International Journal of Pure and Applied Mathematics
Volume 27 No. 1 2006, 103-107

A NOTE ON MULTIPLICATION MODULES

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Abstract: Let R be a commutative ring with identity and M be an R-module. Then M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$. This paper is devoted to the study some of properties of prime submodules of multiplication modules.

AMS Subject Classification: 16D80, 16D10
Key Words: prime submodules, multiplication modules

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Throughout this paper, R will denote a commutative ring with identity and M a unital R-module. Let N be a submodule of M. Then we define $(N : M) = \{r \in R : rM \subseteq N\}$. Note that $(N : M)$ is an ideal of R, in fact, $(N : M)$ is the annihilator of the R-module M/N. The submodule N of M is called prime if $N \neq M$ and, given $r \in R$ and $m \in M$ such that $rm \in N$, either $m \in N$ or $r \in (N : M)$. For the basic properties of prime submodules, see [8], for example.

Let M be an R-module. M is called a multiplication module provided for every submodule N of M there exists an ideal I of R such that $N = IM$. The purpose of this paper is to explore some basic facts of prime submodules of multiplication modules.

Theorem 1. (see [8], Theorem 2.3) *Let M be an R-module, $L_1, L_2, ...,$ L_n a finite number of submodules of M, and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. Assume that at most two of the L_i' s are not prime, and*

Received: February 23, 2006

that $(L_j : M) \not\subseteq (L_k : M)$ *whenever* $j \neq k$. *Then* $L \subseteq L_k$ *for some* k.

Lemma 1. (see [5]) *Let* M *be a multiplication module and let* N *be a submodule of* M. *Then* $N = (N : M)\, M$.

Let $L_1, L_2, ..., L_n$ be submodules of an R-module M. We call a covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ efficient if no L_k is superfluous.

Theorem 2. (The Prime Avoidance Theorem For Multiplication Modules) *Let* M *be a multiplication* R*-module,* $L_1, L_2, ..., L_n$ *a finite number of submodules of* M*, and* L *a submodule of* M *such that* $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. *Assume that at most two of the* L_i*' s are not prime. Then* $L \subseteq L_k$ *for some* k.

Proof. We may assume that the covering is efficient. Then $(L_j : M) \not\subseteq (L_k : M)$ whenever $j \neq k$. Otherwise, if $(L_j : M) \subseteq (L_k : M)$, then $L_j = (L_j : M)\, M \subseteq (L_k : M)\, M = L_k$. This is a contradiction. Consequently, $L \subseteq L_k$ for some k by Theorem 1. □

Theorem 3. (see [5], Corollary 2.11) *The following statements are equivalent for a proper submodule* N *of* M :

(i) N *is a prime submodule of* M.

(ii) $(N : M)$ *is a prime ideal of* R.

(iii) $N = PM$ *for some prime ideal* P *of* R *with* ann$(M) \subseteq P$.

Lemma 2. *Let* M *be a multiplication* R*-module. If* R *is a Noetherian ring, then* M *is a Noetherian module.*

Proof. Let $N_1 \subseteq N_2 \subseteq ... \subseteq N_k \subseteq ...$ be any ascending sequence of submodules of M. Then $(N_1 : M) \subseteq (N_2 : M) \subseteq ... \subseteq (N_k : M) \subseteq ...$ is an ascending sequence of ideals of R. By hypothesis, there exists a positive integer t such that $(N_t : M) = (N_{t+1} : M) = (N_{t+2} : M) = ...$. Therefore, $(N_t : M)\, M = (N_{t+1} : M)\, M = (N_{t+2} : M)\, M = ...$. Since M is a multiplication module, $N_t = N_{t+1} = N_{t+2} = ...$. Consequently, M is a Noetherian module. □

Recall that a ring R is called a ZPI-ring if every ideal of R can be written as a product prime ideals of R.

