International Journal of Pure and Applied Mathematics
Volume 26 No. 4 2006, 465-478

A PSEUDODIFFERENTIAL REGULARIZATION METHOD

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Abstract: In this paper a new regularization and renormalization method for ultraviolet infinities appearing in quantum field theory is presented. This procedure depends on a family of elliptic pseudodifferential operators. It is also shown that for a particular family of such operators the dimensional renormalization result can be recovered.

AMS Subject Classification: 81T99, 35S05
Key Words: dimensional regularization method, pseudodifferential operators, complex power methods

1. Introduction

In the treatment of the ultraviolet divergent integrals appearing in various fundamental Feynman diagrams, many regularization and renormalization procedures have been developed. We stand for regularization a procedure that expresses divergent integrals as limits of convergent ones depending on a parameter. In order to eliminate the infinity part of such limits, the counterterms of the same order have to be added, this process is known as renormalization. Different authors develop these techniques, see for instance, [1], [4], [6], [9], [10].

Among other regularization techniques, the dimensional regularization me-

Received: November 7, 2005

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thod has demonstrated to be a powerful tool. The basic idea of this method is to introduce in the given divergent integral a complex parameter D as the dimension of the integration domain obtaining an analytic function of this parameter in some complex domain. More details will be given in Definition 2.7.

In the next section, we state a regularization method by mean of a perturbation of the integrand with the symbol of an analytic family of elliptic pseudodifferential operators. The parameter will essentially be the order of such operators. Later, we shall connect this method with the dimensional regularization one. The last section is devoted to the consideration of some known examples appearing in the study of some Feynmann diagrams in order to see how this method works.

2. Pseudodifferential Regularization and Renormalization

2.1. General Considerations

First, in order to give a self-contained explanation, we recall some basic concepts and properties of pseudodifferential operators. For more details see for instance [8].

In this work we shall consider pseudodifferential operators T of order $m \in \mathbb{C}$ defined on $C_0^\infty(X)$, the space of the smooth functions over an open set $X \subset \mathbb{R}^n$ with compact support. It is given by

$$Tu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_T(x,\xi)\widehat{u}(\xi)d\xi,$$

where

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx$$

is the Fourier transform of u.

The symbol $\sigma_T(x,\xi)$ of the operator T belongs to the Hörmander class $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$. Let us recall that the Hörmander class $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ is defined by all the smooth functions $a(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{\mathrm{Re}\, m - \rho|\alpha| + \delta|\beta|}, \quad (x,\xi) \in X \times \mathbb{R}^n,$$

with $0 \leq \delta < \rho \leq 1$. As usual, for $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$ and $\partial_y^\alpha = \partial_{y_1}^{\alpha_1}\partial_{y_2}^{\alpha_2} \ldots \partial_{y_n}^{\alpha_n}$, for $y = (y_1, y_2, \ldots y_n)$.

The Schwartz kernel K_T of T, defined by $\langle K_T, u \times v\rangle = \langle Tu, v\rangle$ in the distributional sense, is locally given by

$$K_T(x, x-u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi.u}\, \sigma_T(x,\xi)\, d\xi, \qquad u \neq 0. \tag{1}$$

Actually, $K_T(x, x-u) = \mathcal{F}_\xi^{-1}\left(\sigma_T(x,\xi)\right)(u)$, where $\mathcal{F}_\xi^{-1}$ is the inverse Fourier transformation in the variable ξ.

Among all the well known properties of the pseudodifferential operators, the one it is going to be recalled is how to define the symbol of the composition between two operators of this kind.

Given T and S with symbols $\sigma_T(x,\xi)$ and $\sigma_S(x,\xi)$ respectively, the symbol of the composition operator $T \circ S$ is defined by

$$\sigma_{T\circ S}(x,\xi) = \sum_{|\alpha|\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_T(x,\xi) \partial_x^\alpha \sigma_S(x,\xi). \tag{2}$$

