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l^q-VALUED EXTENSION OF THE FRACTIONAL MAXIMAL OPERATORS FOR NON-DOUBLING MEASURES VIA POTENTIAL OPERATORS

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Abstract: In this paper we consider fractional maximal operators with Radon measure on $\mathbf{R}^d$. We do not pose any assumption on the measure μ except that the measure is non-zero. For the proof we construct Riesz-like potential operators for μ.

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1. Introduction

In this paper we will extend the boundedness of the modified maximal operators on the non-homogeneous space. Let μ be a Radon measure on $\mathbf{R}^d$. Throughout this paper by "ball" we mean a ball with positive radius, and if μ is finite, we regard $\mathbf{R}^d$ as a special ball with radius ∞. Denote $B(x,r)$ as an open ball in $\mathbf{R}^d$ with center x and radius $r > 0$ and for a ball $B = B(x,r)$, we set $r(B) := r$ and $\kappa B := B(x, \kappa r)$. Set the centered maximal operator by

$$\tilde{M} f(x) := \sup_{B=B(x,r)\in\mathcal{B}(\mu)} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y).$$

Here and in what follows we denote $\mathcal{B}(\mu)$ as the totality of the balls with positive μ-measure. $\tilde{M}$ is weak-$(1,1)$ bounded with the aid of Besicovitch's Covering

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Lemma. The proof can be found in [10]. If we set the centered fractional maximal operator by

$$\tilde{M}_\alpha f(x) := \sup_{B=B(x,r)\in\mathcal{B}(\mu)} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad 0 \le \alpha < 1, \tag{1}$$

then the similar proof shows that $\tilde{M}_\alpha$ is weak-$(1,(1-\alpha)^{-1})$ bounded.

What happens for the uncentered maximal operator? We set the uncentered Hardy-Littlewood maximal operator M as

$$Mf(x) := \sup_{x\in B\in\mathcal{B}(\mu)} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y).$$

If the measure is doubling, that is, $\mu(2B) \le C\,\mu(B)$ for any ball B with center in $\mathrm{supp}\,(\mu)$, then M is weak-$(1,1)$ bounded. But if the measure is not doubling, then it can happen that M is not weak-$(1,1)$ bounded (see [2]). It is the same that M_α is not weak-$(1,(1-\alpha)^{-1})$ bounded in general if we replace the uncentered fractional modified maximal operator, where

$$M_\alpha f(x) := \sup_{x\in B\in\mathcal{B}(\mu)} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad 0 \le \alpha < 1.$$

Again if the measure is doubling, we have weak-$(1,(1-\alpha)^{-1})$ boundedness of M_α.

To overcome this difficulty, in [3], [7], [12], [13] and [17] the modified maximal operator was considered. If we define modified maximal operators by

$$M_{0,\kappa} f(x) := \sup_{x\in B\in\mathcal{B}(\mu)} \frac{1}{\mu(\kappa B)} \int_B |f(y)|\, d\mu(y), \quad \kappa > 1\,,$$

or more generally

$$M_{\alpha,\kappa} f(x) := \sup_{x\in B\in\mathcal{B}(\mu)} \frac{1}{\mu(\kappa B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad \kappa > 1, \quad 0 \le \alpha < 1, \tag{2}$$

then $M_{\alpha,\kappa}$ is weak-$(1,(1-\alpha)^{-1})$ bounded. These operators enjoy the boundedness properties with the aid of Besicovitch's Covering Lemma. For our later purpose we use somehow stronger version, whose proof is similar to [13, Theorem 1.5]. For convenience we prove Lemma 1 in Appendix.

Lemma 1. *Let $\kappa > 1$ be a fixed number. Suppose that $\mathcal{B} = \{B_\lambda\}_{\lambda\in\Lambda}$ is a family of the balls with bounded radii : We assume $\sup_{\lambda\in\Lambda} r(B_\lambda) < \infty$. Then we can take $\{B_\lambda\}_{\lambda\in\Lambda_1}, \ldots, \{B_\lambda\}_{\lambda\in\Lambda_N}$, subfamilies such that the following conditions hold. Here, $N = N_\kappa$ depends only on $\kappa > 1$.*

1. *If* $\lambda, \lambda' \in \Lambda_j, j = 1, \ldots, N_\kappa$ *are different, then*

$$\kappa B_\lambda \cap \kappa B_{\lambda'} = \emptyset. \tag{3}$$

2. *For all* $\lambda \in \Lambda$ *there exists* $i(\lambda) \in \bigcup_{j=1}^N \Lambda_j$ *such that*

$$B_\lambda \subset \kappa B_{i(\lambda)}. \tag{4}$$

The aim of this paper is to obtain the Fefferman-Stein type extension of $M_{\alpha,\kappa}$ and $\tilde{M}_\alpha$ with $0 < \alpha < 1$ and $\kappa > 1$ for a general non-zero Radon measure μ via Riesz-Potential-like operators for μ. In [1] D. Adams defined the fractional integral operator for Lebesgue measure by

$$(-\Delta)^{-\alpha/2} f(x) := \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi^{\alpha-\frac{d}{2}}\Gamma\left(\frac{d-\alpha}{2}\right)} \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy,\ 0 < \alpha < d.$$

$(-\Delta)^{-\alpha/2}$ is known to be $L^p(\mathbf{R}^d)$-$L^q(\mathbf{R}^d)$ bounded, if $p, q > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. It is easily seen that $\tilde{M}_\alpha f(x), M_{\alpha,\kappa} f(x) \le C(-\Delta)^{-d\alpha/2}|f|(x)$ for some big constant $C > 0$. We define a fractional integral operator of this type for general measure μ.