Corollary 1. *Let* R *be a* ZPI*-ring. If* M *is a multiplication, then* M *is a Noetherian module.*

Proof. Let R be a ZPI-ring. Then, R is a Noetherian ring, see [4], Theorem 9.10. Therefore, M is a Noetherian module by Lemma 2. □

Theorem 4. (see [5], Theorem 3.1) *Let R be a commutative ring with identity and M a faithful (i.e,* $\text{Ann}(M) = 0$*) multiplication R-module. Then the following statements are equivalent.*

(i) *M is finitely generated.*

(ii) *If A and B are ideals of R such that $AM \subseteq BM$ then $A \subseteq B$.*

(iii) *For each submodule N of M there exists a unique ideal I of R such that $N = IM$.*

(iv) *$M \neq AM$ for any proper ideal A of R.*

(v) *$M \neq PM$ for any maximal ideal P of R.*

Definition 1. Let R be a ring and M be an R-module. M is called a ZPI module if every submodule N of M such that $N \neq M$ either is prime or has a prime factorization $N = P_1P_2...P_nN^*$ where $P_1, P_2, ..., P_n$ are prime ideals of R and N^* is a prime submodule in M.

Theorem 5. (see [9]) *Let R be a ZPI-ring and M a multiplication R-module. Let* $\text{Ann}(M) = 0$. *Then M is a ZPI module.*

Proof. M is a Noetherian R-module by Lemma 2. Let N be a submodule of M such that $N \neq M$. Since M is a multiplication R-module, $N = IM$ for some ideal of R. Since R is a ZPI-ring, $I = P_1P_2...P_n$ and so $N = P_1P_2...P_nM$, where $P_1, P_2, ..., P_n$ are prime ideals of R.If $n = 1$, then N is a prime submodule of M. If $n > 1$, then N has a prime factorization by Theorem 3. □

Theorem 6. (see [9]) *Let R be a ring and M be a finitely generated multiplication R-module. If M is a ZPI module and* $\text{Ann}(M) = 0$, *then R is a ZPI-ring.*

Proof. Let I be any ideal of R such that $I \neq R$. Then IM is a proper submodule of M by Theorem 4. Since M is a ZPI module, $IM = P_1P_2...P_nN^*$, where $P_1, P_2, ..., P_n$ are prime ideals of R and N^* is a prime submodule of M. Since M is a multiplication module, $N^* = (N^* : M) M$ where $(N^* : M)$ is a prime ideal of R. Then $IM = P_1P_2...P_n (N^* : M) M$. Since $\text{Ann}(M) = 0, I = P_1P_2...P_n (N^* : M)$ by Theorem 4. Consequently, R is a ZPI-ring. □

Theorem 7. (see [6], Lemma 1.1) *In a Noetherian arithmetical ring R every nonzero ideal of R is of the form $P_1^{v_1}P_2^{v_2}...P_n^{v_n}$ where $P_1, P_2, ..., P_n$ are uniquely determined comaximal prime ideals of R and $v_1, v_2, ..., v_n$ are positive integers, and $P_i^{v_i} \neq 0$.*

Theorem 8. *Let R be a ZPI-ring. Let M be a multiplication R-module. Let $0 \neq N = P_1P_2...P_nM$, where $P_1, P_2, ..., P_n$ are comaximal prime ideals of R. If* $\text{Ann}(M) = 0$, *then the prime ideals are uniquely determined by N.*

Proof. M is a Noetherian R-module by Lemma 2. Since R is a ZPI-ring, R is arithmetical Noetherian ring. Since M is a multiplication module, $N = (N : M)M$ by Lemma 1. Then, $(N : M) = Q_1^{v_1} Q_2^{v_2} ... Q_m^{v_m}$ for some comaximal prime ideals $Q_1, Q_2, ..., Q_m$ of R and positive integers $v_1, v_2, ..., v_m$ by Thoeorem 7. Then $N = P_1 P_2 ... P_n M = Q_1^{v_1} Q_2^{v_2} ... Q_m^{v_m} M$. Since $\mathrm{Ann}(M) = 0$, $P_1 P_2 ... P_n = Q_1^{v_1} Q_2^{v_2} ... Q_m^{v_m}$ by Theorem 4.

