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STABILIZED LINEARLY IMPLICIT SIMPSON-TYPE SCHEMES FOR NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: The classical Simpson rule is an optimal fourth order two-step integration scheme for first-order initial-value problems; however, it is unconditionally unstable. An A-stabilized version of Simpson rule was given by Chawla et al [3] and an L-stable version was given by Chawla et al [2]. These rules are functionally implicit, and when applied for the time integration of nonlinear differential equations, require an iterative method such as Newton's method for the solution of resulting nonlinear systems at each time step of integration. In the present paper, we present a new class of *linearly implicit* Simpson-type rules which are A- and L-stable. For the time integration of nonlinear differential equations, our linearly implicit schemes obviate the need to solve a nonlinear system at each time step of integration. The obtained schemes are computationally illustrated for stability and accuracy by considering a nonlinear initial value problem in ODEs and the diffusion equation with a nonlinear reaction term.

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1. Introduction

For the numerical integration of first order initial-value problems:

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$$y' = f(t, y), \quad y(t_0) = \eta, \tag{1.1}$$

the classical Simpson rule:

$$y_{n+2} = y_n + \frac{h}{3}\left[f_n + 4f_{n+1} + f_{n+2}\right], \tag{1.2}$$

is an optimal fourth order two-step scheme; however, it is unconditionally unstable (see, e.g. Lambert [6]). Chawla et al [3] had given an A-stabilized version of Simpson rule (called ASIMP) which, compressed to a single interval, is described by

$$\begin{gathered}\bar{y}_{n+1/2} = \frac{1}{2}(y_n + y_{n+1}) + \frac{h}{8}(f_n - f_{n+1}), \quad \bar{f}_{n+1/2} = f\left(t_{n+1/2}, \bar{y}_{n+1/2}\right), \\ y_{n+1} = y_n + \frac{h}{6}\left(f_n + 4\bar{f}_{n+1/2} + f_{n+1}\right).\end{gathered} \tag{1.3}$$

Later, Chawla et al [2] had obtained a one-parameter family of L-stabilized Simpson rules (called LSIMP(α_0)) described in the following:

$$\begin{aligned}\tilde{y}_{n+1/2} &= \frac{1}{4}(y_n + 3y_{n+1}) - \frac{h}{4}f_{n+1}, \quad \tilde{f}_{n+1/2} = f\left(t_{n+1/2}, \tilde{y}_{n+1/2}\right), \\ \bar{y}_{n+1/2} &= \alpha_0 y_n + (1-\alpha_0)\, y_{n+1} \\ &\quad + \frac{h}{24}\left[(1+4\alpha_0)\, f_n + 8(2\alpha_0 - 1)\,\tilde{f}_{n+1/2} + (4\alpha_0 - 5)\, f_{n+1}\right], \\ \bar{f}_{n+1/2} &= f\left(t_{n+1/2}, \bar{y}_{n+1/2}\right), \\ y_{n+1} &= y_n + \frac{h}{6}\left(f_n + 4\bar{f}_{n+1/2} + f_{n+1}\right).\end{aligned} \tag{1.4}$$

While for $\alpha_0 = 1/2$, (1.4) reduces to the ASIMP, the LSIMP(α_0)) are L-stable for $\alpha_0 < 1/2$. These L-stable Simpson-type rules have been employed to obtain high-accuracy time integration schemes for parabolic equations by Chawla et al [4]. However, these rules are functionally implicit, and when applied for the time integration of nonlinear differential equations, require an iterative method such as Newton's method for the solution of resulting nonlinear systems at each time step of integration.

In the present paper we present a new class of *linearly implicit* Simpson-type rules which are A- and L-stable. For the time integration of nonlinear differential equations, our linearly implicit schemes obviate the need to solve a nonlinear system at each time step of integration. The obtained schemes are computationally illustrated for stability and accuracy by considering a nonlinear initial value problem in ODEs and the diffusion equation with a nonlinear reaction term.