Suppose that μ satisfies the growth condition $\mu(B(x,r)) \le c_0 r^n$. Then in [5] García-Cuerva and Gatto defined a fractional integral operator J_α by

$$J_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y). \tag{5}$$

Carderón-Zygmund theory for J_α with growth measure has been developed. García-Cuerva and Gatto showed the $L^p(\mu)$-$L^q(\mu)$ boundedness of J_α if $p, q > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The $L^p(\mu)$-$L^q(\mu)$ boundedness of I_α in more general form was firstly proved by V. Kokilashvili in $\mathbf{R}^d$ $(1 < p < q < \infty)$ in [8]. In general non-homogeneous spaces in general setting $(1 < p < q < \infty)$ it is proved in [9]. See also the monograph by D. Edmunds, V. Kokilashvili and A. Meskhi [4]. Chen and Sawyer showed the boundedness of the commutator $[a, J_\alpha]$, where a is an RBMO function defined by X. Tolsa [17]. For the multilinear version we refer [6]. The example of Calderón-Zygmund theory without growth condition can be found in [7] and [11].

However, the operator J_α is not sufficient for our purpose: Even if μ satisfies the growth condition, J_α can not dominate $\tilde{M}_\alpha$ nor $M_{\alpha,\kappa}$. Thus we shall define linear operators which dominate $\tilde{M}_\alpha$ and $M_{\alpha,\kappa}$ respectively. Our linear operators will be of the form

$$\tilde{I}_\alpha f(x) = \int_{\mathbf{R}^d} \tilde{k}_\alpha(x,y) f(y)\, d\mu(y) \text{ and } I_{\alpha,\kappa} f(x) = \int_{\mathbf{R}^d} k_{\alpha,\kappa}(x,y) f(y)\, d\mu(y).$$

$\tilde{I}_\alpha$ and $I_{\alpha,\kappa}$ are different from those appearing in the fore-mentioned papers. Here $k_{\alpha,\kappa}$ and k_α are positive μ-measurable functions. They will satisfy pointwise estimates

$$\tilde{M}_\alpha f(x) \le C\, \tilde{I}_\alpha |f|(x) \quad \text{and} \quad M_{\alpha,\kappa} f(x) \le C\, I_{\alpha,\kappa}|f|(x).$$

In what follows we will distinguish the centered-type operators from the uncentered-type operators by $\tilde{}$ and subindices: The operators with κ such as $M_{\alpha,\kappa}, I_{\alpha,\kappa}, \ldots$ are related with the uncentered maximal operators. Meanwhile the operators related to the centered maximal operators are denoted with $\tilde{}$. For example $\tilde{M}$ is a centered non-modified maximal operator.

We will work on the modified Morrey space defined in [14]. For details we refer [14]. For distinction of weak-type spaces defined in [15] we name them strong-type Morrey spaces.

For $L^q_{loc}(\mu)$ function f the (strong-type) $\mathcal{M}^p_q(\mu)$ norm is given by

$$\|f \,:\, \mathcal{M}^p_q(k,\mu)\| := \sup_{B \in \mathcal{B}(\mu)} \mu(kB)^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}.$$

Set $\mathcal{M}^p_q(k,\mu)$ as a set of μ-measurable functions with the norm above finite. It is known that if $k_1, k_2 > 1$, then $\mathcal{M}^p_q(k_1,\mu) \sim \mathcal{M}^p_q(k_2,\mu)$. We also have the weak-type Morrey space whose semi-norm is given by

$$\|f \,:\, \mathcal{M}^p_q(k,\mu)\|_w := \sup_{\lambda>0} \sup_{B \in \mathcal{B}(\mu)} \mu(kB)^{\frac{1}{p}-\frac{1}{q}} \left(\lambda^q \mu\{x \in B \,:\, |f(x)| > \lambda\} \right)^{\frac{1}{q}}.$$

The definition can be found in [15]. Set $\mathcal{M}^p_q(k,\mu)_w$ as a set of μ-measurable functions with the semi-norm above finite. In what follows for definiteness we write $\mathcal{M}^p_q(\mu) := \mathcal{M}^p_q(2,\mu)$ and $\mathcal{M}^p_q(\mu)_w := \mathcal{M}^p_q(2,\mu)_w$. Like the strong-type Morrey spaces the number 2 does not affect the definition of the spaces: It suffices to be strictly greater than 1. We note that $\mathcal{M}^p_p(\mu)_w$ is weak-$L^p(\mu)$ space.

We shall prove the following theorems.

Theorem 2. *Let $0 < \alpha < 1$ and $\tilde{M}_\alpha$ be the centered maximal operator given by* (1).

1. *Suppose that $1 < p < \alpha^{-1}$, $1 \le r \le \infty$ and that $\frac{1}{q} = \frac{1}{p} - \alpha$. Let $\{f_j\}_{j=1}^\infty$ be a system of μ-measurable functions. Then*

$$\left\| \left(\sum_{j=1}^\infty \tilde{M}_\alpha f_j{}^r \right)^{\frac{1}{r}} \,:\, L^q(\mu) \right\| \le C \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{\frac{1}{r}} \,:\, L^p(\mu) \right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^{\infty}$.

2. *Suppose that $1 \le r \le \infty$. Let $\{f_j\}_{j=1}^{\infty}$ be a system of μ-measurable functions. Then*

$$\mu\left\{x \in \mathbf{R}^d \,:\, \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r\right)^{\frac{1}{r}} > \lambda\right\} \le \left(\frac{C}{\lambda}\int_{\mathbf{R}^d}\left(\sum_{j=1}^{\infty}|f_j(x)|^r\right)^{\frac{1}{r}} d\mu(x)\right)^{\frac{1}{1-\alpha}}.$$

Here, C is independent of $\{f_j\}_{j=1}^{\infty}$.

