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RIGHT CODIMENSIONAL SMOOTH POINTS OF BRILL-NOETHER LOCI OF $\mathcal{M}_g$

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Abstract: Fix integers g, t, r and d such that $r > 0$, $g \ge t + 2 \ge 3$, $2 \le d \le 2g - 4$ and $\rho(g, r, d) = -t$. For all integers $u > 0, v > 0$ set $\mathcal{M}_g(u, v) := \{C \in \mathcal{M}_g : C \text{ has a } g^u_v\}$. Here we prove the existence of $X \in \mathcal{M}_g$ with the following properties:

(i) $X \in \mathcal{M}_g(r, d)$ and $\mathcal{M}_g(r, d)$ is smooth and of dimension $3g - 3 - t$ at X;

(ii) X has no g^y_x such that $\rho(g, y, x) < 0$ and $0 < x < d$;

(iii) X has no g^u_v such that $v < 2d$ and $\rho(g, u, v) \le -t - 1$.

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1. Introduction

Here we use the theory of limit linear series due to D. Eisenbud and J. Harris ([3], [4]) to prove an existence theorem and a smoothness theorem for certain Brill-Noether loci of the moduli space $\mathcal{M}_g$ of smooth genus g curves. Simultaneously, we prove a non-existence theorem for certain linear series on the general curve in the "good" component of these Brill-Noether loci. More precisely, we prove the following result.

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Theorem 1. *Fix integers g, t, r and d such that $r > 0$, $g \geq t+2 \geq 3$, $2 \leq d \leq 2g-4$ and $\rho(g,r,d) = -t$. For all integers $u > 0, v > 0$ set $\mathcal{M}_g(u,v) := \{C \in \mathcal{M}_g : C$ has a $g^u_v\}$. Then there exists $X \in \mathcal{M}_g$ with the following properties:*

(i) $X \in \mathcal{M}_g(r,d)$ and $\mathcal{M}_g(r,d)$ is smooth and of dimension $3g-3-t$ at X;

(ii) X has no g^y_x such that $\rho(g,y,x) < 0$ and $0 < x < d$;

(iii) X has no g^u_v such that $v < 2d$ and $\rho(g,u,v) \leq -t-1$.

Here $\rho(a,b,c) := a-(b+1)(a+b-c)$ denotes the Brill-Noether number. Theorem 1 should have been proved twenty years ago, but as far as we know, it is not contained in the literature on the subject (see [5], [1], [7] and [6] for related works).

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. Proof of Theorem 1

Notation 1. Let C be a smooth curve of genus q, $P_1, \dots, P_n \in X$ distinct points and A a g^r_d on C. Let $0 \leq a^A_0(P_i) < \cdots < a^A_r(P_i) \leq d$ be the vanishing sequence of A at P_i. Set $\alpha^A_j(P_i) := a^A_j(P_i) - j$. Hence $0 \leq \alpha^A_0(P_i) \leq \cdots \leq \alpha^A_r(P_i) \leq d-r$ is the ramification sequence of A at P_i. Set $\rho(A, P_1, \dots, P_n) := \rho(q,r,d) - \sum_{i=1}^n \sum_{j=0}^r \alpha^A_j(P_i)$, i.e. set

$$\rho(A, P_1, \dots, P_n) := (r+1)d - rq - r(r+1) - \sum_{i=1}^{n}\sum_{j=0}^{r} a^A_j(P_i) - n(r+1)r/2\,.$$

Notice that if $q = 1$ and $P, Q \in C$, $P \neq Q$, we have

$$\rho(A,P) = (r+1)d - (r+3)r/2 - \sum_{j=0}^{r} a^A_j(P)\,, \qquad (1)$$

$$\rho(A,P,Q) = (r+1)d - r - \sum_{j=0}^{r} a^A_j(P) - \sum_{j=0}^{r} a^A_j(Q)\,. \qquad (2)$$

Let Y be a genus q nodal curve of compact type and L a refined limit g^r_d on Y. For any irreducible component T of Y, let L_T denote the aspect of L

on T. Fix any $P \in \mathrm{Sing}(Y)$ and let T, C be the irreducible components of Y containing P. Since L is refined, we have

$$a_j^{L_T}(P) + a_{r-j}^{L_C}(P) = d \tag{3}$$

for all $j \in \{0, \dots, r\}$.