In this article we only consider classical pseudodifferential operators. T will be called a *classical* pseudodifferential operator if its symbol $\sigma_T(x,\xi)$ has an asymptotic expansion in homogeneous functions $\sum_{j=0}^{\infty} \varphi(\xi)\, \sigma_{m-j}(x,\xi)$. It means that

$$\sigma_T(x,\xi) = \sum_{j=0}^{N} \varphi(\xi)\, \sigma_{m-j}(x,\xi) + r_N(x,\xi), \quad r_N(x,\xi) \in S_{1,0}^{m-N-1} \tag{3}$$

for each $N \in \mathbb{N}$, where $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 0$ in a neighborhood of $\xi = 0$, $\varphi(\xi) = 1$ for $|\xi| \geq 1$. The functions $\sigma_{m-j}(x,\xi)$ are positive homogeneous of degree $m-j$ in ξ, that is

$$\sigma_{m-j}(x, t\xi) = t^{m-j}\, \sigma_{m-j}(x,\xi),$$

for all $t > 0$ and $|\xi| \neq 0$. This fact will be denoted by

$$\sigma_T(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x,\xi).$$

According to (1) and (3), for an integer N large enough, the Schwartz kernel of a classical pseudodifferential operator can be written as

$$K_T(x, x-u) = \sum_{j=0}^{N} K_{-m-n+j}(x,u) + R_N(x,u), \tag{4}$$

where

$$K_{-m-n+j}(x,u) = \mathcal{F}_{\xi}^{-1}\left(\sigma_{m-j}(x,\xi)\right)(u) \tag{5}$$

and

$$R_N(x,u) = \mathcal{F}_{\xi}^{-1}\left(\tilde{r}_N(x,\xi)\right)(u), \tag{6}$$

with $\tilde{r}_N(x,\xi) = \sigma_T(x,\xi) - \sum_{j=0}^{N} \sigma_{m-j}(x,\xi)$.

Notice that the distributions $K_{-m-n+j}(x,u)$ are homogeneous in $u \neq 0$ of degree $-n-m+j$ since they are the inverse Fourier transforms of homogeneous functions of degree $m-j$.

It is well known that when the operator T is good enough, the value of K_T at the diagonal can be computed evaluating $K_T(x, x-u)$ at $u=0$. For instance, that is the case when $\mathrm{Re}\,(m) < -n$. Nevertheless, in general, this is not possible because it would lead to compute a divergent integral. When the order m of the operator T belongs to $\mathbb{C} \setminus \mathbb{Z}$, it is possible to assign a value at the diagonal to the kernel of T by means of the definition of the canonical trace density given in Kontsevich and Vishik (see [5]). This extended value is given by

$$K_T(x,x) := R_N(x,0), \tag{7}$$

if $N \in \mathbb{N}$ is sufficiently large. It is enough to take N greater or equal than $n + [\mathrm{Re}\,(m)]$, where $[x]$ denotes the greater entire less or equal than x. Let us notice that the value of $R_N(x,0)$ does not change under a shift $N \to N+k$, $k \in \mathbb{N}$. Obviously, when the operator is good enough, this definition agrees with the value of its kernel at the diagonal.

The spirit of this extension is to profit from the homogeneity in u of the terms $K_{-m-n+j}(x,u)$. Due to this homogeneity, these terms have null or infinite value at $u=0$. So (7) means that we trough out the infinite terms in the expansion (4) of the kernel.

2.2. Pseudodifferential Method for Regularization and Renormalization

As it was mentioned in the introduction, we are specially interested in the regularization of ultraviolet divergent integrals appearing when we deal with Feynman diagrams. Usually, these integrals are defined in a Minkowski space which, after a change of variables, can be expressed in an Euclidean space; see for instance [9]. In many situations these integrals can be written as

$$I(f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi)\, d\xi, \tag{8}$$

where $f \in S^m_{1,0}$ can be thought as a classical symbol of a pseudodifferential operator A. Consequently, we assume that

$$Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\widehat{u}(\xi) d\xi, \tag{9}$$

where $f(\xi) = \sigma_A(\xi) \sim \sum_{k\geq 0} a_{m-k}(\xi)$. Let us note that the symbols we will deal with only depend on ξ. Now, by (1), the Schwartz kernel of the operator A is

$$K_A(x, x-u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi . u}\, d\xi, \qquad u \neq 0.$$

From now on, taking into account that this kernel is independent of x, it will be denoted $K_A(u)$.