2. Stabilized Linearly Implicit Simpson-Type Schemes for Nonlinear ODEs

We consider a linearization of the Simpson-type rules (1.4) in the form:

$$\begin{aligned}&\Big[1+a_0\left\{hf^y\left(t_n+\alpha_0 h,y_n+\beta_0 hf_n\right)\right\}+a_1\left\{hf^y\left(t_n+\alpha_1 h,y_n+\beta_1 hf_n\right)\right\}^2\\&\qquad\qquad+a_2\left\{hf_n^y\right\}^3\Big]\frac{\Delta y_n}{h}\\&=\frac{1}{6}\left[f_n+4f\left(t_{n+1/2},y_n+\frac{h}{2}f_n\right)+f\left(t_{n+1},y_n+hf_n\right)\right]\\&\quad+\left\{hf_n^y\right\}\left[b_0f_n+b_1f\left(t_n+\gamma_1 h,y_n\right)\right]+b_2\left\{hf^y\left(t_n+\gamma_2 h,y_n+\delta_2 hf_n\right)\right\}f_n\\&\quad+b_3\left\{hf_n^y\right\}^2f\left(t_n+\sigma_3 h,y_n\right)+t_n\left(h\right).\end{aligned}\tag{2.1}$$

With the help of the following results:

$$y''=f^t+ff^y,\quad y'''=f^{tt}+2ff^{ty}+f^2f^{yy}+f^tf^y+f\left(f^y\right)^2,$$

$$\begin{aligned}y^{(4)}&=D_t^3f+3ff^{tty}+3f^tf^{ty}+5ff^yf^{ty}+3f^2f^{tyy}\\&\quad+3ff^tf^{yy}+4f^2f^yf^{yy}+f^3f^{yyy}+f^yf^{tt}+f^t\left(f^y\right)^2+f\left(f^y\right)^3,\end{aligned}$$

it can be shown that

$$t_n\left(h\right)=hC_1\left(f_n\right)+h^2C_2\left(f_n\right)+h^3C_3\left(f_n\right)+O\left(h^4\right),\tag{2.2}$$

where

$$C_1\left(f\right)=\left(a_0-b_0-b_1-b_2\right)ff^y,$$

$$\begin{aligned}C_2\left(f\right)&=\left(a_0\alpha_0-b_2\gamma_2\right)ff^{ty}+\left(a_0\beta_0-b_2\delta_2\right)f^2f^{yy}\\&\quad+\left(\frac{1}{6}+\frac{1}{2}a_0-b_1\gamma_1\right)f^tf^y+\left(\frac{1}{6}+\frac{1}{2}a_0+a_1-b_3\right)f\left(f^y\right)^2,\end{aligned}$$

$$\begin{aligned}C_3\left(f\right)&=\frac{1}{2}\left(a_0\alpha_0^2-b_2\gamma_2^2\right)ff^{tty}+\frac{1}{2}\left(\frac{1}{4}+a_0\alpha_0\right)f^tf^{ty}\\&\quad+\left(\frac{5}{24}+\frac{1}{2}a_0\alpha_0+\frac{1}{3}a_0+2a_1\alpha_1\right)ff^yf^{ty}+\left(a_0\alpha_0\beta_0-b_2\gamma_2\delta_2\right)f^2f^{tyy}\\&\quad+\frac{1}{2}\left(\frac{1}{4}+a_0\beta_0\right)ff^tf^{yy}+\left(\frac{1}{6}+\frac{1}{2}a_0\beta_0+\frac{1}{6}a_0+2a_1\beta_1\right)f^2f^yf^{yy}\end{aligned}$$

$$+\frac{1}{2}\left(a_0\beta_0^2-b_2\delta_2^2\right)f^3f^{yyy}+\frac{1}{2}\left(\frac{1}{12}+\frac{1}{3}a_0-b_1\gamma_1^2\right)f^yf^{tt}$$
$$+\left(\frac{1}{24}+\frac{1}{6}a_0+\frac{1}{2}a_1-b_3\sigma_3\right)f^t\left(f^y\right)^2+\left(\frac{1}{24}+\frac{1}{6}a_0+\frac{1}{2}a_1+a_2\right)f\left(f^y\right)^3.$$

The resulting order conditions are given in the following.

