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CONVERGENCE RATES OF EMPIRICAL BAYES TEST FOR ONE-SIDE TRUNCATION PARAMETERS WITH ASYMMETRIC LOSS FUNCTIONS

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Abstract: This paper is to investigate the convergence rates of empirical Bayes test of parameters in one-side truncated distribution family under the asymmetric loss function of the form $L(\theta, \theta_0) = k_1(\theta - \theta_0)^2 I_{(\theta<\theta_0)} + [k_2(\theta - \theta_0)^2 + k_3(\theta - \theta_0)] I_{(\theta\geq\theta_0)} k_i \geq 0,\ i = 1, 2, 3$. The kernel estimation is used to construct the empirical Bayes decision function and its convergence rates $O(n^{-\lambda/2})$ is presented. Finally an illustrative example is given to verify the conditions in theorem.

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1. Introduction

Recently many investigators examined the empirical Bayes tests for parameters of distribution families and its convergence rates under the linear loss or linear weighted loss functions [6], [7], which may be inappropriate in some practical problems such as reliability and predictions [13], [1], [4], [3]. Varian [9] intro-

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duced a kind of asymmetric loss function called linex loss function, that has been applied greatly in practical problems such as reliability and predictable problems, for instance, see Zellner [13] for the Bayes analysis of several statistical estimation and prediction problems, Basu and Ebrahimi [1] applying the linex loss in lifetime testing and reliability estimation, Huang [3] for empirical Bayes testing procedures in a class of nonexponential families, Huang and Liang [4] for the empirical Bayes estimation of the truncation parameter with linex loss, Shi and Xu [8], [12], [11] for EB estimation of two-side truncated parameters in the case of NA samples with linex loss. These investigations describe the application of asymmetric loss functions are of importance in practical science.

In this paper, an empirical Bayes (EB) test for parameters in one-side truncated distribution family is investigated under a kind of asymmetric loss functions introduced by Canfield [2], and the convergence rate is to be obtained as $O(^{-\lambda/2})$.

Consider a kind of one-side truncation distribution family with probability density function (pdf) of the form:

$$f(x\,|\theta) = u(x)\varphi(\theta)I_{(a<\theta<x<b)}, \tag{1.1}$$

where $u(x)$ is positive, integral on (a,b), $\varphi(\theta) = [\int_\theta^b u(x)dx]^{-1}, -\infty \le a < \theta < b \le +\infty$. In this paper we only consider (1.1) and the other form $f_1(x\,|\theta) = u_1(x)\varphi_1(\theta)I_{(a<x<\theta<b)}$ can be similarly investigated, where $u_1(x)$ is positive, integral on$(a,b), \varphi_1(\theta) = [\int_a^\theta u_1(x)dx]^{-1}, -\infty \le a < \theta < b \le +\infty$.

Let pdf of r.v.X be the form of (1.1) and the prior distribution function of θ is assumed to be $G(\theta)$ with pdf $g(\theta)$. Then the marginal density function of X can be written as,

$$f(x) = \int_a^x f(x\,|\theta\,)dG(\theta) = u(x)\Phi_1(x), \tag{1.2}$$

where $\Phi_1(x) = \int_a^x \varphi(\theta)dG(\theta)$.

The hypothesis to be tested is

$$H_0 : \theta < \theta_0 \leftrightarrow H_1 : \theta \ge \theta_0. \tag{1.3}$$

Now let the asymmetric loss function be

$$L(\theta, d) = L(\theta, d_0) + L(\theta, d_1),$$

where $L(\theta, d_0) = k_1(\theta-\theta_0)^2 I_{(\theta<\theta_0)}$, $L(\theta, d_1) = [k_2(\theta-\theta_0)^2 + k_3(\theta-\theta_0)]I_{(\theta\ge\theta_0)}$; $k_j > 0, j = 1,2,3$, d_i indicates accepting H_i $(i = 0,1\,)$ and $D = \{d_0, d_1\}$ is the decision space.