For integrable functions f, for instance $f \in S^m_{1,0}, \mathrm{Re}\,(m) < -n$, this kernel can be computed at $u = 0$ and its value is

$$K_A(0) = I(f).$$

Obviously, under the hypotheses on f, this kernel takes the same value at each point of the diagonal.

On the other hand, if it is possible to extend the kernel of the operator A at the diagonal, given by a value which is not depending on x, we can assign this value to the integral $I(f)$. For instance, in the case of $f \in S^m_{1,0}$ with the parameter $m \in \mathbb{C} \setminus \mathbb{Z}$, by (4), (5), (6), the kernel of A has the following expansion

$$K_A(u) = \sum_{j=0}^{N} K_{-m-n+j}(u) + R_N(u) \tag{10}$$

and then, by (7), it is possible to define

$$I(f) := R_N(0). \tag{11}$$

It is no possible to apply this definition when $f \in S^m_{1,0}$ with $m \in \mathbb{Z}$. Indeed, the different regularization and renormalization methods deal with this problem.

Thinking the integral (8) as the value at the diagonal of a kernel of a pseudodifferential operator allows to take advantage of the theory of this kind of operators. The procedure we next develop is based in a one parameter family of pseudodifferential operators.

Let us consider a holomorphic family $\mathcal{B} = \{B(s)\}_{s\in\Omega}$ of classical elliptic pseudodifferential operators of order $d(s)$ such that $0 \in \Omega \subset \mathbb{C}$, $B(0) = I$ is the identity operator and $d(s)$ is a holomorphic function on Ω with $d'(0) \neq 0$. Assume that their symbols do not depend on x and have the asymptotic expansion

$$\sigma_{B(s)}(\xi) \sim \sum_{j\geq 0} b_{d(s)-j}(\xi),$$

where $b_{d(s)-j}(\xi) \in S^{d(s)-j}_{1,0}$ are homogeneous functions of degree $d(s)-j$ in $|\xi| \geq 1$.

Notice that we take a family of symbols not depending on x because our final purpose is to giving meaning to the integral defined by (8) which is independent of x.

It is well known that their kernels $K_{B(s)}(u)$ are holomorphic functions of $s \in \Omega$ and has an entire extension for fixed u, $u \neq 0$ (see [5], [8]). For $u = 0$ it can be extended meromorphically to the whole complex plane with at most simple poles at $s \in \mathbb{C}$ such that $d(s) \in \mathbb{Z}$, see footnote[1].

Now, for A defined by (9), we specifically take the holomorphic family $B(s) \circ A$, with $s \in \Omega$. Following the composition rule (2) we have

$$\sigma_{B(s)\circ A}(\xi) \sim \sum_{h\geq 0} c_{d(s)+m-h}(\xi),$$

where $c_{d(s)+m-h}(\xi) = \sum_{j+k=h} b_{d(s)-j}(\xi)\, a_{m-k}(\xi)$ are homogeneous functions of degree $d(s)+m-h$ in $|\xi| \geq 1$.

Due to (4), (5) and (6) we have for $u \neq 0$

$$K_{B(s)\circ A}(u) = \sum_{h=0}^{N} K_{-d(s)-m-n+h}(u) + R_N(u,s),$$

$$K_{-d(s)-m-n+h}(u) = \mathcal{F}^{-1}_{\xi}\left(c_{d(s)+m-h}(\xi)\right)(u)$$

[1]These properties of the kernel and its extension at the diagonal given in (7) have demonstrated to be a powerful tool in pseudodifferential operators techniques. For instance, in [3] we can find an application of this extension to another kind of situations.

and

$$R_N(u,s) = \mathcal{F}_\xi^{-1}\left(\tilde{r}_N(\xi,s)\right)(u),$$

with $\tilde{r}_N(\xi,s) = \sigma_{B(s)\circ A}(\xi) - \sum_{j=0}^{N} \tilde{c}_{d(s)+m-j}(\xi)$, where $\tilde{c}_{d(s)+m-j}(\xi)$ is the homogeneous extension up to the origin of $c_{d(s)+m-j}(\xi)$.

