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COMPLEX CONJUGATE DIVISORS ON REAL HYPERELLIPTIC CURVES

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Abstract: Here we study divisor of the form $A + \bar{A}$ on special linear systems on a real hyperelliptic curve.

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1. Real Hyperelliptic Curves

Let X be a smooth proper geometrically integral curve over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \emptyset$. Let g be the genus of X. J.P. Monnier studied in [3] the following invariant $M(X)$ of the real curve X. Let $M(X)$ be the smallest integer $m \geq 1$ such that any real divisor D of degree $2m$ such that the degree of D on each connected component of $X(\mathbb{R})$ is even is linearly equivalent to a totally non-real effective divisor. By [3], Section 2, $M(X)$ is a well-defined integer, $M(X) = 1$ if $g = 0, 1$, and $g \leq M(X) \leq 2g$ if $g \geq 2$. From now on we will assume $g \geq 2$. The example given in [3], Proposition 2.6, to show that $M(X) > g - 1$ is associated to a non-spanned real line bundle. It is very natural to consider the same problem for spanned linear line bundles. It is very natural to consider the same topic for a fixed, but important, line bundle, e.g for ω_X. Hence we introduce the following notation. Let $^-$ denote the complex conjugation. Let L be a real line bundle on X. Let $M(L)$ (resp. $\tilde{M}(L)$) denote the subset of the linear system $|L|$ formed by all divisors $A + \bar{A}$ for some effective divisor A (resp. with the additional condition that the support of A is disjoint from $X(\mathbb{R})$. If X is realand

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$X(\mathbb{R}) \neq \emptyset$, then ω_X is defined over $\mathbb{R}$ and the degree of every real canonical divisor is even on each irreducible component of $X(\mathbb{R})$ ([1], Corollary 4.3, or [2], Proposition 2.1).

Here we consider the case of hyperelliptic curves. Recall that if X is hyperelliptic and $X(\mathbb{R}) \neq \emptyset$, then the g^1_2 on X is defined over $\mathbb{R}$ ([1], Corollary 4.3, or [2], Proposition 2.1) and that for every spanned line bundle L on X such that $h^1(X, L) \neq 0$, there is an integer k such that $1 \leq k \leq g-1$, $\deg(L) = 2k$ and $L \cong kg^1_2$. Furthermore, $\omega_X \cong (g-1)g^1_2$.

Remark 1. Let L be a real line bundle. Since $A + \bar{A} \in |L^{\otimes 2}|$ for all $A \in |L|$, we have $M(L^{\otimes 2}) \neq \emptyset$. Now assume L spanned. Since there is $A \in |L|$ whose support is contained in $X(\mathbb{C})\backslash X(\mathbb{R})$, we have $\tilde{M}(L^{\otimes 2}) \neq \emptyset$. It is easy (for $g \geq 2$) to find real non-spanned L such that $M(L^{\otimes 2}) = \emptyset$. For instance, take $L = \mathcal{O}_X(P)$ for a general $P \in X(\mathbb{R})$.

Theorem 1. *Let X be a real hyperelliptic curve of genus $g \geq 2$ such that $X(\mathbb{R}) \neq \emptyset$. The following conditions are equivalent:*

(i) $M(\omega_X) = \emptyset$.

(ii) $\tilde{M}(\omega_X) = \emptyset$.

(iii) *g is even and X has no real Weierstrass point.*

Theorem 2. *Let X be a real hyperelliptic curve of genus $g \geq 2$ such that $X(\mathbb{R}) \neq \emptyset$. Fix an integer k such that $1 \leq k \leq g$. The following conditions are equivalent:*

(i) $M(kg^1_2) = \emptyset$.

(ii) $\tilde{M}(kg^1_2)\emptyset$.

(iii) *k is odd and X has no real Weierstrass point.*

Proposition 1. *Let X be a real hyperelliptic curve of genus $g \geq 2$ such that $X(\mathbb{R}) \neq \emptyset$. The following conditions are equivalent:*

(i) $M(g^1_2) = \emptyset$;

(ii) $\tilde{M}(g^1_2) = \emptyset$;

(iii) *X has no real Weierstrass point.*

Proof. Let $f : X \to \mathbb{P}^1$ be the degree two morphism associated to the linear system $|g^1_2|$. $M(g^1_2) = \emptyset$ if and only if there is no $P \in \mathbb{P}^1(\mathbb{R})$ such that $f^{-1}(P) = Q + \bar{Q}$ for some $Q \in X(\mathbb{C})$. Since $X(\mathbb{R}) \neq \emptyset$, X has no real Weierstrass point if it has a Weierstrass equation $y^2 = P(x)$ with $P(x)$ real degree $2g+2$ polynomial with no real root. □

Proof of Theorem 1. By Remark 1 $\tilde{M}(\omega_X) \neq \emptyset$ if $g-1$ is even. Use Proposition 1 and that every divisor of ω_X is the sum of $g-1$ divisors of g^1_2. □

Proof of Theorem 2. By Remark 1 $\tilde{M}(kg^1_2) \neq \emptyset$ if k is even. Use Proposition 1 and that if $k \leq g$ every divisor of kg^1_2 is the sum of k divisors of g^1_2. □

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