Consequently, $\{P_1, P_2, ... P_n\} = \{Q_1, Q_2, ..., Q_m\} \, (n = m)$ by Theorem 7. □

Theorem 9. (see [5], Corollary 2.7) *Artinian Multiplication modules are cyclic.*

Definition 2. An R-module M is called prime if 0_M is a prime submodule of M.

Theorem 10. *Let M be an Artinian faithful multiplication R-module. If R is a domain, then M is a prime module.*

Proof. Let M be an Artinian faithful multiplication R-module. Then M is a cyclic R-module by Theorem 9, say $M = Rm$. Since M is a faithful module, $\mathrm{Ann}(M) = \mathrm{Ann}(m) = 0$. Then $M = Rm \cong R$. Since R is a domain, M is a prime module. □

Let N and K be submodules of a multiplication R-module M with $N = I_1 M$ and $K = I_2 M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1 I_2 M$. Then by [1, Theorem 3.4], the product of N and K is independent of N and K. Clearly NK is a submodule of M and $NK \subseteq N \cap K$ (see, for more detail, $[5] - [7]$).

Theorem 11. *Let M be a multiplication R-module and $N_1, N_2, ..., N_k$ be submodules of M. Let N be prime submodule of M. Then the following statements are equivalent.*

(i) $N_j \subseteq N$ *for some* j *with* $1 \leq j \leq k$.

(ii) $\bigcap_{i=1}^{k} N_i \subseteq N$

(iii) $\prod_{i=1}^{k} N_i \subseteq N$

Proof. $(i) \Rightarrow (ii)$: Clear.

$(ii) \Rightarrow (iii)$: Since $\prod_{i=1}^{k} N_i \subseteq \bigcap_{i=1}^{k} N_i$, $\prod_{i=1}^{k} N_i \subseteq N$ by (ii).

$(iii) \Rightarrow (i)$: We have $N_i = I_i M$ for some ideals $I_i\,(1 \leq i \leq k)$ of R. Then $N_1 N_2 ... N_k = I_1 I_2 ... I_k M \subseteq N$ and so $I_1 I_2 ... I_k \subseteq (N : M)$. Since $(N : M)$ is a prime ideal of R, $I_j \subseteq (N : M)$ for some $j\,(1 \leq j \leq k)$. Therefore, $N_j = I_j M \subseteq N$ for some $j\,(1 \leq j \leq k)$. □

Corollary 2. *Let M be a multiplication R-module and $N_1, N_2, ..., N_k$ be submodules of M. Let N be a prime submodule of M. Let $N = \bigcap_{i=1}^{n} N_i$. Then $N = N_j$ for some j with $1 \leq j \leq n$.*

References

[1] R. Ameri, On the prime submodules of multiplication modules, *International Journal of Mathematics and Mathematical Sciences*, **27** (2003), 1715-1724.

[2] S. Ebrahimi Atani, Submodules of multiplication modules, *Taiwanese J. Math.*, No. 3 (2005), 385-396.

[3] S. Ebrahimi Atani, On Prime and primary submodules, *Chiang Mai J. Sci*, **32**, No. 1 (2005), 5-9.

[4] S. Ebrahimi Atani, The product of multiplication submodules, *Honam Mathematical Journal*, **27**, No. 1 (2005), 1-8.

[5] Z.A. Elbast, P.F. Smith, Multiplication modules, *Comm. in Algebra*, **16** (1988), 755-779.

[6] V. Erdogdu, Coprimely packed rings, *Journal of Number Theory*, **28**, No. 1 (1988), 1-5.

[7] M.D. Larsen, P.J. Mc Carthy, *Multiplicative Theory Ideals*, Academic Press (1971).

[8] C.P. Lu, Unions of Prime submodules, *Houston Journal of Mathematics*, **23**, No. 2 (1997), 203-213.

[9] Ü. Tekir, A note on Dedekind and ZPI modules, *Algebra Colloquium*, **13**, No. 1 (2006), 41-45.