International Journal of Pure and Applied Mathematics
Volume 27 No. 1 2006, 75-80

A GEOMETRIC CHARACTERISTIC OF QUASICONFORMAL MAPPINGS

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Abstract: Let $f : R^n \to R^n$ be a homeomorphism, in this paper, the authors prove that f is a quasiconformal mapping if and only if $f(D)$ is a distance Cigar domain for any distance Cigar domain D in R^n.

AMS Subject Classification: 30C65
Key Words: quasiconformal mapping, homeomorphism, distance Cigar domain

1. Introduction

We shall assume throughout this paper that D is a proper subdomain of Euclidean n-space $R^n (n \geq 2)$.

We say that D is an uniform domain if there exists a constant $a \geq 1$ such that each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$, for which

$$l(\gamma) \leq a|x_1 - x_2|, \tag{1.1}$$

$$\min_{j=1,2} l\Big(\gamma(x_j, x)\Big) \leq ad(x, \partial D) \quad \text{for all} \quad x \in \gamma, \tag{1.2}$$

where $l(\gamma)$ denotes the Euclidean length of γ, $\gamma(x_j, x)$ is the part of γ between x_j and x, and $d(x, \partial D)$ is the Euclidean distance from x to ∂D.

Uniform domains were first introduced in [8] by O. Martio and J. Sarvas in connection with approximation and injectivity properties of functions defined

Received: February 8, 2006

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in R^n. Later, P.W. Jones [6] proved that D is a uniform domain if and only if D has the BMO extension properties, F.W. Gehring and B.G. Osgood [1] proved that uniform domains are invariant under qusiconformal mapping in R^n. In 1992, X.G. He [4] proved the following result.

Theorem A. *Let $f : R^n \to R^n$ be a homeomorphism, if $f(D)$ is a uniform domain for any uniform domain D in R^n, then f must be a quasiconformal mapping.*

Since the rectifiability of a curve is not invariant under quasiconformal mappings, hence R. Näkki and J. Väisälä [10] extended the concept of uniform domains when they researched the properties of quasiconformal mappings on John domains, they defined and studied the following distance Cigar domains.

We say that D is a distance Cigar domain or $c-$distance Cigar domain if there exists a constant $c > 0$ such that each pair of points $x_1, x_2 \in D$ can be joined by a continuous arc $\gamma \subset D$ for which

$$\min_{j=1,2} |x - x_j| \le cd(x, \partial D) \quad \text{for all} \quad x \in \gamma. \tag{1.3}$$

In recently, distance Cigar domains have been used extensively in the research fields of the Sobolev space [5], the integrability of superharmonic functions [2], the Hardy-Littlewood maximal operators [3], the fluid dynamics [7] and the black hole theory [9].

The main purpose of this paper is to use distance Cigar domain to depict the geometric characteristics of quasiconformal mappings, we obtain the following result.

Theorem. *If $f : R^n \to R^n$ is a homeomorphism, then f is a quasiconformal mapping if and only if $f(D)$ is a distance Cigar domain for any distance Cigar domain $D \subset R^n$.*

For convenience we shall adopt the notation and terminology as in paper [11], R^n denotes the n-dimensional Euclidean space. For $x \in R^n$ and $0 < r < \infty$, let $B^n(x, r) = \{z \in R^n : |z - x| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$,$B^n = B^n(1)$, and $\overline{B^n}(x, r)$ be the closure of $B^n(x, r)$. Suppose that f is a homeomorphism in R^n, let $L(x, f, r) = \sup\limits_{|y-x|=r} |f(y) - f(x)|$, and $l(x, f, r) = \inf\limits_{|y-x|=r} |f(y) - f(x)|$. ω_{n-1} and Ω_n denote the $n-1$ and n dimensional *Lebesgue* measure of S^{n-1} and B^n respectively.

2. Two Lemmas and the Proof of Theorem

We shall first establish and introduce the following two lemmas, they are the key of the proof of the theorem.

Lemma 1. *If $f : R^n \to R^n$ is a $K-$quasiconformal mapping, and $c \geq 1$ is a constant, then for all $x \in R^n$ and $0 < r < +\infty$, we have*

$$l\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)/r \geq \frac{1}{a}, \tag{2.1}$$

where $x^{'} = f(x)$, f^{-1} is the inverse of f, and $a = a(n, K, c)$ is a constant which depends only on n, K and c.

Proof. Let Γ be the curve family which joins $S^{n-1}\Big(x, l(x^{'}, f^{-1}, r)/c\Big)$ and $\partial\Bigg(f^{-1}\Big(B^n(x^{'}, r)\Big)\Bigg)$ in $f^{-1}\Big(B^n(x^{'}, r)\Big)\backslash\overline{B}^n\Big(x, l(x^{'}, f^{-1}, r)/c\Big)$, $\Gamma^{'} = f(\Gamma)$. The comparison principle of module and the result in [11], Section 7.5, yield

$$M(\Gamma) \geq \omega_{n-1}\Big(\log\frac{cL(x^{'}, f^{-1}, r)}{l(x^{'}, f^{-1}, r)}\Big)^{1-n} \tag{2.2}$$

and

$$M(\Gamma^{'}) \leq \omega_{n-1}\Bigg(\log\frac{r}{L\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)}\Bigg)^{1-n}. \tag{2.3}$$

By the properties of quasiconformal mapping in [8], we have

$$KM(\Gamma^{'}) \geq M(\Gamma) \geq M(\Gamma^{'})/K \tag{2.4}$$

and

$$\frac{L(x^{'}, f^{-1}, r)}{l(x^{'}, f^{-1}, r)} \leq c^{'} = c'(n, K), \tag{2.5}$$

where $c^{'}(n, K)$ is a constant which depends only on n and K.