Suppose that $\delta(x) = P$ (accept $H_0\,|X = x$) is the randomized decision function, then one can obtain the Bayes risk of $\delta(x)$ as,

$$R(G,\delta) = \int_a^b \int_a^x \{L(\theta,d_0)\delta(x) + L(\theta,d_1)(1-\delta(x))\} f(x\,|\theta)dG(\theta)dx$$
$$= \int_a^b \int_a^x \{L(\theta,d_0) - L(\theta,d_1)\}\delta(x) f(x\,|\theta)dG(\theta)dx$$
$$+ \int_a^b \int_a^x L(\theta,d_1) f(x\,|\theta)dG(\theta)dx = \int_a^b r(x)\delta(x)dx + C_G, \quad (1.4)$$

where

$$C_G = \int_a^b \int_a^x L(\theta,d_1) f(x\,|\theta)dG(\theta)dx,$$

$$r(x) = \int_a^x \{L(\theta,d_0) - L(\theta,d_1)\} f(x\,|\theta)dG(\theta)$$
$$= \int_a^x L(\theta,d_0) f(x\,|\theta)dG(\theta) - \int_a^x L(\theta,d_0) f(x\,|\theta)dG(\theta)$$
$$= \int_a^{\theta_0} k_1(\theta-\theta_0)^2 f(x\,|\theta)dG(\theta) - \int_{\theta_0}^x [k_2(\theta-\theta_0)^2 + k_3(\theta-\theta_0)] f(x\,|\theta)dG(\theta)$$
$$= \{2k_2[\Phi_2(x) - \Phi_2(\theta_0)] + (k_3 - 2k_2\theta_0)[\Phi_3(x) - \Phi_3(\theta_0)]\}\,u(x)$$
$$- 2k_1[\Phi_2(\theta_0) - \theta_0\Phi_3(\theta_0)]u(x), \quad (1.5)$$

with $\Phi_2(x) = \int_a^x t\Phi_1(t)dt$, $\Phi_3(x) = \int_a^x \Phi_1(t)dt$.

From (1.4) one can obtain the Bayes test test function as:

$$\delta_G(x) = \begin{cases} 1, & r(x) \le 0, \\ 0, & r(x) > 0. \end{cases} \quad (1.6)$$

Then the Bayes risk of $\delta_G(x)$ can be written as:

$$R(G,\delta_G) = \int_a^b r(x)\delta_G(x)dx + C_G\,. \quad (1.7)$$

2. Main Results

Note that in (1.7) Bayes risk $R(G, \delta_G)$ can be obtained accurately when prior distribution $G(\theta)$ is known and $\delta(x) = \delta_G(x)$. However, in practical problems usually $G(\theta)$ is unknown and then $\delta_G(x)$ cannot be applied. Now the empirical Bayes approach is introduced as follows.

The same as usual empirical Bayes construction, suppose $(X_1, \theta_1), \cdots, (X_n, \theta_n)$, (X, θ) are independently identically distribution (iid) random variables. $X_1, \cdots, X_n$ (past samples) and X (present sample) have the same pdf of form (1.2); $\theta_1, \cdots, \theta_n$ and θ have the same pdf $g(\theta)$.

Let $F(x)$ be the distribution function of r.v.X, and its empirical distribution function is presented as:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(a<X_i \leq x)}.$$

So the estimators of $\Phi_2(x)$ and $\Phi_3(x)$ can be expressed as:

$$\Phi_{2n}(x) = \frac{1}{n} \sum_{i=1}^{n} X_i \frac{1}{u(X_i)} I_{(a<X_i \leq x)}, \ \Phi_{3n}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{u(X_i)} I_{(a<X_i \leq x)}. \tag{2.1}$$

Therefore the empirical Bayes test of θ can be defined as:

$$\delta_n(x) = \begin{cases} 1, & r_n(x) \leq 0, \\ 0, & r_n(x) > 0, \end{cases} \tag{2.2}$$

where

$$r_n(x) = \{2k_{2n}[\Phi_{2n}(x) - \Phi_{2n}(\theta_0)] + (k_3 - 2k_2\theta_0)[\Phi_{3n}(x) - \Phi_3(\theta_0)]\}\, u(x) \\ - 2k_1[\Phi_2(\theta_0) - \theta_0\Phi_3(\theta_0)]u(x).$$

The over all Bayes risk of $\delta_n(x)$ can be obtained as:

$$R(G, \delta_n) = E_n \int_a^b r(x)\delta_n(x)dx + C_G = \int_a^b r(x)E_n[\delta_n(x)]dx + C_G, \tag{2.3}$$

where E_n indicates the expectation with respect to joint distribution of $(X_1, \cdots, X_n)$.