$$a_0=b_0+b_1+b_2, \tag{2.3}$$

$$a_0\alpha_0=b_2\gamma_2, \tag{2.4}$$

$$a_0\beta_0=b_2\delta_2, \tag{2.5}$$

$$\frac{1}{6}+\frac{1}{2}a_0=b_1\gamma_1, \tag{2.6}$$

$$\frac{1}{6}+\frac{1}{2}a_0+a_1=b_3, \tag{2.7}$$

$$a_0\alpha_0^2=b_2\gamma_2^2, \tag{2.8}$$

$$\frac{1}{4}+a_0\alpha_0=0, \tag{2.9}$$

$$\frac{5}{24}+\frac{1}{2}a_0\alpha_0+\frac{1}{3}a_0+2a_1\alpha_1=0, \tag{2.10}$$

$$a_0\alpha_0\beta_0=b_2\gamma_2\delta_2, \tag{2.11}$$

$$\frac{1}{4}+a_0\beta_0=0, \tag{2.12}$$

$$\frac{1}{6}+\frac{1}{2}a_0\beta_0+\frac{1}{6}a_0+2a_1\beta_1=0, \tag{2.13}$$

$$a_0\beta_0^2=b_2\delta_2^2, \tag{2.14}$$

$$\frac{1}{12}+\frac{1}{3}a_0=b_1\gamma_1^2, \tag{2.15}$$

$$\frac{1}{24}+\frac{1}{6}a_0+\frac{1}{2}a_1=b_3\sigma_3, \tag{2.16}$$

$$\frac{1}{24}+\frac{1}{6}a_0+\frac{1}{2}a_1+a_2=0. \tag{2.17}$$

Before we obtain a solution of the order equations, we consider stability of the schemes (2.1) to also take into account any resulting stability conditions. For the purpose, we consider the test equation:

$$y'=-\lambda y, \quad \lambda > 0 \text{ constant}. \tag{2.18}$$

By applying the scheme in (2.1) to the test equation in (2.18), and setting $H = \lambda h$, we obtain

$$y_{n+1} = R(H)\, y_n, \tag{2.19}$$

where

$$R(H) = \frac{1 - (1 + a_0) H + \left(\frac{1}{2} + a_0 + a_1\right) H^2 - (a_2 + b_3) H^3}{1 - a_0 H + a_1 H^2 - a_2 H^3}.$$

For L-stability we need require that for $H > 0$, $|R(H)| < 1$ and $\lim\limits_{H\to\infty} |R(H)| = 0$. These result in the following stability conditions:

$$b_3 = -a_2, \quad \frac{1}{2} + a_0 \leq 0, \quad a_2 < 0, \quad \frac{1}{2} + a_0 + 2a_1 \geq 0. \tag{2.20}$$

Note that if $a_2 = 0$ then the method is only A-stable.

We now consider solution of the order conditions. Equations (2.4), (2.5), (2.8), (2.11) and (2.14) imply that

$$b_2 = a_0, \quad \gamma_2 = \alpha_0, \quad \delta_2 = \beta_0. \tag{2.21}$$

Then, equations (2.9) and (2.12) give

$$\beta_0 = \alpha_0. \tag{2.22}$$

With the first equation in (2.20), from equations (2.16) and (2.17) we have

$$\sigma_3 = 1. \tag{2.23}$$

Now, (2.7) and (2.17) admit a one free parameter (s) solution given by

$$a_0 = \frac{4s - 5}{6}, \quad a_1 = \frac{11 - 16s}{36}, \quad a_2 = \frac{2s - 1}{18}. \tag{2.24}$$

Then, from equation (2.9) we have

$$\alpha_0 = \frac{3}{2(5 - 4s)}, \tag{2.25}$$

while from equations (2.6) and (2.15) we obtain

$$\gamma_1 = \frac{8s - 7}{3(4s - 3)}, \quad b_1 = \frac{(4s - 3)^2}{4(8s - 7)}, \tag{2.26}$$

and then (2.3) gives

$$b_0 = \frac{(4s-3)^2}{4\,(7-8s)}. \tag{2.27}$$

Finally, from equations (2.10) and (2.13) we obtain, respectively,

$$\alpha_1 = \frac{7-8s}{2\,(11-16s)}, \quad \beta_1 = \frac{7-8s}{4\,(11-16s)}. \tag{2.28}$$

This completes determination of the parameters of the schemes in (2.1). It is now easy to see that the L-stability conditions in (2.20) are all satisfied if $s < 1/2$; for $s = 1/2$, the scheme is only A-stable.

