper essential extensions. Following this, we define a generalization of this notion to subrings of a given ring and consider separately the rings which are essential factor extensions of all those subrings which are 'large enough', in the sense that we consider only those subrings R of S which are such that $S \subseteq K(R)$. Following this, we also obtain the basic properties of essential factor extensions.

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All rings in this paper, unless otherwise mentioned, will be Noetherian integral domains. Further, if R is an integral domain, we will denote its quotient field by $K(R)$ and the group of units of R by $U(R)$.

Our first aim is to characterize UFD's as a class of rings having no "proper essential factor extensions", the term "essential" being chosen purposely in order to emphasize the similarity with the notion of essential extensions of a

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module and the corresponding characterization of an injective module. We define:

Definition. Let R be a Noetherian integral domain and let $K(R)$ denote the field of fractions of R. Then a Noetherian integral domain S such that $R \subseteq S \subseteq K(R)$ is said to be an *essential factor extension* of R if:

(1) Every prime element of R remains prime in S.

(2) For any prime q of S, there exists $u \in U(S)$, i.e. a unit u of S, such that $q = up$ for some prime p of R.

(3) If R is a proper subring of S, there exists an element in S of form $\frac{1}{r}$ for some $r \in R - U(R)$, i.e. some nonunit of R becomes unit in S.

Remarks. The three conditions imposed on an intermediate ring S for it to be an essential factor extension of R serve different purposes; the condition (1) captures the essence of the idea that S is "essential" over R. The condition (2) makes sure that the primes in S are still directly "dependent" on R. The condition (3) is vital in the proof of the main theorem, i.e. Theorem 3.

It is well known that any Noetherian integral domain R is atomic, i.e. any element of R can be expressed as a finite product of irreducibles in R. We now consider the following subsets of R: Let $S_p(R)$ denote the set of elements in R that can be expressed as a product of primes in R and let $S_\pi(R)$ denote the set of *nonunit* elements of R that are not divisible by any prime in R. It is clear that both S_p and S_π are multiplicatively closed.

Theorem 1. *Let R be a Noetherian integral domain. Then each nonunit element x of R can be expressed as a product yz, where $y \in S_p$ and $z \in S_\pi$. The decomposition (y, z) is unique up to associates.*

Proof. Choose an element $x \in R$. For each prime p of R dividing x, take the largest power of p dividing x. Then $x = p^r x_1$, where p does not divide x_1. We proceed thus with a prime dividing x_1. Since R is a Noetherian integral domain, there are only finitely many primes dividing x and hence $\exists$ a positive integer k and distinct primes $p_1, ..., p_k$ such that $x = {p_1}^{r_1} ... {p_k}^{r_k} z$, where z has no prime divisors. If we let $y = {p_1}^{r_1} ... {p_k}^{r_k}$, we see that $x = yz$, where y is a product of primes, i.e. $y \in S_p(R)$ and z has no prime factors, i.e. $z \in S_\pi(R)$.

Now suppose that x has two such representations $y_1 z_1$ and $y_2 z_2$. Then, $y_1 z_1 = y_2 z_2$. Consider any prime p dividing y_1. Then p divides $y_2 z_2$. It cannot divide z_2, since $z_2 \in S_\pi(R)$ and hence must divide y_2, which is a product of primes. Hence, p is associate to some prime dividing y_2. Considering the largest power of p dividing x (and hence y_1), we see that y_1 and y_2 differ by a unit. Thus, z_1 and z_2 must differ by a unit as well. This proves the uniqueness of the decomposition as stated above. □

Corollary 2. *Let R and S_π be as above. Then ${S_\pi}^{-1}R$ is a UFD. Further, since $R \subseteq {S_\pi}^{-1}R \subseteq K(R)$ (where $K(R)$ is the field of fractions of R), ${S_\pi}^{-1}R$ is an essential factor extension of R.*

Proof. Since R is Noetherian, ${S_\pi}^{-1}R$ is Noetherian. Thus, ${S_\pi}^{-1}R$ is an atomic domain. Consider an irreducible element of ${S_\pi}^{-1}R$, say $\frac{a}{s}$, where $s \in S_\pi(R)$. Since s is a unit in ${S_\pi}^{-1}R$, $\frac{a}{1}$ is irreducible in ${S_\pi}^{-1}R$. Then, a must be irreducible in R. Since $a \notin S_\pi R$, a is a prime in R. It is now easy to see that $\frac{a}{1}$ and hence $\frac{a}{s}$ is prime in ${S_\pi}^{-1}R$. Hence, ${S_\pi}^{-1}R$ is a unique factorization domain.