By (7), for each $s \in \Omega$ such that $d(s)+m \notin \mathbb{Z}$, the punctual value of the extension of the kernel of $B(s)\circ A$ at the diagonal was defined as

$$K_{B(s)\circ A}(0) = R_N(0,s), \tag{12}$$

where $N = N(s) \in \mathbb{N}$ is greater or equal than $n + [\mathrm{Re}\,(d(s))] + m$.

Remark 2.1. In spite of the existence of $N = N(s)$ for each $s \in \Omega$ such that $d(s)+m \notin \mathbb{Z}$, it is possible to find N uniformly for $s \in \tilde{\Omega}$ where $\tilde{\Omega}$ is an open subset of Ω. After the meromorphic extension of the kernel is obtained, we are allowed to choose a particular open subset $S \subset \mathbb{C}$ in order to make the computation in an easy way. Usually, S is an open strip such that $\Omega \cap S \neq \emptyset$ and we take $\tilde{\Omega} = \Omega \cap S$.

Now, we are in conditions to define the pseudodifferential regularization and renormalization procedures.

Definition 2.2. Let us take f, A and $\mathcal{B} = \{B(s)\}_{s\in\Omega}$ under the hypotheses of this section, then the $\mathcal{B}$ - *pseudodifferential regularization* assigns to the integral

$$I(f) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)\, d\xi$$

the family of s-values

$$I_{B(s)}(f) := K_{B(s)\circ A}(0) \tag{13}$$

for each $s \in \Omega$.

Remark 2.3. As we have recalled, the function $I_{B(s)}(f)$ can be extended meromorphically to the whole complex plane with at most simple poles at $s \in \mathbb{C}$ such that $\mathrm{Re}\,(d(s)) + m \in \mathbb{Z}$ and $\mathrm{Re}\,(d(s)) + m \geq -n$.

Definition 2.4. For a given family $\{I_{B(s)}(f)\}_{s\in\Omega}$ as in (13), $\mathcal{B}$-*renormalization* assigns to the integral $I(f)$ the numerical value

$$\mathrm{Ren}_{\mathcal{B}}\, I(f) := \underset{s\to 0}{\mathrm{FP}}\, I_{B(s)}(f),$$

where the symbol $\underset{s\to 0}{\mathrm{FP}}\, I_{B(s)}(f)$ denotes the finite part of the function $I_{B(s)}(f)$.

Remark 2.5. The *finite part* of a function $g(z)$ at $z = z_0$ is usually understood as the limit for $z \to z_0$ of $g(z)$ minus the counterterms corresponding to the pole of $g(z)$ at z_0. Particularly, here we deal with meromorphic functions with simple poles, then if

$$g(z) = a(z - z_0)^{-1} + b + c(z - z_0) + d(z - z_0)^2 + \dots$$

in a neighborhood of z_0,

$$\operatorname*{FP}_{z \to z_0} g(z) := \lim_{z \to z_0} \left(g(z) - \frac{a}{z - z_0} \right) = \frac{d}{dz}((z - z_0)g(z)) \Bigg|_{z=z_0} = b.$$

Remark 2.6. Notice that if the original integral $I(f)$ is convergent, $I_{B(s)}(f)$ is analytic at $s = 0$. Then $I_{B(s)}(f)\big|_{s=0} = I(f) = \mathrm{Ren}_{\mathcal{B}}\, I(f)$.

2.3. Connection to Dimensional Regularization Method

Now, we would like to make evident the relationship between pseudodifferential regularization and dimensional regularization procedures in $\mathbb{R}^4$. To this issue, we first recall the dimensional regularization method. As it was mentioned in the introduction, it gives a parametric expression of the integral

$$I(f) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} f(\xi) d\xi,$$

for a radial function f.

This method introduces, in a naive way, the complex parameter D considering this integral in D dimensions; following [9] the complex parametric integral is taken as

$$I_D(f) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} f(\xi) d\xi.$$

The idea is to take polar coordinates in an "Euclidean space with complex dimension", then a formula for the area of the unit sphere $|S^{D-1}|$ in $\mathbb{R}^D$ is needed. So, remember that for $N \in \mathbb{N}$ the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ is given by $|S^{N-1}| = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$, where Γ is the usual Euler function. Then, considering $|S^{D-1}| = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$, and taking formal polar coordinates the method can be defined as follows.