The finally resulting scheme (2.1) is summarized in the following.

$$\alpha_0 = \frac{3}{2\,(5-4s)}, \quad \alpha_1 = \frac{7-8s}{2\,(11-16s)}, \quad b_0 = \frac{(4s-3)^2}{4\,(7-8s)}, \quad \gamma_1 = \frac{8s-7}{3\,(4s-3)},$$

$$a_0 = \frac{4s-5}{6}, \quad a_1 = \frac{11-16s}{36}, \quad a_2 = \frac{2s-1}{18},$$

$$\begin{aligned}
&\Bigg[1 + a_0\left\{hf^y\left(t_n+\alpha_0 h, y_n+\alpha_0 hf_n\right)\right\} + a_1\left\{hf^y\left(t_n+\alpha_1 h, y_n+\frac{1}{2}\alpha_1 hf_n\right)\right\}^2 \\
&\quad + a_2\left\{hf_n^y\right\}^3\Bigg]\frac{\Delta y_n}{h} = \frac{1}{6}\left[f_n + 4f\left(t_{n+1/2}, y_n+\frac{h}{2}f_n\right) + f\left(t_{n+1}, y_n+hf_n\right)\right] \\
&\quad + b_0\left\{hf_n^y\right\}\left[f_n - f\left(t_n+\gamma_1 h, y_n\right)\right] + a_0\left\{hf^y\left(t_n+\alpha_0 h, y_n+\alpha_0 hf_n\right)\right\}f_n \\
&\quad - a_2\left\{hf_n^y\right\}^2 f\left(t_{n+1}, y_n\right).
\end{aligned} \tag{2.29}$$

These schemes are L-stable for $s < 1/2$ and only A-stable for $s = 1/2$. We call the one-parameter family of L-stable schemes in (2.29) *linearly implicit L-stable Simpson-type rules* (LI-LSIMP(s)) for first order initial-value problems (1.1), and the A-stable scheme a *linearly implicit A-stable Simpson-type rule* (LI-ASIMP).

It may be interesting to note the LI-ASIMP scheme is a simpler version of (2.29) described by

$$\begin{aligned}
&\left[1 - \frac{1}{2}hf^y\left(t_{n+1/2}, y_n+\frac{h}{2}f_n\right) + \frac{1}{12}\left\{hf^y\left(t_{n+1/2}, y_n+\frac{h}{4}f_n\right)\right\}^2\right]\frac{\Delta y_n}{h} \\
&\quad = \frac{1}{6}\left[f_n + 4f\left(t_{n+1/2}, y_n+\frac{h}{2}f_n\right) + f\left(t_{n+1}, y_n+hf_n\right)\right]
\end{aligned}$$

$$+\frac{1}{12}\left\{hf_n^y\right\}\left[f_n-f\left(t_{n+1},y_n\right)\right]-\frac{1}{2}\left\{hf^y\left(t_{n+1/2},y_n+\frac{h}{2}f_n\right)\right\}f_n. \quad (2.30)$$

This LI-ASIMP scheme is a linearized *linearly implicit* version of the ASIMP given in (1.3).

3. Stabilized Linearly Implicit Simpson-Type Schemes for Nonlinear Parabolic Equations

We consider the diffusion equation with a nonlinear reaction term:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial x^2}+p\left(u\right), \quad 0<x<\ell,\ t>0, \quad (3.1)$$

where ν is the diffusivity constant, with initial condition:

$$u\left(x,0\right)=f\left(x\right), \quad (3.2a)$$

and Dirichlet boundary conditions:

$$u\left(0,t\right)=a\left(t\right), \quad u\left(\ell,t\right)=b\left(t\right). \quad (3.2b)$$

We first obtain a semidiscretization of (3.1).