Theorem 3. *Let $0 < \alpha < 1$ and $M_{\alpha,\kappa}$ be the uncentered maximal operator given by (2). Let $\{f_j\}_{j=1}^{\infty}$ be a system of μ-measurable functions.*

1. *Suppose that $1 < q \le p < \alpha^{-1}$, $1 \le r \le \infty$, $\frac{1}{s} = \frac{1}{p} - \alpha$ and that $\dfrac{t}{s} = \dfrac{q}{p}$. Then*

$$\left\|\left(\sum_{j=1}^{\infty} M_{\alpha,\kappa} f_j{}^r\right)^{\frac{1}{r}} : \mathcal{M}_t^s(\mu)\right\| \le C \left\|\left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}} : \mathcal{M}_q^p(\mu)\right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^{\infty}$.

2. *Suppose that $1 \le p < \alpha^{-1}$, $1 \le r \le \infty$, and that $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$\left\|\left(\sum_{j=1}^{\infty} M_{\alpha,\kappa} f_j{}^r\right)^{\frac{1}{r}} : \mathcal{M}_{s/p}^s(\mu)\right\|_w \le C \left\|\left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}} : \mathcal{M}_1^p(\mu)\right\|.$$

Here, C is independent of $\{f_j\}_{j=1}^{\infty}$.

The centered maximal operator $\tilde{M}$ is not bounded from $\mathcal{M}_q^p(\mu)$ to $\mathcal{M}_q^p(\mu)_w$. Let us note that Theorem 3.2 is obtained in [14] for $r > 1$. We have obtained the theorem using the vector-valued maximal inequality of Fefferman-Stein type for $M_{0,\kappa}$ with $\kappa > 1$ for Morrey spaces. In [13] and [14] we have proved the vector-valued maximal inequality of Fefferman-Stein type for $M_{0,\kappa}$

with $\kappa > 1$. However, the one for $\tilde{M}$ is still missing so that we cannot obtain Theorem 2. The method by the vector-valued maximal function inequality of Fefferman and Stein has another disadvantage. We cannot recover the case $r = 1$. If $r = 1$, then the vector-valued maximal function inequality of Fefferman and Stein fails. For details of this fact we refer [16]. Our present method does not suffer from this shortcoming. Suppose that, for example, the inequality $\|\tilde{I}_\alpha f \,:\, L^q(\mu)\| \le C \|f \,:\, L^p(\mu)\|$ is obtained for $1 < p < \alpha^{-1}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Then we obtain Theorem 2.1 as follows: By l^r-$l^{r'}$ duality and the pointwise estimate $\tilde{M}_\alpha f_j(x) \le C\, \tilde{I}_\alpha |f_j|(x)$, we have

$$\left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r\right)^{\frac{1}{r}} \le C \left(\sum_{j=1}^{\infty} \tilde{I}_\alpha |f_j|(x)^r\right)^{\frac{1}{r}} = C \sup_{\substack{a=a(x)\,:\,\|a\,:\,l^{r'}\|\le 1 \\ a(x)=\{a_j(x)\}_{j=1}^{\infty}\in l^{r'}}} \sum_{j=1}^{\infty} a_j(x) \tilde{I}_\alpha |f_j|(x).$$

In what follows let us write sup instead of $\sup_{\substack{a=a(x)\,:\,\|a\,:\,l^{r'}\|\le 1 \\ a(x)=\{a_j(x)\}_{j=1}^{\infty}\in l^{r'}}}$. Since $\tilde{I}_\alpha$ is a linear operator, we can proceed further.

$$\begin{aligned}\left(\sum_{j=1}^{\infty} \tilde{M}_\alpha f_j(x)^r\right)^{\frac{1}{r}} &= C \sup \sum_{j=1}^{\infty} a_j(x) \int_{\mathbf{R}^d} \tilde{k}_\alpha(x,y) |f_j(y)|\, d\mu(y) \\ &= C \sup \int_{\mathbf{R}^d} \tilde{k}_\alpha(x,y) \left(\sum_{j=1}^{\infty} a_j(x) |f_j(y)|\right) d\mu(y) \\ &\le C \int_{\mathbf{R}^d} \tilde{k}_\alpha(x,y) \left(\sum_{j=1}^{\infty} |f_j(y)|^r\right)^{\frac{1}{r}} d\mu(y) = C \left[\tilde{I}_\alpha \left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}}\right](x).\end{aligned}$$

As a consequence we obtain

$$\left\| \left(\sum_{j=1}^{\infty} \tilde{M}_\alpha {f_j}^r\right)^{\frac{1}{r}} \,:\, L^q(\mu)\right\|$$

$$\leq \ C \left\| I_\alpha \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \ : \ L^q(\mu) \right\| \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \ : \ L^p(\mu) \right\|.$$

The rest of Theorem 2 and Theorem 3 can be obtained similarly.

Thus the rest of this paper is devoted to constructing $\tilde{I}_\alpha$ and $I_{\alpha,\kappa}$ and to studying their properties. To establish the boundedness of these operators we use Lemma 4 and Lemma 5. The proof can be obtained in the standard way by Besicovitch's covering lemma and interpolation. For details we refer [10].

Lemma 4. *Let $\tilde{M}$ be a centered maximal operator. Then $\tilde{M}$ is $L^p(\mu)$-bounded for $p > 1$ and weak-$(1,1)$ bounded.*

Unfortunately the centered maximal operator $\tilde{M}$ is not bounded on $\mathcal{M}^p_q(\mu)$ in general. In Section 4 we give an example for which $\tilde{M}$ is not weak-$\mathcal{M}^2_1(\mu)$ bounded.

As for the uncentered maximal operator we use the following result.