Remark 1. Let C be an elliptic curve and $P, Q \in C$ such that $P \neq Q$. Take any complete degree $d \geq 3$ linear series on C. We have $a_i^L(P) + a_{d-1-i}^L(Q) \leq d$ for all $i \in \{0, \dots, d-1\}$ and we have equality for one index i if and only if $L \cong \mathcal{O}_C(iP + (d-i)Q)$. Taking a linear projection we get $a_i^A(P) + a_{r-i}^A(Q) \leq d$ for every g_d^r A on C.

Lemma 1. *Fix integers $d > r > 0$. Let C be an elliptic curve and $P, Q \in C$, $P \neq Q$, such that $\mathcal{O}_C(P-Q)$ has exact order d in $\mathrm{Pic}^0(C)$. There is a unique g_d^r, say A, with vanishing orders $a_i^A(P), a_j^A(Q)$ such that $d-1 \leq a_i^A(P) + a_{r-i}^A(Q) \leq d$ for all $i \in \{0, \dots, r\}$ and such that $\rho(A, P, Q) = -1$. We have $a_{r-i}^A(Q) = d - 1 - a_i^A(P)$ for $i \neq 0, r$, $a_0^A(Q) = d - a_r(P)$ and $a_r^A(Q) = d - a_0^A(P)$. The line bundle associated to A is $\mathcal{O}_C(dP) \cong \mathcal{O}_C(dQ)$.*

Proof. Since $\rho(A, P, Q) = -1$, there are exactly two indices $i \in \{0, \dots, r\}$ such that $a_i^A(P) + a_{r-i}^A(Q) = d$ (use (2)). Any equality of this type forces the line bundle associated to A to be isomorphic $\mathcal{O}_C(xP + (d-x)Q)$ for some $0 \leq x \leq d$, and in this case we have $a_x^A(P) + a_{r-x}^A(Q) = d$. Since we have two such equalities and $\mathcal{O}_C(P-Q)$ has exact order d in $\mathrm{Pic}^0(C)$, we need to have $a_{r-i}^A(Q) = d-1-a_i^A(P)$ for $i \neq 0, r$, $a_0^A(Q) = d - a_r^A(P)$ and $a_r^A(Q) = d - a_0^A(P)$. The uniqueness of the g_d^r is just [4], Proposition 5.2. The existence of at least one such g_d^r follows from the criterion given in [4], Proposition 5.2. □

Remark 2. Let C be an elliptic curve and $P, Q \in C$, $P \neq Q$. Assume $\mathcal{O}_C(P-Q)$ either not torsion in $\mathrm{Pic}^0(C)$ or with exact order d in $\mathrm{Pic}^0(C)$. Fix integers x, y such that $0 < y < x < d$. There is no g_x^y on C such that $\rho(g_x^y, P, Q) < 0$. For a similar observation, see [8], Proposition 2.8.

Remark 3. Let C be an elliptic curve and $P \in C$. By (1) there is no g_d^r on C such that $\rho(g_d^r, P) < 0$. Let A be any g_d^r on C such that $\rho(A, P) = 0$. By (1) we have $a_0^A(P) = d-r-1$ and $a_i^A(P) = d-r+i$ for all $i \in \{1, \dots, r\}$. Hence $A = (d-r-1)P + |(r+1)P|$. As in the proof of Lemma 1 we get $L \cong \mathcal{O}_C(dP)$

Lemma 2. *Let C be an elliptic curve, $P, Q \in C$, $P \neq Q$, and a g_v^u A on C. Assume one of the following conditions:*

(i) $\mathcal{O}_C(P-Q)$ not torsion in $\mathrm{Pic}^0(C)$;

(ii) $\mathcal{O}_C(P-Q)$ *has exact order* d *in* $\mathrm{Pic}^0(C)$ *and* $v < 2d$.