Further, let p be a prime of R. Consider $\frac{p}{1}$ and let $\frac{p}{1} | \frac{a}{s_1}\frac{b}{s_2}$. Then, $\exists\ s_3 \in S_\pi(R)$ such that $p|abs_3$ (in R). Since $s_3 \in S_\pi(R)$, p does not divide s_3. Thus, $p|ab$ and hence $p|a$ or $p|b$(p being prime in R). Thus, $\frac{p}{1}$ is prime in ${S_\pi}^{-1}R$. This verifies condition (1) for ${S_\pi}^{-1}R$ to be an essential factor extension of R. In the first part of the proof, we showed that $\frac{a}{s}$ is a prime in ${S_\pi}^{-1}R$ iff a is prime in R. Condition (2) and (3) are therefore satisfied. Hence, ${S_\pi}^{-1}R$ is an essential factor extension of R. ☐

We have now reached the main result of the paper.

Theorem 3. (Main Result) *Let R be a Noetherian integral domain. Then R has no proper essential factor extensions if and only if R is a unique factorization domain.*

Proof. "Only if ". Suppose that R is not a unique factorization domain. Then, there must exist irreducible(s) in R, which are not prime. In that case, we see that ${S_\pi}^{-1}R$ is a proper essential factor extension of R.

"If part". Conversely, suppose that R is a UFD. Suppose that there exists a ring S which is a proper essential factor extension of R. Then, there exists a nonunit element $x \in R$ such that $\frac{x}{1}$ is a unit in S (using condition (3) in the definition of essential factor extension). Since R is a UFD, there exists a prime p in R such that $p|x$. Since $R \subset S$ is an essential factor extension, p remains a prime in S. Hence p cannot divide x in S, because x is a unit in S. This is a contradiction. ☐

Now suppose that we have an extension of rings $R \subseteq S$, where both R and S are Noetherian integral domains. We want to extend the concept of essential factor extension and with it, the concept of unique factorization domain to this situation.

In this case, we shall restrict ourselves to those (nonunit) irreducibles in R which are units in S. In view of the original definition of essential factor extensions, where R is considered as a subring of its quotient field $K(R)$, it is natural to start out by looking at irreducibles in R that are units in S. This is because in the original case where $S = K(R)$, all (nonzero) elements, and with them, all irreducibles in R become units in S.

Let us define the following terminology:

$Irr_S(R)$: The set of irreducible elements in R that become units in S.

$Fact_S(R)$: The set of elements of R that can be written as a product of the elements of $Irr_S(R)$.

$Fact_{Sp}(R)$: The set of elements of $Fact_S(R)$ that can written as a product of primes of R that are also elements of $Irr_S(R)$.

$Fact_{S\pi}(R)$: The set of elements in $Fact_S(R)$ that are not divisible by any primes of R that occur in $Irr_S(R)$.

We apologize to the reader for this cumbersome notation. Let us say that a Noetherian integral domain T such that $R \subseteq T \subseteq S$ is an *essential factor extension of R in S* if

(1) Every prime of R that becomes a unit in S remains prime in T.

(2) Given a prime p_T of T such that $p_T \in Irr_S(R)$, there exists a unit u in T and a prime p_R of R such that $p_R \in Irr_S(R)$ such that $p_T = up_R$.

(3) If T contains R properly, there is at least one element in $Irr_S(R)$ that is a unit of T.