For a positive integer N, consider the rectangular grid $\left(x_i,t_j\right)$, $x_i=ih$, $i=0\left(1\right)N+1$, $t_j=jk$, with spatial increment $h=\ell/\left(N+1\right)$ and temporal increment $k>0$. In the following, we set $\rho=\nu k/h^2$, and let $u_{i,j}=u\left(x_i,t_j\right)$, etc. Discretizing the spatial derivatives in (3.1) by the central difference formulas, we obtain

$$\frac{\partial}{\partial t}u_i\left(t\right)=\frac{\nu}{h^2}\left[u\left(x_{i+1},t\right)-2u\left(x_i,t\right)+u\left(x_{i-1},t\right)\right]+p\left(u_i\left(t\right)\right), \quad i=1\left(1\right)N. \quad (3.3)$$

Let

$$\mathbf{u}\left(t\right)=\begin{bmatrix} u_1\left(t\right)\\ \cdot\\ \cdot\\ \cdot\\ u_N\left(t\right)\end{bmatrix}, \quad J=\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1\\ & & & -1 & 2\end{bmatrix},$$

$$\mathbf{c}\left(t\right)=\begin{bmatrix} a\left(t\right)\\ 0\\ \cdot\\ 0\\ b\left(t\right)\end{bmatrix}, \quad \mathbf{p}\left(\mathbf{u}\left(t\right)\right)=\begin{bmatrix} p\left(u_1\left(t\right)\right)\\ \cdot\\ \cdot\\ \cdot\\ p\left(u_N\left(t\right)\right)\end{bmatrix},$$

then using the boundary conditions in (3.2b) we can write the system (3.3) as

$$\frac{\partial}{\partial t}\mathbf{u}\left(t\right)=\frac{1}{k}\phi\left(t,\mathbf{u}\right), \tag{3.4}$$

where we have set

$$\phi\left(t,\mathbf{u}\right)=\rho\left[\mathbf{c}\left(t\right)-J\mathbf{g}\left(\mathbf{u}\right)\right]+\mathbf{p}\left(\mathbf{u}\right),$$

with the initial condition

$$\mathbf{u}\left(0\right)=\left[f\left(x_1\right),...,f\left(x_N\right)\right]^\top.$$

Note that the Jacobian of ϕ is given by

$$\phi^{\mathbf{u}}\left(\mathbf{u}\right)=-\rho J+\mathbf{p}^{\mathbf{u}}\left(\mathbf{u}\right).$$

We first note that by applying the classical trapezoidal formula for the time integration of (3.4) we obtain

$$\mathbf{u}_{j+1}=\mathbf{u}_j+\frac{1}{2}\left[\phi\left(t_j,\mathbf{u}_j\right)+\phi\left(t_{j+1},\mathbf{u}_{j+1}\right)\right]. \tag{3.5}$$

This is the Crank-Nicolson (C-N) scheme (see [5]) for parabolic equations (3.1). Note that C-N is a functionally implicit scheme and the nonlinear system in (3.5) need to be solved at each time step of integration by an iterative method such as Newton's method; see, e.g., Morton and Mayers [7], Smith [8] and Thomas [9].

Again, applying the LSIMP(α_0) schemes (1.4) for the time integration of (3.4), we obtain

$$\widetilde{\mathbf{u}}_{j+1/2}=\frac{1}{4}\left(\mathbf{u}_j+3\mathbf{u}_{j+1}\right)-\frac{1}{4}\phi\left(t_{j+1},\mathbf{u}_{j+1}\right),$$

$$\begin{aligned}\bar{\mathbf{u}}_{j+1/2}&=\alpha_0\mathbf{u}_j+\left(1-\alpha_0\right)\mathbf{u}_{j+1}\\&\quad+\frac{1}{24}\left[\begin{array}{c}\left(1+4\alpha_0\right)\phi\left(t_j,\mathbf{u}_j\right)+8\left(2\alpha_0-1\right)\phi\left(t_{j+1/2},\widetilde{\mathbf{u}}_{j+1/2}\right)\\+\left(4\alpha_0-5\right)\phi\left(t_{j+1},\mathbf{u}_{j+1}\right)\end{array}\right],\end{aligned}$$

$$\mathbf{u}_{j+1}=\mathbf{u}_j+\frac{1}{6}\left[\phi\left(t_j,\mathbf{u}_j\right)+4\phi\left(t_{j+1/2},\bar{\mathbf{u}}_{j+1/2}\right)+\phi\left(t_{j+1},\mathbf{u}_{j+1}\right)\right]. \tag{3.6}$$