Lemma 5. *Suppose that $\kappa > 1$. $M_{0,\kappa}$ is the κ-times uncentered modified maximal operator.*

1. (see [15, Theorem 3.3]) *Let $1 \leq p < \infty$. Then $\|M_{0,\kappa} f \ : \ \mathcal{M}^p_1(\mu)\|_w \leq C \|f \ : \ \mathcal{M}^p_1(\mu)\|$, where C is independent of f.*
2. (see [14, Theorem 2.3]) *Let $1 < q \leq p < \infty$. Then $\|M_{0,\kappa} f \ : \ \mathcal{M}^p_q(\mu)\| \leq C \|f \ : \ \mathcal{M}^p_q(\mu)\|$, where C is independent of f.*

2. Centered Fractional Maximal Operators

In this section we consider centered fractional maximal operators. For this purpose we will define auxiliary numbers $r_k(x)$.

Definition 6. Given a Radon measure μ and a point $x \in \mathbf{R}^d$, we define

$$r_k(x) := \sup\{r \geq 0 \ : \ \mu(B(x,r)) < 2^k\}$$

for $k \in \mathbf{Z}$ with $k > \log_2 \mu(\{x\})$.

We note that $r_k(x) > 0$ for all $k > k(x)$ and that $r_k(x) \uparrow \infty$ as $k \to \infty$ by definition.

Now we will define a potential operator for centered maximal operators.

Definition 7. Let $0 < \alpha < 1$. Then for μ-measurable positive function f we set

$$\tilde{I}_\alpha f(x) := \sum_{k=[\log_2 \mu(\{x\})]+1}^{\infty} \frac{1}{2^{k(1-\alpha)}} \int_{B(x,r_k(x))} f(y)\, d\mu(y).$$

Here $[\cdot]$ denotes the Gauss sign. If we define the integral kernel by

$$\tilde{k}_\alpha(x,y) := \sum_{k=[\log_2 \mu(\{x\})]+1}^{\infty} \frac{\chi_{B(x,r_k(x))}(y)}{2^{k(1-\alpha)}},$$

then we can write

$$\tilde{I}_\alpha f(x) = \int_{\mathbf{R}^d} \tilde{k}_\alpha(x,y) f(y)\, d\mu(y).$$

Before going into details let us see how our kernel looks like for Lebesgue measure.

Example 8. If the measure μ is Lebesgue measure, then $\tilde{I}_\alpha$ is pointwise-comparable to $(-\Delta)^{-\frac{d\alpha}{2}}$ in the following sense. For all $x \in \mathbf{R}^d$ the estimate

$$C_0\,(-\Delta)^{-d\alpha/2} f(x) \le \tilde{I}_\alpha f(x) \le C_1\,(-\Delta)^{-d\alpha/2} f(x)$$

holds for all positive Lebesgue measurable function f.

Returning to general Radon measures, we note that the measurability of the function $\tilde{I}_\alpha f(x)$ follows from lower-semicontinuity of $x \in \mathbf{R}^d \mapsto r_k(x) \in \mathbf{R}$.

The following estimate is the key of the boundedness of $\tilde{I}_\alpha$.

Proposition 9. *Let* $1 \le p < \alpha^{-1}$, $q > 1$, $0 < \alpha < 1$ *and* $\frac{1}{q} = \frac{1}{p} - \alpha$. *Then*

$$\tilde{I}_\alpha f(x) \le C\, \tilde{M} f(x)^{\frac{p}{q}} \cdot \|f \,:\, L^p(\mu)\|^{1-\frac{p}{q}}$$

for all positive μ*-measurable function* f*.*

Proof of Proposition 9. Fix $x \in \mathbf{R}^d$. Set $K := \{k \in \mathbf{Z} \,:\, r_k(x) > r_{k-1}(x)\}$. We shall estimate

$$I_k := \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x,r_l(x))} |f(y)|\, d\mu(y)$$

for $k \in K$. First we note that $\mu(B(x,r_k(x))) \le 2^{k_j}$ by definition. Consequently Hölder's inequality yields

$$\begin{aligned} I_k &= \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x,r_k(x))} |f(y)|\, d\mu(y) \\ &\le \frac{C}{2^{k(1-\alpha)}} \int_{B(x,r_k(x))} |f(y)|\, d\mu(y) \le C\, 2^{-\frac{k}{s}} \|f \,:\, L^p(\mu)\|. \end{aligned} \tag{6}$$

In the same way, using maximal operator $\tilde{M}$, we have

$$I_k = \sum_{\substack{l \in \mathbf{Z} \\ r_l(x) = r_k(x)}} \frac{1}{2^{l(1-\alpha)}} \int_{B(x, r_k(x))} |f(y)| \, d\mu(y) \leq C \, 2^{\alpha k} \tilde{M} f(x). \tag{7}$$

Combine (6) and (7) to obtain

$$I_k \leq C \, \min\{2^{-\frac{k}{s}} \| f \, : \, L^p(\mu) \|, 2^{\alpha k} \tilde{M} f(x)\}.$$

Since $\tilde{I}_\alpha f(x) = \sum_{k \in K} I_k$, we have the desired result. □

Since $\tilde{M}$ is $L^p(\mu)$-bounded for $p > 1$, we have the following corollary.

Corollary 10. *Let $\frac{1}{q} = \frac{1}{p} - \alpha$ and $1 < p < \alpha^{-1}$. Then*

$$\| \tilde{I}_\alpha f \, : \, L^q(\mu) \| \leq C \, \| f \, : \, L^p(\mu) \|$$

for all positive μ-measurable function f. In particular I_α can be extended linearly to a bounded operator from $L^p(\mu)$ to $L^q(\mu)$.

From this corollary, as is noted in Introduction, we can obtain Theorem 2.

Finally before investigating the uncentered maximal operator, if the measure μ satisfies the growth condition $\mu(B(x, r)) \leq c_0 \, r^n$ with $0 < n \leq d$, let us see that $\tilde{I}_\alpha$ dominates $J_{n\alpha}$ defined by (5).