The first thing, of course, is to note that the previously defined notion of essential factor extension in S coincides with the notion of essential factor extensions in the quotient field when $S = K(R)$. Further, let us say that R is *uniquely factorizable in S* if any factorization of an element of R in terms of irreducibles from $Irr_S(R)$ is unique. It is important to note that by this definition, since $Irr_S(S) = \phi$ for all rings S (irreducibles of S cannot be units in S), a given Noetherian integral domain S is always uniquely factorizable in itself. Thus, the statement "S is uniquely factorizable in S" is always (vacuously) true and therefore *does not imply* that S is a UFD.

Proceeding in an analogous manner, we obtain the following theorem.

Theorem 4. *The following are true for a ring extension $R \subseteq S$, where both R and S are Noetherian integral domains:*

(1) Both $Fact_{Sp}(R)$ and $Fact_{S\pi}(R)$ are multiplicatively closed sets in R and hence the ring $(Fact_{S\pi}(R))^{-1}R$ may be considered as a subring of S.

(2) The ring $(Fact_{S\pi}(R))^{-1}R$ is is an essential factor extension of R in S. Let us denote this ring $(Fact_{S\pi}(R))^{-1}R$ by T. Then, any factorization of an element of T in terms of irreducibles (of T) from $Irr_S(T)$ is unique.

(3) The ring R is uniquely factorizable in S if and only if it has no essential factor extensions in S.

Proof. (1) is obvious.

(2) In a manner analogous to Corollary 2, we can easily show that all primes of R that lie in $Irr_S(R)$ will remain prime in $(Fact_{S\pi}(R))^{-1}R$. Since $(Fact_{S\pi}(R))^{-1}R$ is a localization of R, the properties (2) and (3) for essential factor extensions in S are trivially satisfied. Let us denote $(Fact_{S\pi}(R))^{-1}R$ by T. Since all non prime irreducibles in R become units in T, the elements of $Irr_S(T)$ are all primes of T. Hence, T is uniquely factorizable in S.

(3) Suppose that R is uniquely factorizable in S and yet has a *proper* essential factor extension T in S. Then, there exists an element, say x in $Irr_S(R)$ such that x is a unit in T. Since R is uniquely factorizable in S, x, being in $Irr_S(R)$, must be a prime of R. Thus, x must be a prime of T as well, in which case we have that x is both a prime and a unit in T, a contradiction.

Conversely, suppose that R is *not* uniquely factorizable in S. Then, it is easy to see, from part (2) that $(Fact_{S\pi}(R))^{-1}R$ becomes a proper essential factor extension of R in S. □

We have said before that the term “essential” was chosen to emphasize the similarity of these results with the system of results for essential extensions of modules and injective hulls. The following theorem further illustrates this rather strange analogy, for we shall now consider a concept that would correspond to the concept of a uniform module. A uniform module is that which is an essential extension of all its non zero submodules. “Analogously”, we define the following.

Definition. Let S be a Noetherian integral domain. Consider all Noetherian integral subrings R of S such that $S \subseteq K(R)$.Then, S is *uniform extension ring* if it is an essential factor extension of R in S for every such subring R.

Theorem 5. *Let S be a Noetherian integral domain Then the following are equivalent:*

(1) S is a uniform extension ring.

(2) S does not contain a proper subring R (such that $S \subseteq K(R)$) that is uniquely factorizable in S as defined above.

Proof. (1) ⇒ (2). Assume that S is a uniform extension ring. Suppose that R is a proper subring of S such that $S \subseteq K(R)$ and R is uniquely factorizable in S. Since S is a uniform extension ring, S is an essential factor extension of R in S. This contradicts (3) of Theorem 4. Hence (1) ⇒ (2).

(2) ⇒ (1). Assume (2). Suppose that S is not a uniform extension ring, i.e. there exists a subring R (with the property that $S \subseteq K(R)$) such that S is not an essential factor extension of R in S. Consider the subring $(Fact_{S\pi}(R))^{-1}R$ of S.