These LSIMP(α_0) schemes for parabolic equation (3.1) are also functionally implicit and, for nonlinear problems, the nonlinear systems in (3.6) have to be solved by an iterative method, such as Newton's method, at each time step of integration.

Next, by applying the LI-LSIMP(s) schemes given in (2.29) for the time integration of (3.4), we obtain

$$\alpha_0 = \frac{3}{2(5-4s)}, \quad \alpha_1 = \frac{7-8s}{2(11-16s)}, \quad b_0 = \frac{(4s-3)^2}{4(7-8s)}, \quad \gamma_1 = \frac{8s-7}{3(4s-3)},$$

$$a_0 = \frac{4s-5}{6}, \quad a_1 = \frac{11-16s}{36}, \quad a_2 = \frac{2s-1}{18}, \; \phi_j = \phi(t_j, \mathbf{u}_j), \quad \phi_j^{\mathbf{u}} = (t_j, \mathbf{u}_j),$$

$$\begin{aligned}\Big[I + a_0\phi^{\mathbf{u}}(t_j + \alpha_0 k, \mathbf{u}_j + \alpha_0\phi_j) + a_1\left\{\phi^{\mathbf{u}}\left(t_j + \alpha_1 k, \mathbf{u}_j + \frac{1}{2}\alpha_1\phi_j\right)\right\}^2 \\ + a_2\left\{\phi_j^{\mathbf{u}}\right\}^3\Big]\Delta\mathbf{u}_j = \frac{1}{6}\left[\phi_j + 4\phi\left(t_{j+1/2}, \mathbf{u}_j + \frac{1}{2}\phi_j\right) + \phi(t_{j+1}, \mathbf{u}_j + \phi_j)\right] \\ + b_0\phi_j^{\mathbf{u}}\left[\phi_j - \phi(t_j + \gamma_1 k, \mathbf{u}_j)\right] + a_0\phi^{\mathbf{u}}(t_j + \alpha_0 k, \mathbf{u}_j + \alpha_0\phi_j)\phi_j \\ - a_2\left\{\phi_j^{\mathbf{u}}\right\}^2\phi(t_{j+1}, \mathbf{u}_j),\end{aligned} \quad (3.7)$$

where I denotes identity matrix. We call (3.7) a one-parameter family (for $s < 1/2$) of LI-LSIMP(s) schemes for parabolic equations (3.1).

3.1. Unconditional Stability

For homogeneous linear problems (3.1) with $p(u) = 0$ and homogeneous boundary conditions in (3.2b), the LI-LSIMP(s) schemes (3.7) can be written as

$$\mathbf{u}_{j+1} = Q\mathbf{u}_j, \quad j = 0, 1, 2..., \quad (3.8)$$

where the amplification matrix for the difference schemes is given by

$$Q = \left[I - a_0\rho J + a_1(\rho J)^2 - a_2(\rho J)^3\right]^{-1} \times \left[I - (1 + a_0)\rho J + \left(\frac{1}{2} + a_0 + a_1\right)(\rho J)^2\right],$$

Since J has positive eigenvalues (see Smith [8]), it follows that all the eigenvalues of Q are located within the unit circle. Therefore, the LI- LSIMP(s) schemes (3.7) are unconditionally stable for all $\rho > 0$.

N	LSIMP(0)	LI-LSIMP(0)	order
4	4.5(-4)	5.1(-3)	
8	3.5(-5)	3.2(-4)	3.99
16	2.4(-6)	2.0(-5)	4.00
32	1.6(-7)	1.2(-6)	4.06
64	1.0(-8)	7.4(-8)	4.02
128	6.5(-10)	4.6(-9)	4.01

Table 1: Absolute errors and order for $t = 4$

4. Numerical Experiments

We next illustrate the accuracy and stability of the obtained LI-LSIMP(s) schemes, and compare their performance with the functionally implicit schemes LSIMP(s) by considering two nonlinear problems from first order initial value problems and parabolic equations. The main point of these numerical experiments is that the obtained LI-LSIMP(s) schemes do provide accuracy and stability comparable with the functionally implicit schemes without the need to solve a nonlinear system at each time step of integration.