Definition 2.7. (see [9]) For $f(\xi)$ a radial function in $\mathbb{R}^4$, dimensional regularization assigns to the integral

$$I(f) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} f(\xi) d\xi$$

the family of D-values

$$I_D(f) := \frac{1}{2^{D-1}\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})} \int_0^\infty r^{D-1} f(r) dr, \tag{14}$$

where D is a complex parameter.

The dimensional renormalization assigns to $I(f)$ the numerical value

$$\operatorname{Ren} I(f) := \operatorname*{FP}_{D\to 4} I_D(f).$$

Remark 2.8. As it was noticed in [9], $I_D(f)$ is a Mellin transform. Then, it is an analytic function of D for a wide class of functions f in an appropriate domain.

Proposition 2.9. *Let $f(\xi)$ be a radial function on $\mathbb{R}^4$. Suppose that $f(\xi)$ is the symbol of a classical pseudodifferential operator A of order $m < 0$. Then, for the holomorphic pseudodifferential operators family $\mathcal{B} = \{B(s)\}_{s\in\Omega}$ given by*

$$B(s) = C(s)(-\Delta)^s \quad \text{with} \quad C(s) = \frac{(4\pi)^{-s}}{\Gamma(2+s)} \tag{15}$$

and Δ is the Laplacian operator, the pseudodifferencial regularization gives the parametric family defined by (14). That is,

$$I_{B(s)}(f) = I_D(f),$$

where the complex dimensional parameter is $D = 4 + 2s$.

Proof. For each $s \in \Omega$, the symbol of $B(s)$ is given by $\sigma_{B(s)}(\xi) = C(s)|\xi|^{2s}$, then as it was defined the $\mathcal{B}$-pseudodifferential regularization method in (13) it results

$$\begin{aligned} I_{B(s)}(f) = K_{B(s)\circ A}(0) &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \sigma_{B(s)}(\xi) f(\xi) d\xi \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} C(s)|\xi|^{2s} f(\xi) d\xi. \end{aligned} \tag{16}$$

Now, taking polar coordinates (remark that it can be done by choosing an appropriate Ω) and by the definition of $C(s)$ in (15), we have

$$I_{B(s)}(f) = \frac{|S^3|}{(2\pi)^4} \int_0^\infty \frac{(4\pi)^{-s}}{\Gamma(2+s)} r^{2s+3} f(r) dr = \frac{2\pi^2}{(2\pi)^4}$$
$$\times \int_0^\infty \frac{(4\pi)^{-s}}{\Gamma(2+s)} r^{2s+3} f(r) dr = \frac{1}{2^{3+2s}\pi^{2+s}\Gamma(2+s)} \int_0^\infty r^{2s+3} f(r) dr. \quad (17)$$

Introducing the change of parameter, $D = 4 + 2s$, it can be written

$$I_{B(s)}(f) = \frac{1}{2^{D-1}\pi^{\frac{D}{2}}\Gamma(\frac{D}{2})} \int_0^\infty r^{D-1} f(r) dr = I_D(f),$$

where $I_D(f)$ was defined by (14). □

Corollary 2.10. *The renormalized values obtained from the dimensional and the pseudodifferential regularization methods are related by*

$$\operatorname*{FP}_{D\to 4} I_D(f) = \operatorname*{FP}_{s\to 0} I_{B(s)}(f).$$

3. Examples

In this section we consider two well known examples of ultraviolet divergent integrals. In both integrals we apply the pseudodifferential regularization method showing some technical features concerning to the domain Ω of the family $B(s)$ and the calculation of $K_{B(s)\circ A}(0)$ defined by (12). As $B(s)$ we will consider the family of operators defined in the previous proposition.

Proposition 3.1. *Let a be a non negative number. Given the following generic Feynman integrals in Euclidean space*

$$I = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{|\xi|^2 + a} d\xi, \qquad J = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{1}{(|\xi|^2 + a)^2} d\xi,$$

the values of its dimensional regularization and renormalization can be respectively stated as follows:

1. $I_{B(s)}(f) = \frac{|S^3|}{2(2\pi^4)} C(s) a^{s+1} \left[\frac{1}{s} + \sum_{j=1}^{\infty} \frac{(-1)^j}{s+j} + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j-s} \right],$

$\mathrm{Ren}_{\mathcal{B}}\, I(f) = \frac{a}{16\pi^2} \left[-\ln\left(\frac{4\pi}{a}\right) + \gamma - 1 \right]$, *where* $f(\xi) = (|\xi|^2 + a)^{-1}$.