This ring is an essential factor extension of R in S and hence $(Fact_{S\pi}(R))^{-1}R$ is a proper subring of S (if S, were equal to $(Fact_{S\pi}(R))^{-1}R$, it would be an essential factor extension of R; which contradicts the assumption). Further, $(Fact_{S\pi}(R))^{-1}R$ is uniquely factorizable in S. This contradicts (2). □

In Theorems 6, 8 and 9 and also in Corollary 7, we have the following setup: R, S and T are Noetherian integral domains such that $R \subseteq S \subseteq T \subseteq K(R)$. Also, S and T are *localizations* of R, i.e. there exist multiplicatively closed sets M_S and M_T in R such that $S = (M_S)^{-1}R$ and $T = (M_T)^{-1}R$. This has the following consequence: *If x is any element of S(resp T), then there exists $r \in R$ such that $r \in U(S)$ (resp $r \in U(T)$) such that $rx \in R$.* Also, this means that if S contains R properly, there must exist a nonunit element of R that becomes a unit in S. *This means that condition (3) for S to be an essential factor extension of R is always satisfied.*

Theorem 6. *Let R, S, T be Noetherian integral domains and such that T is an essential factor extension of R (in $K(R)$). Further, let S and T be localizations of R. Then, if S is such that $R \subseteq S \subseteq T$, then T is also an essential factor extension of S (in $K(T) = K(R)$).*

Proof. Consider a prime p in S. Since $S \subseteq K(R)$, there exists $x \in U(S)$ and $x \in R$ such that $px \in R$. Since x is a unit of S, px is a prime in S and hence in R as well. Since T is an essential factor extension of R, px is a prime of T. Since x is a unit of S, it is a unit of T as well. Since px is a prime of T and x is a unit of T, p is a prime of T. This verifies condition (1) for T to be an essential factor extension of S. Now, let p_T be a prime of T. Then, there exists a prime p_R of R and a unit $u \in U(T)$ such that $p_T = up_R$. Since, p_R is a prime in R, it is a prime in T and hence in S. Thus, $p_T = up_R$, $u \in U(T)$ and p_R a prime of S. Hence, T is an essential factor extension of S. This verifies the second condition for T to be an essential factor extension of S. To prove the third, we note that since R is a subring of S and T is a localization of R containing S, T is also a localization of S. The fact that S contains a nonunit element that becomes a unit in T, is now trivial. □

Corollary 7. *Let R be a Noetherian integral domain and let $T \supset R$ be an essential factor extension of R (in $K(R)$). Let S be a Noetherian integral domain such that $R \subseteq S \subset T$. Further, let S and T be localizations of R. Then S cannot be a unique fatorization domain.*

Proof. The previous theorem shows that T is a proper essential factor extension of S as well, which means that S cannot be a unique factorization domain. □

Hence, in some sense, ${S_\pi}^{-1}R$ (notation as defined before) is a minimal unique factorization domain containing R (in $K(R)$).

Theorem 8. *Let $R \subseteq S \subseteq T \subseteq K(R)$ be Noetherian integral domains. Further, let S and T be localizations of R. Then, if T is an essential factor extension of R, so is S.*

Proof. Let p be a prime of R. Then, p is a prime in T, it must be a prime in S(verifies the first condition for S to be an essential factor extension of R). Further, let p_S be a prime of S. Then, since S is a localization of R, there exists a unit u_S of S such that $p_S u_S \in R$. Then, $p_S u_S$ being associate to a prime of S, is itself a prime of S and hence is a prime of R. Then, $p_S = {u_S}^{-1}(p_S u_S)$. Since $p_S u_S$ is a prime of R, we have proved the second condition for S to be an essential factor extension. Since S has been said to be a localization of R, the third condition for S to be an essential factor extension holds trivially. □

Theorem 9. *Let $R \subseteq S \subseteq T \subseteq K(R)$ be Noetherian integral domains. Further, let S and T be localizations of R. If T is an essential factor extension of S and S is a essential factor extension of R, then T is an essential factor extension of R.*

Proof. Let p be a prime of R. Then, since S is an essential factor extension of R and T is an essential factor extension of S, we see that p is prime in S and hence in T. Now, consider a prime p_T of T. Then, $\exists\ u_T \in U(T)$ and a prime p_S of S such that $p_T = u_T p_S$. Again, there exists $u_S \in U(S)$ and a prime p_R of R such that $p_S = u_S p_R$. Thus, $p_T = u_T u_S p_R$. Since $u_T u_S \in U(T)$, we are through. □

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