Then $\rho(A,P,Q) \ge -1$.

Proof. Let $L \in \mathrm{Pic}^v(C)$ denote the line bundle associated to A. Assume $\rho(A,P,Q) \ge -2$. Assume the existence of 3 integers, say i,j,k, such that $0 \le i < j < k \le r$ and $a_i^A(P)+a_{r-i}^A(Q) = a_j^A(P)+a_{r-j}^A(Q) = a_k^A(P)+a_{r-k}^A(Q) = v$. Since $A \subseteq H^0(C,L)$ and $\deg(L) = v$, we get $L \cong \mathcal{O}_C(a_i^A(P)P + a_{r-i}^A(Q)Q) \cong \mathcal{O}_C(a_j^A(P)P + a_{r-i}^A(Q)Q) \cong \mathcal{O}_C(a_k^A(P)P + a_{r-i}^A(Q)Q)$. Since $a_i^A(P) < a_j^A(P) < a_k^A(P)$, we get that $\mathcal{O}_C(P-Q)$ has order in $\mathrm{Pic}^0(C)$ dividing both $a_j^A(P) - a_i^A(P)$ and $a_k^A(P) - a_j^A(P)$, while at least one of these positive integers is at most $v/2$. □

Proof of Theorem 1. Fix the integer $t > 0$ such that $\rho(g,r,d) = -t$. By assumption we have $g \ge t+2$. Let Y be the following genus g stable curve of compact type. $Y = Y_1 \cup \cdots \cup Y_g$ is a chain of g smooth elliptic curves Y_i, $1 \le i \le g$, i.e. $Y_i \cap Y_j \ne \emptyset$ if and only if $|i-j| \le 1$. Hence $\sharp(Y_i \cap \mathrm{Sing}(Y)) = 1$ for $i = 1, g$, while $\sharp(Y_i \cap \mathrm{Sing}(Y)) = 2$ for $2 \le i \le g-1$. Set $P_i := Y_i \cap Y_{i+1}$, $1 \le i \le g-1$. If $2 \le i \le t$ we assume that $\mathcal{O}_{Y_i}(P_{i-1} - P_i)$ has exact order d in $\mathrm{Pic}^0(Y_i)$. If $t+2 \le i \le g-1$ we assume that $\mathcal{O}_{Y_i}(P_{i-1} - P_i)$ is not a torsion element in $\mathrm{Pic}^0(Y_i)$.

(a) To prove part (i) it is sufficient to construct a refined dimensionally proper g_d^r, L, on Y ([2], Theorem 3.4). To give L it is sufficient to give its aspects L_{Y_i}, $1 \le i \le g$, and to check the compatibility conditions (use (3)) at each P_i, $1 \le i \le g-1$. As associated line bundle $\mathbb{L}_{Y_i}$ we will take $\mathcal{O}_{Y_i}(P_i)$ for $1 \le i \le g-1$ and $\mathbb{L}_{Y_g} := \mathcal{O}_{Y_g}(P_{g-1})$. We want to have $\rho(L_{Y_1}, P_1) = \rho(L_{Y_g}, P_{g-1}) = 0$, $\rho(L_{Y_i}; P_{i-1}, P_i) = -1$ for $2 \le i \le t+1$ and $\rho(L_{Y_i}; P_{i-1}, P_i) = 0$ for $t+2 \le i \le g-1$. Set $L_{Y_1} := d-r-1+|(r+1)P_1|$. Hence $a_j^{L_{Y_1}}(P_1) = d-r-1+j$ for $0 \le j \le r-1$, $a_r^{L_{Y_1}}(P_1) = d$ and $\rho(L_{Y_1}, P_1) = 0$ (use (1)). Fix an integer x such that $2 \le x \le g-1$ and assume defined L_{Y_i} for all $i \in \{1, \dots, x-1\}$. By (3) we are forced to require $a_j^{L_{Y_x}}(P_{x-1}) = d - a_{r-j}^{L_{Y_{x-1}}}(P_{x-1})$ for all j. To get $\rho(L_{Y_i}; P_{i-1}, P_i) = -1$ for $2 \le i \le t+1$ and $\rho(L_{Y_i}; P_{i-1}, P_i) = 0$ for $t+2 \le i \le g-1$ we need to obtain