Proposition 11. *Assume that the measure μ satisfies the growth condition $\mu(B(x, r)) \leq c_0 r^n$ with $0 < n \leq d$ and that $0 < \alpha < 1$. Then for μ-a.e. $x \in \mathbf{R}^d$*

$$J_{n\alpha} f(x) \leq C \, \tilde{I}_\alpha f(x),$$

where f is a positive μ-measurable function and C is independent of f.

Proof of Proposition 11. It suffices to compare the corresponding kernels. That is, we have only to show that

$$\frac{1}{|x - y|^{n(1-\alpha)}} \leq C k_\alpha(x, y), \ \mu\text{-a.e. } (x, y) \in \mathbf{R}^d \times \mathbf{R}^d \tag{8}$$

for some $C > 0$.

We may assume that $x \neq y$, since μ does not charge any point in $\mathbf{R}^d$ under the growth condition. Let k be an integer such that $r_{k-1}(x) < |x - y| \leq r_k(x)$. Then

$$|x - y|^{n(1-\alpha)} \geq {c_0}^{\alpha - 1} \mu(B(x, |x - y|))^{1-\alpha} \geq C \, 2^{k(1-\alpha)},$$

which proves (8). □

As a corollary of Proposition 11, we note that this proposition allows us to transplant Corollary 10 and Theorem 2 to $J_\beta, 0 < \beta < n$.

3. Uncentered Fractional Maximal Operators

3.1. A Family of Balls

In this subsection we will consider a family of balls that will be needed to define another fractional maximal operator. The following lemma shows that we can drop the assumption of bounded radii in Lemma 1 for some special family of balls.

Lemma 12. *Let $b > a > 0$ be fixed positive numbers. Suppose that μ is a Radon measure and*

$$\mathcal{B}(\mu)_{a,b} := \{B \in \mathcal{B}(\mu) \,:\ \ a \le \mu(\kappa^2 B) \le b\} \ne \emptyset.$$

Then there exists $N(= N_\kappa)$ subfamilies $\mathcal{B}(\mu)_{a,b,1}, \dots, \mathcal{B}(\mu)_{a,b,N}$ such that

$$\{\kappa B \,:\, B \in \mathcal{B}(\mu)_{a,b,j}\} \text{ is disjoint for all } j = 1, \dots, N\,, \tag{9}$$

and for all $B \in \mathcal{B}(\mu)_{a,b}$ we can find $B' \in \bigcup_{j=1}^{N} \mathcal{B}(\mu)_{a,b,j}$ such that $B \subset \kappa B'$. Here N_κ does not depend on a nor b.

Proof of Lemma 12. We may assume that $\mu(\mathbf{R}^d) > b$. If $\mu(\mathbf{R}^d) \le b$, we have only to take $\mathcal{B}(\mu)_{a,b,j} = \{\mathbf{R}^d\}, j = 1, \dots, N$. Furthermore we may assume that $\bigcup_{B \in \mathcal{B}(\mu)_{a,b}} \kappa B$ is connected and non-empty. Otherwise we have only to apply this lemma to each connected component and collect the subfamilies from each component.

Let us call a family of balls $\{B_j\}_{j=0}^{I} \subset \mathcal{B}(\mu)_{a,b}$ an I-chain if $\kappa B_j \cap \kappa B_{j-1} \ne \emptyset$ for all $j = 1, \dots, I$. Fix $B^* \in \mathcal{B}(\mu)_{a,b}$. Since we assume that $\bigcup_{B \in \mathcal{B}(\mu)_{a,b}} \kappa B$ is connected, for all B there exists an I-chain $\{B_j\}_{j=0}^{I} \subset \mathcal{B}(\mu)_{a,b}$ such that $B_0 = B^*$ and $B_I = B$.

Define $i(B)$ as the smallest integer of such I. Let $\mathcal{B}_I := \{B \in \mathcal{B}(\mu)_{a,b} \,:\, i(B) = I\}$. Then by induction and the assumption $\mu(\mathbf{R}^d) > b$, it is easy to see that $\mathcal{B}_I$ is a family of bounded radii.

Now we apply Lemma 1to $\mathcal{B}_I$. We obtain $\mathcal{B}_I^{(1)}, \dots, \mathcal{B}_I^{(N)}$ with (3) and (4) for $\mathcal{B}_I$. The definition of $i(B)$ prohibits $\kappa B \cap \kappa B' \ne \emptyset$ with $B \in \mathcal{B}_I^{(j)}$ and $B' \in \mathcal{B}_{I+2}^{(j')}$ for any $j, j' = 1, \dots, N$ and $I \in \mathbf{N}$. Thus our desired family can be obtained by putting

$$\mathcal{B}(\mu)_{a,b,j} := \bigcup_{I \equiv 1 \text{mod } 2} \mathcal{B}_I^{(j)},\ \mathcal{B}(\mu)_{a,b,j+N} := \bigcup_{I \equiv 0 \text{mod } 2} \mathcal{B}_I^{(j+N)}$$

for $j \le N$. □

Definition 13. Let $k \in \mathbf{Z}$. We apply Lemma 12 to the family with $a = 2^{k-1}, b = 2^k$ and $\kappa > 1$ to obtain a (at most) countable balls $\{B_j^{(k)}\}_{j \in J_k}$ such that:

1. $\sum_{j \in J_k} \chi_{\kappa B_j^{(k)}}(x) \le N_\kappa$.

2. For any ball B with $2^{k-1} \le \mu(\kappa^2 B) \le 2^k$ we can find $B_j^{(k)}, j \in J_k$ such that $B \subset \kappa B_j^{(k)}$.

Throughout Section 3 we fix $\{B_j^{(k)}\}_{j \in J_k}$ for each $k \in \mathbf{Z}$.