The sequence of $\{\delta_n(x)\}$ is said to be asymptotic optimal empirical Byes tests with respect to $\mathcal{F}$ if $\lim_{n\to\infty} R(G, \delta_n) = R(G, \delta_G)$ for any $G(\theta) \in \mathcal{F}$; the sequence of $\{\delta_n(x)\}$ is said to be asymptotic optimal empirical Bayes tests with convergence rate q with respect to $\mathcal{F}$ if $R(G,\delta_n)-R(G,\delta_G)=O(n^{-q})$, $(q>0)$, where $\mathcal{F}$ is the prior distribution family of θ.

Now we claim the following theorem.

Theorem 2.1. *Let $\delta_n(x)$ be defined by (2.2), and M is a positive constant, $0<\lambda\le 1$. If the following conditions are satisfied:*

$$(1)\qquad \int_a^b u(x)[w_j(x)]^{\lambda/2}dt \le M<\infty;$$

$$(2)\qquad \int_a^b u(x)[w_j(x)]^{\lambda/2}[\Phi_i(x)]^{1-\lambda}dt \le M<\infty,\quad i,j=2,3.$$

Then

$$R(G,\delta_n)-R(G,\delta_G)=O(n^{-\lambda/2})$$

with

$$w_2(x)=\int_a^x t^2\frac{\Phi_1(t)}{u(t)}dt, w_3(x)=\int_a^x \frac{\Phi_1(t)}{u(t)}dt.$$

3. Proof of Theorem 2.1

In this section it is assumed that M is a positive constant in different cases even in the same expression.

The following lemmas are introduced to complete the proof.

Lemma 3.1. (see [10]) *Let $R(G,\delta_G)$ and $R(G,\delta_n)$ be defined by (1.7) and (2.3) respectively, then on arrives at:*

$$0\le R(G,\delta_n)-R(G,\delta_G)\le \int_a^b |r(x)|\,P\left\{\,|r_n(x)-r(x)|\ge |r(x)|\,\right\}dx.$$

Proof.

$$\begin{aligned}0\le R(G,\delta_n)-R(G,\delta_G)&=\int_a^b |r(x)|\,\{E_n[\delta_n(x)]-\delta(x)\}dx\\ &=\int_a^b |r(x)|\,[P\{r_n(x)\le 0\}-\delta_G(x)]dx=\int_a^b |r(x)|\,B_n(x)dx,\end{aligned}$$

where $B_n(x)=P\{r_n(x)\le 0\}-\delta_G(x)$.

When $r(x)\le 0$, then $B_n(x)=P\{r_n(x)\le 0\}-1=-P\{r_n(x)>0\}$,

$$R(G,\delta_n)-R(G,\delta_G)=\int_a^b |r(x)|\,P\{r_n(x)>0\}dx.$$

When $r(x) > 0$, then $B_n(x) = P\{r_n(x) \le 0\}$,

$$R(G, \delta_n) - R(G, \delta_G) = \int_a^b |r(x)|\, P\{r_n(x) \le 0\} dx.$$

Note that

$$P\{r_n(x) > 0\} \le P\{\,|r_n(x) - r(x)| \ge |r(x)|\,\}, \text{when } r_n(x) > 0 \text{ and } r(x) \le 0;$$

$$P\{r_n(x) \le 0\} \le P\{\,|r_n(x) - r(x)| \ge |r(x)|\,\}, \text{when } r_n(x) \le 0 \text{ and } r(x) > 0.$$

Therefore we obtain

$$0 \le R(G, \delta_n) - R(G, \delta_G) \le \int_a^b |r(x)|\, P\{\,|r_n(x) - r(x)| \ge |r(x)|\,\} dx. \qquad \square$$