Problem 1. (A Nonlinear ODE) We first consider a first order nonlinear initial-value problem in ODEs:

$$y' + \frac{1}{5}ty = e^{-\frac{1}{5}t^2}y^{-1}, \quad 0 < t < 4, \quad y(0) = 1, \tag{4.1}$$

with the exact solution given by

$$y(t) = \sqrt{2t+1}e^{-\frac{1}{10}t^2}. \tag{4.2}$$

We computed approximations for the solution for time $t = 4$ by LSIMP(0) and LI-LSIMP(0) for various step lengths and these are shown in Table 1. These approximations do confirm fourth order of the LI-LSIMP schemes. Note also that the LI-LSIMP approximations are close to those provided by LSIMP, without the need to solve nonlinear systems at each time step of integration. To illustrate the proximity of these approximations, we show the results for $N = 4$ in Figure 1.

Problem 2. (Nonlinear Reaction-Diffusion) We consider the diffusion equation with a nonlinear reaction term:

$$u_t = u_{xx} + u^2(1-u), \quad 0 < x < 10, \ t > 0, \tag{4.3}$$

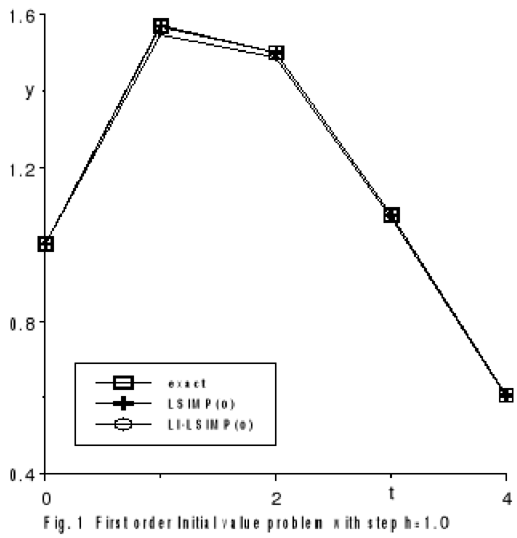

Figure 1:

with the initial condition and Dirichlet boundary conditions taken from the exact solution (see Berzins et al [1]):

$$u\,(x,t) = \frac{1}{1+e^{\sigma(x-\sigma t)}}, \quad \sigma = \frac{1}{\sqrt{2}}. \tag{4.4}$$

We computed approximations for the solution for time $t = 15$ by the LSIMP(0) and LI-LSIMP(0) schemes by taking $h = 0.5$ and time step $k = 0.6$. The absolute errors in the computed approximations are 6.1(-2) for LSIMP and 4.2(-4) for LI-LSIMP. These approximations are displayed in Figure 2.

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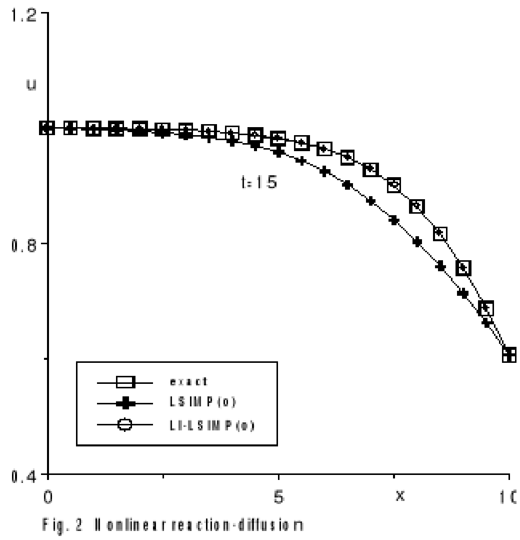

Figure 2:

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