3.2. Fractional Integral Operators for Uncentered Maximal Operators

Using $\{B_j^{(k)}\}_{j \in J_k, k \in \mathbf{Z}}$ we will construct our potential operators.

Definition 14. Let $0 < \alpha < 1$. Let f be a positive μ-measurable function. Then we define

$$I_{\alpha,\kappa} f(x) := \sum_{j,k \in \mathbf{Z}} \left(\frac{1}{2^{k(1-\alpha)}} \int_{\kappa B_j^{(k)}} f(y)\, d\mu(y) \right) \chi_{\kappa B_j^{(k)}}(x).$$

We can write $I_{\alpha,\kappa} f(x) = \int_{\mathbf{R}^d} k_{\alpha,\kappa}(x,y) f(y)\, d\mu(y)$ in terms of integral kernel. Here, $k_{\alpha,\kappa}$ is defined as

$$k_{\alpha,\kappa}(x,y) := \sum_{j,k \in \mathbf{Z}} \frac{\chi_{\kappa B_j^{(k)}}(x) \chi_{\kappa B_j^{(k)}}(y)}{2^{k(1-\alpha)}}.$$

$I_{\alpha,\kappa}$ is also pointwise comparable to the operator $(-\Delta)^{-\alpha d/2}$, if μ is Lebesgue measure on $\mathbf{R}^d$. The similar two-sided estimate like Example 8 holds.

Next we shall prove the boundedness of this fractional integral operator. Unlike the centered maximal operator, we use the uncentered maximal operator $M_{0,\kappa}$.

Proposition 15. *Let $1 \le p < \infty$, $s > 1$, $0 < \alpha < 1$ and $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$I_{\alpha,\kappa} f(x) \le C\, M_{0,\kappa} f(x)^{\frac{p}{s}} \| f \,:\, \mathcal{M}_1^p(\mu) \|^{1-\frac{p}{s}}$$

for all positive μ-measurable function f.

Proof of Proposition 15. We will estimate $\frac{1}{2^{k(1-\alpha)}}\int_{\kappa B_j^{(k)}} f(y)\,d\mu(y)$ in two ways again. One is obtained by the definition of Morrey norms. Note that $2^{k-1} \le \mu(\kappa^2 B_j^{(k)}) \le 2^k$. Hence,

$$\begin{aligned}\frac{1}{2^{k(1-\alpha)}}\int_{\kappa B_j^{(k)}} f(y)\,d\mu(y) &\le \frac{\mu(\kappa^2 B_j^{(k)})^{1-\frac{1}{p}}}{2^{k(1-\alpha)}} \cdot \mu(\kappa^2 B_j^{(k)})^{\frac{1}{p}-1} \\ &\times \left(\int_{\kappa B_j^{(k)}} f(y)\,d\mu(y)\right) \le 2^{-\frac{k}{s}+1}\|f\,:\,\mathcal{M}^p_1(\mu)\|. \quad (10)\end{aligned}$$

The estimate using the uncentered modified maximal operator is the same as that via the centered modified maximal operator.

$$\frac{1}{2^{k(1-\alpha)}}\int_{\kappa B_j^{(k)}} f(y)\,d\mu(y) \le \mu(\kappa^2 B_j^{(k)})^{\alpha} M_{0,\kappa}f(x) \le 2^{k\alpha}M_{0,\kappa}f(x). \qquad (11)$$

We summarize (10) and (11):

$$\frac{1}{\mu(\kappa^2 B_j^{(k)})^{1-\alpha}}\int_{\kappa B_j^{(k)}} f(y)\,d\mu(y) \le 2\,\min\{2^{-\frac{k}{s}}\|f\,:\,\mathcal{M}^p_1(\mu)\|, 2^{k\alpha}M_{0,\kappa}f(x)\}.$$

Let us recall that by our construction, $\sum_{j\in J_k}\chi_{\kappa B_j^{(k)}}(x) \le N_\kappa$ for all $x \in \mathbf{R}^d$, where N_κ is independent of j, k and x.

Consequently we have

$$\begin{aligned} I_{\alpha,\kappa}f(x) &\le C\,N_\kappa \sum_{k\in\mathbf{Z}} \min\{2^{-\frac{k}{s}}\|f\,:\,\mathcal{M}^p_1(\mu)\|, 2^{k\alpha}M_{0,\kappa}f(x)\} \\ &\le C\,M_{0,\kappa}f(x)^{\frac{p}{s}}\|f\,:\,\mathcal{M}^p_1(\mu)\|^{1-\frac{p}{s}}. \end{aligned}$$

This is the desired conclusion. □

As a corollary of Proposition 15, we obtain the boundedness of I_α.

Theorem 16. *Let $\kappa > 1$, $0 < \alpha < 1$, $1 \le q \le p < \alpha^{-1}$, $s > 1$, $\frac{q}{p} = \frac{t}{s}$ and $\frac{1}{s} = \frac{1}{p} - \alpha$. Then*

$$\|I_{\alpha,\kappa}f\,:\,\mathcal{M}^s_t(\mu)\| \le C\,\|f\,:\,\mathcal{M}^p_q(\mu)\|.$$

In particular $I_{\alpha,\kappa}$ can be extended to a bounded linear operator from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$.

Theorem 17. *Let* $\kappa > 1$, $0 < \alpha < 1$, $1 \le p < \alpha^{-1}$, $s > 1$ *and* $\frac{1}{s} = \frac{1}{p} - \alpha$. *Then*

$$\|I_{\alpha,\kappa} f \,:\, \mathcal{M}^s_{s/p}(\mu)\|_w \le C \, \|f \,:\, \mathcal{M}^p_1(\mu)\|.$$

Finally for the proof of Theorem 3 it suffices to prove the following estimate: We can dominate $M_{\alpha,\kappa}$ by $I_{\alpha,\kappa^{\frac{1}{2}}}|f|$ pointwise.