Lemma 3.2. (see [5]) *Let $\Phi_{2n}(x)$ and $\Phi_{3n}(x)$ be defined by (2.1), then for any $\lambda \in (0, 1]$ we have*

$$E_n |\Phi_{in}(x) - \Phi_i(x)|^{2\lambda} \le n^{-\lambda}[w_i(x)]^{\lambda}, \quad i = 2, 3.$$

Proof. We only prove the case $i = 2$ and it is similar for the case $i = 3$. From the definition of $\Phi_{2n}(x)$ we can easily get that $\Phi_{2n}(x)$ is the unbiased estimator of $\Phi_2(x)$, that is $E_n[\Phi_{2n}(x)] = \Phi_2(x)$.

Let $\xi_i = X_i \frac{1}{u(X_i)} I_{(a<X_i \le x)}$, $i = 1, 2, \cdots, n$, then $\xi_1, \cdots \xi_n$ are samples and $\Phi_{2n}(x) = \frac{1}{n} \sum_{i=1}^{n} \xi_i$.

So

$$\begin{aligned} E_n |\Phi_{2n}(x) - \Phi_2(x)|^2 &= E_n \,|\, \Phi_{2n}(x) - E_n[\Phi_{2n}(x)] \,|^2 \\ &= \mathrm{Var}[\Phi_{2n}(x)] = \frac{1}{n}\mathrm{Var}\xi_1 \le \frac{1}{n} E_n \xi_1^2 = \frac{1}{n} \int_a^x t^2 \frac{\Phi_1(t)}{u(t)} dt \le n^{-1} w_2(x). \end{aligned}$$

From Jensen inequality we get

$$E_n |\Phi_{2n}(x) - \Phi_2(x)|^{2\lambda} \le \left\{ E_n |\Phi_{2n}(x) - \Phi_2(x)|^2 \right\}^{\lambda} \le n^{-\lambda}[w_2(x)]^{\lambda}.$$

This completes the proof. $\square$

Proof of Theorem 2.1. From Lemma 3.1, Markov inequality and C_r inequality one can obtain that

$$0 \le R(G,\delta_n) - R(G,\delta_G) \le \int_a^b |r(x)|\, P\{\, |r_n(x) - r(x)| \ge |r(x)|\,\}\, dx$$
$$\le \int_a^b |r(x)|^{1-\lambda}\, E_n\, |r_n(x) - r(x)|^{\lambda}\, dx \le M \int_a^b |r(x)|^{1-\lambda}\, [u(x)]^{\lambda}$$
$$\times \left\{ E_n\, |\Phi_{2n}(x) - \Phi_2(x)|^{\lambda} + E_n\, |\Phi_{3n}(x) - \Phi_3(x)|^{\lambda} \right\} dx$$
$$\le M n^{-\lambda/2} \int_a^b |r(x)|^{1-\lambda}\, [u(x)]^{\lambda} \left\{ [w_2(x)]^{\lambda/2} + [w_3(x)]^{\lambda/2} \right\} dx \le M n^{-\lambda/2}$$
$$\times \int_a^b u(x) \left\{ [\Phi_2(x)]^{1-\lambda} + [\Phi_3(x)]^{1-\lambda} + 1 \right\} \left\{ [w_2(x)]^{\lambda/2} + [w_3(x)]^{\lambda/2} \right\} dx$$
$$\le M n^{-\lambda/2} \left\{ \sum_{i,j=2}^{3} \int_a^b u(x) [\Phi_i(x)]^{1-\lambda} [\Phi_j(x)]^{1-\lambda} dx + \sum_{i=2}^{3} \int_a^b u(x) [w_i(x)]^{\lambda/2} dx \right\} \le M n^{-\lambda/2},$$

where the last inequality can be obtained from conditions of Theorem 2.1. □

Remark. Note that in Theorem 2.1, $0 < \lambda \le 1$ and when λ approaches to 1 arbitrarily, the empirical Bayes test $\{\delta_n(x)\}$ of parameter θ has the convergence rate 1/2.