Combining (2.2), (2.3), (2.4) and (2.5), we get

$$\frac{L\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)}{r} \geq \frac{1}{(cc^{'})^{K^{\frac{1}{n-1}}}}. \tag{2.6}$$

The same reason to obtain (2.5) implies

$$\frac{L\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)}{l\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)} \leq c^{'}. \tag{2.7}$$

The above (2.6) and (2.7) gives

$$\frac{l\Big(x, f, l(x^{'}, f^{-1}, r)/c\Big)}{r} \geq \frac{1}{c^{'}(cc^{'})^{K^{\frac{1}{n-1}}}} = \frac{1}{a}\ . \qquad \square$$

Lemma 2. (see [4]) *Let $f : R^n \to R^n$ be a homeomorphism, if there exists a constant $c > 0$ such that for any $x \in R^n$ and $r > 0$ yields*

$$\Bigg[\mathrm{dia}\Bigg(f\Big(B^n(x,r)\Big)\Bigg)\Bigg]^n \leq cm\Bigg[f\Big(B^n(x,r)\Big)\Bigg],$$

then f is a quasiconformal map, where $\mathrm{dia}[f(B^n(x,r))]$ and $m[f(B^n(x,r))]$ are the Euclidean diameter and $n-$dimensional Lebesgue measure of $f(B^n(x,r))$ respectively.

Theorem. *If $f : R^n \to R^n$ is a homeomorphism, then f is a quasiconformal mapping if and only if $f(D)$ is a distance Cigar domain for any distance Cigar domain $D \subset R^n$.*

Proof. ($\Rightarrow$) Let D be a $c-$distance Cigar domain, $D^{'} = f(D)$. For any $y_1, y_2 \in D^{'}$, if we take $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$, then $x_1, x_2 \in D$ and there exists a continuous curve $\gamma \subset D$ and γ joins x_1 and x_2 for which (1.3) holds.

Since f is a quasiconformal mapping, hence there exists $K \geq 1$ such that f is a $K-$quasiconformal mapping. $f(\gamma) \subset D^{'}$ is a continuous arc and $f(\gamma)$ joins y_1 and y_2. For any $y \in f(\gamma)$, if we take $x = f^{-1}(y)$, then $x \in \gamma$ and

$$\min_{j=1,2} |x - x_j| \leq cd(x, \partial D). \tag{2.8}$$

Without loss of generality, we may assume that $|x - x_1| \leq cd(x, \partial D)$, this implies $B^n(x, \frac{|x-x_1|}{c}) \subset D$, and then we get

$$B^n\Big(f(x), l(x, f, \frac{|x - x_1|}{c})\Big) \subset D^{'}. \tag{2.9}$$

By $|x - x_1| \geq l(y, f^{-1}, |y - y_1|)$, we can obtain

$$l\Big(x, f, \frac{|x - x_1|}{c}\Big) \geq l\Big(x, f, \frac{l(y, f^{-1}, |y - y_1|)}{c}\Big). \tag{2.10}$$

According to (2.1) in Lemma 1 and (2.10), we have

$$l\Big(x, f, \frac{|x - x_1|}{c}\Big) \geq \frac{|y - y_1|}{a}\ , \tag{2.11}$$

where $a = a(n, K, c)$ is a constant which depends only on n, K and c.

(2.9) and (2.11) imply $B^n\Big(y, \frac{|y-y_1|}{a}\Big) \subset D'$, this gives $|y - y_1| \le ad(y, \partial D')$, and hence $\min\limits_{j=1,2} |y - y_j| \le ad(y, \partial D)$, this shows that D' is a distance Cigar domain.

($\Leftarrow$) For any $x \in R^n$ and $r > 0$, choose $x_1 \in B^n(x, r)$ such that

$$\text{dia}\Big(f(B^n(x,r))\Big) \le 3|f(x) - f(x_1)|. \tag{2.12}$$

For any $y_1, y_2 \in B^n(x, r)$, let α be the close line segment which joins y_1, x and y_2 in this order, it is easy to see that $B^n(x, r)$ is a $1-$distance Cigar domain. By the assumption of the theorem we know that there exists a constant $c > 0$ such that $f(B^n(x, r))$ is a $c-$distance Cigar domain, hence there exists a continuous curve $\gamma \subset f(B^n(x, r))$ joins $f(x)$ and $f(x_1)$ such that

$$\min\Big\{\Big|f(x) - z\Big|, \Big|f(x_1) - z\Big|\Big\} \le cd\Big(z, \partial f(B^n(x,r))\Big) \tag{2.13}$$

for all $z \in \gamma$.

By the continuity of γ we can choose $z_0 \in \gamma$ such that $\Big|f(x) - z_0\Big| = \Big|f(x_1) - z_0\Big|$. (2.12), (2.13) and the triangle inequality imply

$$d\Big(z_0, \partial f(B^n(x,r))\Big) \ge \frac{\Big|f(x) - z_0\Big|}{c} \ge \frac{\Big|f(x) - f(x_1)\Big|}{2c} \ge \frac{\text{dia}\Big(f(B^n(x,r))\Big)}{6c}, \tag{2.14}$$

this gives

$$B^n\left(z_0, \frac{\text{dia}\Big(f(B^n(x,r))\Big)}{6c}\right) \subset f(B^n(x,r))$$

and

$$\left[\text{dia}\Big(f(B^n(x,r))\Big)\right]^n \le \frac{(6c)^n}{\Omega_n} m\Big(f(B^n(x,r))\Big). \tag{2.15}$$

From (2.15) and Lemma 2 we conclude that f is a quasiconformal mapping. This completes the proof of the theorem. □

Acknowledgments

This research is supported by NNSF of P.R. China (10471039).

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