Proposition 18. *Suppose that* $\kappa > 1$, $0 < \alpha < 1$. *Then*

$$M_{\alpha,\kappa^2} f(x) \le 2 I_{\alpha,\kappa}|f|(x)$$

for all $x \in \mathbf{R}^d$ *and for all positive* μ*-measurable function* f.

Proof of Proposition 18. Fix $x \in \mathbf{R}^d$. Let $B \in \mathcal{B}(\mu)$ contain x. It is enough to show

$$\frac{1}{\mu(\kappa^2 B)^{1-\alpha}} \int_B |f(y)| \, d\mu(y) \le 2 I_{\alpha,\kappa}|f|(x).$$

Suppose that $2^{k-1} \le \mu(\kappa^2 B) \le 2^k$. Then by definition of $\mathcal{B}_{2^{k-1},2^k}$ we can take $B^{(k)}_j$ such that

$$B \subset \kappa B^{(k)}_j, \, 2^{k-1} \le \mu(\kappa^2 B^{(k)}_j) \le 2^k.$$

These conditions imply

$$\frac{1}{\mu(\kappa^2 B)^{1-\alpha}} \int_B |f(y)| \, d\mu(y) \le \frac{2}{\mu(\kappa^2 B^{(k)}_j)^{1-\alpha}} \int_{\kappa B^{(k)}_j} |f(y)| \, d\mu(y) \le 2 I_{\alpha,\kappa}|f|(x).$$

As a result Proposition 18 is proved. □

Finally we note that $J_{n\alpha}$ is dominated by $I_{\alpha,\kappa}$. The proof is similar to the proof of Proposition 9. We omit the detail.

Proposition 19. *Let* $0 < \alpha < 1$ *and* $\kappa > 1$. *There exists* $C > 0$ *such that*

$$J_{n\alpha} f(x) \le C \, I_{\alpha,\kappa} f(x)$$

for μ*-almost all* $x \in \mathbf{R}^d$ *and for all positive* μ*-measurable function* f.

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Appendices

Appendix A: Proof of Lemma 1

The proof is similar to that in [13, Theorem 1.5]. For convenience for the readers we supply the outline.

Proof of Lemma 1. The following claim follows easily. □

Claim 20. *Assume additionally that* $\sup_{\lambda\in\Lambda} r(B_\lambda) \le \kappa^{\frac{1}{2}} \inf_{\lambda\in\Lambda} r(B_\lambda)$. *Then Lemma 1 holds.*

The proof is completely the same as [13, Theorem 1.5]. Accepting Claim 20, we will treat general case. Set

$$\mathcal{X}_1 := \left\{ B \in \mathcal{B} \,:\, r(B) \ge \kappa^{-\frac{1}{2}} \sup_{\lambda\in\Lambda} r(B_\lambda) \right\}.$$

Then we can take subfamilies $\{B_\lambda\}_{\lambda\in\Lambda_1^{(1)}}, \ldots, \{B_\lambda\}_{\lambda\in\Lambda_N^{(1)}}$ satisfying the conditions (3) and (4) for $\mathcal{X}_1$. Suppose we have defined $\{B_\lambda\}_{\lambda\in\Lambda_1^{(j)}}, \ldots, \{B_\lambda\}_{\lambda\in\Lambda_N^{(j)}}$. First we set

$$\mathcal{X}_{j+1} \;:=\; \Bigg\{ B \in \mathcal{B} \,:\, \kappa^{-\frac{j+1}{2}} \sup_{\lambda\in\Lambda} r(B_\lambda) \le r(B) \le \kappa^{-\frac{j}{2}} \sup_{\lambda\in\Lambda} r(B_\lambda)$$

$$\text{None of } \kappa B_\lambda, \lambda \in \bigcup_{m=1}^{j} \bigcup_{k=1}^{N} \Lambda_k^{(m)} \text{ contains } \kappa B_\lambda \Bigg\}.$$

Then we can take subfamilies $\{B_\lambda\}_{\lambda \in \Lambda_1^{(j+1)}}, \ldots, \{B_\lambda\}_{\lambda \in \Lambda_N^{(j+1)}}$ satisfying (3) and (4) for $\mathcal{X}_{j+1}$.

Let $B = B_\lambda, \lambda \in \Lambda_j^{(m)}$ and $B' = B_{\lambda'}, \lambda' \in \Lambda_{j'}^{(m')}$ with $j, j' \leq N$ and $m, m' \in \mathbf{N}$. Then in the same way as [13, Theorem 1.5], we see that if m and m' are apart, say $|m - m'| \geq M$, then $\kappa B \cap \kappa B' = \emptyset$. Thus we have only to put

$$\Lambda_{m,j} = \bigcup_{m' \equiv m \bmod M} \Lambda_j^{(m)}$$

for $j = 1, \ldots, N, m = 1, \ldots, M$ and then rearrange them. □

Appendix B: A Note on Centered Maximal Operators and Morrey Spaces

As is announced in Introduction, $\tilde{M}$ is not bounded on Morrey space $\mathcal{M}_q^p(\mu)$, unless $p = q$. Let us see an example of this phenomenon. We are going to construct an example of the measure μ on $\mathbf{R}$ for which $\tilde{M}$ is *not* weak-$\mathcal{M}_1^2(\mu)$ bounded.