4. An Example

Let $f(x\,|\theta) = e^{-(x-\theta)} I_{(0<\theta<x<\infty)}$, then $u(x) = e^{-x}$, $\varphi(\theta) = e^{\theta}$. The prior probability density function is supposed to be $g(\theta) = 2e^{-2\theta} I_{(\theta>0)}$, and then the marginal density function of X can be written as:

$$f(x) = \int_0^x f(x\,|\theta) g(\theta) d\theta = 2e^{-x}(1 - e^{-x}).$$

Straightforward computation yields,

$$\Phi_1(x) = \int_0^x \varphi(\theta) g(\theta)\, d\theta = 2(1 - e^{-x})\Phi_3(x) = \int_0^x 2(1 - e^{-t}) dt$$
$$= 2x - 2(1 - e^{-x}),$$

$$\Phi_2(x) = 2\int_0^x t(1 - e^{-t})\; dt = x^2 - 2(1 - e^{-x}) + 2xe^{-x},$$

$$w_2(x) = \int_0^x t^2 \frac{\Phi_1(t)}{u(t)} dt = x^2 e^x (e^x - 2) - x e^x (e^x - 4) + \frac{1}{2} e^x (e^x - 8) + \frac{7}{2},$$

$$w_3(x) = \int_0^x \frac{\Phi_1(t)}{u(t)} dt = e^{2x} - e^x + 1.$$

One can verify the conditions of Theorem 2.1, when we note that e^x is higher order infinite than $x^n (n \geq 0, x \geq 0)$. So the EB test $\delta_n(x)$ for parameter θ can be defined by (2.2) with convergence rates λ by Theorem 2.1.

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References

[1] A.P. Basu, N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, *J. Statist. Plann. Inference*, **29** (1991), 21-31.

[2] R.V. Canfield, A Bayesian approach to reliability estimation using a loss function, *IEEE Trans. Reliab.*, **19** (1970), 13-16.

[3] S.Y. Huang, Empirical Bayes testing procedures in some nonexponential families using asymmetric linex loss, *J. Statist. Plann. Inference*, **46** (1995), 293-309.

[4] S.Y. Huang, T.C. Liang, Empirical Bayes estimation of the truncation parameter with linex loss, *Statistica Sinica*, **7** (1997), 755-769.

[5] M.V. Johns, Jr., J. Van Ryzin, Convergence rates in empirical Bayes two-action problem: Continuous case, *Ann. Math. Statist.*, **43** (1972), 934-947.

[6] R.J. Karunamuni, Optimal rates of convergence of empirical Bayes tests for the continuous one-parameter exponential family, *Ann. Stasist.*, **24** (1996), 212-231.

[7] Y.M. Ma, N. Balakarishnan, Empirical Bayes estimation for truncation parameters, *J. Statist. Plann. Inference*, **84** (2000), 111-120.

[8] Y.M. Shi, Empirical Bayes estimation for parameter of two-sided truncated distribution under Linex loss function, *Appl. Math. J. Chinese Univ. Ser. A*, **15**, No. 4 (2000), 475-483, In Chinese.

[9] H.R. Varian, A Bayesian approach to real estate assessment, In: *Studies in Bayesian Economics and Statistics in Honour of L.S. Savage* (Ed-s: S.E. Feinberg, A. Zellner), North-Holland, (1975), 195-208.

[10] L.S. Wei, The empirical Bayes test problem for scale exponential family: In the case of NA samples, *Acta Mathematicae Applicatae Sinica*, **23**, No. 3 (2000), 403-412, In Chinese.

[11] Y. Xu, Y.M. Shi, Empirical Bayes test for truncation parameters using linex loss, *Bulletin of the Institute of Mathematics*, Academia Sinica, No. 3 (2004), In Chinese.

[12] Y. Xu, X.L. Shi, Y.M. Shi, The EB estimation of parameter in truncated family with Linex loss using NA samples, *Chinese Journal of Mathematics*, **2** (2004), 124-130, In Chinese.

[13] A. Zellner, Bayesian estimation and prediction using asymmetric loss functions, *J. Amer. Stat. Assoc*, **81** (1986), 446-451.