2. $J_{B(s)}(f) = \frac{|S^3|}{2(2\pi^4)} C(s)(s+1) a^s \left[-\frac{1}{s} - \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j-s} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{s+j} \right],$

$$\operatorname{Ren}_{\mathcal{B}} J(f) = \frac{1}{16\pi^2}\left[-\gamma + \ln\left(\frac{4\pi}{a}\right)\right], \textit{ where } f(\xi) = (|\xi|^2 + a)^{-2}.$$

Here γ *is the Euler's constant,* $C(s) = \dfrac{(4\pi)^{-s}}{\Gamma(2+s)}$ *and the operators* $B(s) = C(s)(-\Delta)^s$.

Proof. 1. By taking into account Definition 2.2, we need to expand $f(\xi) = (|\xi|^2 + a)^{-1}$ in homogeneous functions in $\xi \neq 0$. Then, for $K \in \mathbb{N}$, we have

$$f(\xi) = \sum_{k=0}^{K} (-1)^k a^k |\xi|^{-2-2k} + r_{2K}(\xi),$$

where $r_{2K}(\xi) = (-1)^{K+1} a^{K+1} |\xi|^{-2(K+1)} \left(|\xi|^2 + a\right)^{-1}$.

Taking A the pseudodifferential operator with classical symbol $\sigma_A(\xi) = f(\xi)$, the symbol of $B(s) \circ A$ for $\xi \neq 0$ is

$$\sigma_{B(s)\circ A}(\xi) = \sum_{k=0}^{K} (-1)^k a^k C(s) |\xi|^{2(s-1-k)} + \tilde{r}_{2K}(\xi, s),$$

where

$$\tilde{r}_{2K}(\xi, s) = (-1)^{K+1} a^{K+1} C(s) |\xi|^{2(s-K-1)} \left(|\xi|^2 + a\right)^{-1}.$$

As it was mentioned in Section 2, the kernels of each operator in the holomorphic family $\{B(s) \circ A\}_{s\in\Omega}$ have meromorphic extension at the diagonal with at most simple poles in s such that $2s - 2 \in \mathbb{Z}$. Therefore, so as to compute the kernel we can choose $\Omega = \left\{s \in \mathbb{C} : l < \operatorname{Re}(s) < l + \frac{1}{2}\right\}$ or $\Omega = \left\{s \in \mathbb{C} : l + \frac{1}{2} < \operatorname{Re}(s) < l + 1\right\}$, for $l \in \mathbb{Z}$. For technical reasons it is convenient to take $\tilde{\Omega} = \left\{s \in \mathbb{C} : 0 < \operatorname{Re}(s) < \frac{1}{2}\right\}$, and then $K = 1$, see footnote[2]. Notice that this value of K is uniform for s in such domain, so that $R_2(0, s)$ is the meromorphic extension in the whole complex s-plane of the kernel of $B(s) \circ A$ at the diagonal.

After taking polar coordinates and series expansion, we arrive to

$$\begin{aligned} I_{B(s)}(f) &= R_2(0, s) = \mathcal{F}_\xi^{-1}\left(\tilde{r}_2(\xi, s)\right)(0) \\ &= \frac{|S^3|}{2(2\pi)^4} C(s) a^{s+1} \left[\frac{1}{s} + \sum_{j=1}^{\infty} \frac{(-1)^j}{s+j} + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j-s}\right]. \end{aligned}$$

Now, in order to study the value of the finite part of $I_{B(s)}(f)$ we must find its meromorphic expansion near $s = 0$. First, taking into account the definition

[2] According to (7), for other choices of Ω it is necessary to take a greater value of K.

of $C(s)$ and that $1/\Gamma(s) = s + \gamma s^2 + O(s^3)$, $\Gamma(s+2) = (s+1)s\Gamma(s)$ and $a^s(4\pi)^{-s} = 1 - s\ln(4\pi a^{-1}) + O(s^2)$, we obtain

$$I_{B(s)}(f) = \frac{a\pi^2}{(2\pi)^4}\left[\frac{1}{s} + \left(-\ln\left(\frac{4\pi}{a}\right) + \gamma - 1\right)\right] + O(s).$$

Thus, by Definition 2.4 we have,

$$\mathrm{Ren}_{\mathcal{B}}\, I(f) = \lim_{s\to 0}\left(I_{B(s)}(f) - \frac{a\pi^2}{s(2\pi)^4}\right) = \frac{a}{16\pi^2}\left[-\ln\left(\frac{4\pi}{a}\right) + \gamma - 1\right],$$

arriving to the announced result.