$$\sum_{j=0}^{r} a_j^{L^{Y_{i-1}}}(P_{i-1}) = \sum_{j=0}^{r} a_j^{L^{Y_i}}(P_i) + r - 1 \tag{4}$$

for $2 \le i \le t+1$ and

$$\sum_{j=0}^{r} a_j^{L^{Y_{i-1}}}(P_{i-1}) = \sum_{j=0}^{r} a_j^{L^{Y_i}}(P_i) + r \tag{5}$$

for $t+2 \le i \le g-1$. To apply [4], Proposition 5.2, we will always take $a_j^{L_{Y_i}}(P_i)$, $0 \le j \le r$, satisfying the inequalities

$$d-1 \le a_j^{L_{Y_i}}(P_{i-1}) + a_{r-j}^{L_{Y_i}}(P_i) \le d \tag{6}$$

for all $j \in \{0,\dots,r\}$. Use (4) and (5). If $2 \le i \le t+1$ to get $\rho(L_{Y_i}; P_{i-1}, P_i) = -1$ we need to take $a_j^{L_{Y_i}}(P_{i-1}) + a_{r-j}^{L_{Y_i}}(P_i) = d$ for exactly two indices j, while if $t+2 \le i \le g-1$ to get $\rho(L_{Y_i}; P_{i-1}, P_i) = 0$ we need to take $a_j^{L_{Y_i}}(P_{i-1}) + a_{r-j}^{L_{Y_i}}(P_i) = d$ for exactly one index j. First assume $2 \le i \le t+1$. We take $a_j^{L_{Y_i}}(P_{i-1}) + a_{r-j}^{L_{Y_i}}(P_i) = d$ if and only if $j = 0, d$. The existence criterion given in [4], Proposition 5.2, is satisfied because $\mathbb{L}_{Y_i} \cong \mathcal{O}_{Y_i}(dP_i) \cong \mathcal{O}_{Y_i}(dP_{i-1})$. Now assume $t+2 \le i \le g-1$. $a_j^{L_{Y_i}}(P_{i-1}) + a_{r-j}^{L_{Y_i}}(P_i) = d$ if and only if $j = 0$. The existence crieterion given in [4], Proposition 5.2, is satisfied because $\mathbb{L}_{Y_i} \cong \mathcal{O}_{Y_i}d(P_i)$. Set $Y' := \cup_{i=1}^{g-1} Y_i$ and L' the refined linear series on Y' with L_{Y_i} as its aspects on each Y_i. By the additivity of the Brill-Noether number ([3], Proposition 4.6) we get $\rho(L', P_{g-1}) = \rho(L_{Y_1}, P_1) - t$. Since $\rho(g,r,d) = -t$, we get $a_0^{L_{g-1}}(P_{g-1}) = 0$ and $a_j^{L_{g-1}}(P_{g-1}) = j+1$ for all $j \in \{1,\dots,r\}$. Hence taking $L_{Y_g} := (d-r-1)P_{g-1} + |(r+1)P_{g-1}|$ as aspect on Y_g we give a refined limit linear series on Y, which is smoothable and which appear in the "right" codimension $-\rho(g,r,d)$, because it appear in the riight codimension in the family of all chains of g elliptic curves.

(b) To prove parts (ii) it is sufficient to show that for all integers $u > 0$, $v > 0$ such that either $v < d$ and $\rho(g,u,v) < 0$ or $\rho(g,u,v) \le -t-1$ and $v < 2d$ there is no coarse g_v^u on Y ([2], Proposition 2.1). Use the additivity of the Brill-Noether number ([3], Proposition 4.6) and Remark 2 and Remark 3. □

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