Let $j \in \mathbf{N}$. We set four types of open intervals in $\mathbf{R}$ by

$$A_j := (j, j+1), \quad B_j := \left(j, j + \frac{1}{j}\right), \quad C_j := \left(j, j + \frac{2}{3}\right),$$

$$D_j := \left(j + \frac{2}{3}, j+1\right), \quad j \in \mathbf{N}.$$

First we define a weight function w by

$$w(x) := \chi_{(-\infty,0]}(x) + \sum_{j=1}^{\infty} \chi_{C_j}(x) + \sum_{j=1}^{\infty} \frac{-(j+1)}{(j+1-x)\log^3(j+1-x)} \chi_{D_j}(x).$$

Here $\log^r x := (\log x)^r, x > 0$. We define a measure on $\mathbf{R}$ by setting $\mu(x) = w(x)\mathcal{H}^1(x)$, where $\mathcal{H}^1(x)$ is an 1-dimensional Hausdorff measure.

We shall prove that $\tilde{M}$ is *not* weak-$\mathcal{M}_1^2(\mu)$ bounded by showing that the function defined by

$$f_j(x) = \frac{1}{(x-j)\log^2(x-j)} \chi_{B_j}(x), \quad j \geq 2,$$

is an element in $\mathcal{M}_1^2(\mu)$ such that

$$\|f_j : \mathcal{M}_1^2(\mu)\| = O(\log^{-2} j) \qquad (j \to \infty), \tag{12}$$
$$\|\tilde{M} f_j : \mathcal{M}_1^2(\mu)\|_w \geq c\, j^{\frac{1}{2}} \log^{-1} j \quad (j \geq 2). \tag{13}$$

As for (12), more precisely we shall prove the following claim.

Claim 21. *As $j \to \infty$, we have*

$$\|f_j : \mathcal{M}_1^2(10, \mu)\| = O\left(j^{\frac{1}{2}} \log^{-2} j\right). \tag{14}$$

Hence as $j \to \infty$, it holds that $\|f_j : \mathcal{M}_1^2(2, \mu)\| = O\left(j^{\frac{1}{2}} \log^{-2} j\right)$.

Proof of Claim 21. If I and J are intervals such that I contains J, then $10I \subset 10J$. Thus we see

$$\|f_j : \mathcal{M}_1^2(10, \mu)\| = \sup_{I \in \mathcal{B}(\mu), I \subset B_j} \mu(10I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y).$$

Let I be an interval in $\mathbf{R}$ contained in B_j. We shall show

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) \leq c \log^{-2} j$$

for such I. Here c is independent of j and I. In the sequel in this proof we assume that $j \geq 100$.

Case 1. $j \in 2I$. In this case we note that

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) \leq \mu(10I \cap D_{j-1})^{-\frac{1}{2}} \int_{I \cap B_j} |f(y)|\, d\mu(y).$$

Simple calculation yields

$$\begin{aligned} &\int_{I \cap B_j} |f(y)|\, d\mu(y) \\ &= -\log^{-1} \mathcal{H}^1(I \cap B_j),\ \mu(10I \cap D_{j-1})^{\frac{1}{2}} = -\frac{1}{2} j^{\frac{1}{2}} \log^{-1} \mathcal{H}^1(10I \cap D_{j-1}). \end{aligned}$$

Since $\mathcal{H}^1(I \cap B_j) \leq \mathcal{H}^1(10I \cap D_{j-1}) \leq c\, \mathcal{H}^1(I \cap B_j)$ in this case, we have

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) \leq c\, j^{-\frac{1}{2}}.$$

Thus we have (14).

Case 2. $j \in 2I$. In this case we use

$$\mu(10I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) \le \mu(I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y)$$

and we estimate the right-hand-side. Let $I = (j+a, j+b)$ with $0 < \frac{1}{2}b \le a < b \le \frac{1}{j} \le \frac{1}{100}$. Then we have

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) = \frac{\log b - \log a}{\log a \cdot \log b \cdot (b-a)^{\frac{1}{2}}}.$$

If we take this quantity as a function of b, then simple calculation shows that the maximum can be attained at $b = \min(2a, j^{-1})$. Next fixing $b = \min(2a, j^{-1})$, we regard

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) = \frac{\log b - \log a}{\log a \cdot \log b \cdot (b-a)^{\frac{1}{2}}}.$$

as a function of a, where $0 \le a \le \frac{1}{j}$, that is, $2I \subset B_j$. Then again elementary arithmetic shows that this function attains its maximum at $a = \frac{1}{2j}$. Thus we have

$$\mu(I)^{-\frac{1}{2}} \int_I |f(y)|\, d\mu(y) \le \frac{(\log 2)(2j)^{\frac{1}{2}}}{(\log j)(\log j + \log 2)} \le j^{\frac{1}{2}} \log^{-2} j.$$

Consequently (14) holds for I with $2I \subset B_j$, whether $2I$ contains j or not. □

To disprove the weak-$\mathcal{M}_1^2(\mu)$ boundedness of $\tilde{M}$, it suffices to estimate $\|\tilde{M} f_j : \mathcal{M}_1^2(2,\mu)\|_w$ from below.

Claim 22. *There exists $c > 0$ such that $\|\tilde{M} f_j : \mathcal{M}_1^2(2,\mu)\|_w \ge c\, j^{\frac{1}{2}} \log^{-1} j$ for all $j \ge 2$.*

Proof of Claim 22. Let $E_j = \left(j + \frac{2}{3j}, j + \frac{5}{6j}\right)$. Then

$$\tilde{M} f_j(x) \ge c_0\, j \log^{-1} j$$

on E_j, where c_0 is a constant independent of j. Let $\lambda_j = c_0\, j \log^{-1} j$. Then

$$\begin{aligned} \|\tilde{M} f_j : \mathcal{M}_1^2(2,\mu)\|_w &\ge \mu(2E_j)^{-\frac{1}{2}} \lambda_j \mu\{x \in E_j : \tilde{M} f_j(x) > \lambda_j\} \\ &\ge (3j)^{\frac{1}{2}} \times c_0\, j \log^{-1} j \times (2j)^{-1} = c\, j^{\frac{1}{2}} \log^{-1} j. \end{aligned}$$

The proof of Claim 22 is now complete. □