2. Let us now consider $f(\xi) = (|\xi|^2 + a)^{-2}$, then we write the expansion

$$f(\xi) = \sum_{k=0}^{K}(-1)^k(k+1)a^k|\xi|^{-4-2k} + r_{2K}(\xi),$$

where

$$r_{2K}(\xi) = (-1)^{K+1}|\xi|^{-4}\frac{d}{dx}\left[\frac{x^{K+2}}{1+x}\right]\Bigg|_{x=\frac{a}{|\xi|^2}}.$$

Thus, it results that

$$\sigma_{B(s)\circ A}(\xi) = \sum_{k=0}^{K}(-1)^k(k+1)a^kC(s)|\xi|^{2(s-2-k)} + \tilde{r}_{2K}(\xi, s),$$

with

$$\tilde{r}_{2K}(\xi, s) = (-1)^{K+1}C(s)|\xi|^{2s-4}\frac{d}{dx}\left[\frac{x^{K+2}}{1+x}\right]\Bigg|_{x=\frac{a}{|\xi|^2}}.$$

As in the previous example, the operators $B(s)\circ A$ are a holomorphic family whose kernels have meromorphic extension with, at most, simple poles in s such that $2s \in \mathbb{Z}$, hence we are allowed to take $s \in \mathbb{C}$ in the strip $1 < \mathrm{Re}\,(s) < \frac{3}{2}$. Therefore we can choose $K = 1$, then taking polar coordinates and integrating by parts we obtain

$$\begin{aligned} J_{B(s)}(f) &= \mathcal{K}_{B(s)\circ A}(0) = \mathcal{F}_{\xi}^{-1}\left(\tilde{r}_2(\xi, s)\right)(0) \\ &= -\frac{|S^3|}{2(2\pi)^4}C(s)(s+1)a^s\left[\frac{1}{s} + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{s+j} - \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j-s}\right], \end{aligned}$$

and its meromorphic expansion near $s = 0$ is

$$J_{B(s)}(f) = -\frac{1}{16\pi^2} C(s)(s+1)\frac{a^s}{s} + O(s).$$

Then,

$$\begin{aligned} \operatorname{Ren}_{\mathcal{B}} J(f) &= \operatorname*{FP}_{s\to 0} \left\{ -\frac{1}{16\pi^2} C(s)(s+1)\frac{a^s}{s} \right\} \\ &= \frac{1}{16\pi^2}\left[-\gamma + \ln\left(\frac{4\pi}{a}\right)\right], \end{aligned}$$

as we wanted to see. □

Remark 3.2. Let us remark the profit of the examples given above. In the frame of this paper, the functions of Feynman type (see for instance, [2], [7]) can be defined as

$$f(\xi) = \frac{P(\xi)}{\prod_{j=1}^{L}(l_j(\xi)^2 + m_j^2)}, \tag{18}$$

where $P(\xi)$ is a polynomial function of degree p , $l_j(\xi)$ are lineal functions in ξ for $1 \le j \le L$ and masses $m_j > 0$.

Notice that the application of the regularization methods makes sense when $-4 \le p - 2L < 0$ since if $p - 2L < -4$ the integral

$$I(f) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} f(\xi)d\xi,$$

is absolutely convergent and if $p - 2L \ge 0$ the regularization has no interest.

The functions defined by (18) can be written as a sum of functions that are essentially of the same kind that those developed in the previous examples, then their integrals can be regularized as a combination of the results obtained in Proposition 3.1.

It is also feasible to mention that, with the same technical approach, this type of computations can be carry out for Feynman's functions on an Euclidean space of arbitrary dimension.

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