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NASH MODIFICATIONS OF REAL PROJECTIVE VARIETIES

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Abstract: Let X be an n-dimensional, $n \geq 3$, smooth projective variety defined over $\mathbb{R}$ such that $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$, and $d \geq 3$ an odd integer. Here we show how to construct a degree d finite morphism $f : Y \to X$ defined over $\mathbb{R}$ and such that Y has general type, $\mathrm{Pic}(Y) \cong \mathbb{Z}$, and no non-trivial automorphism.

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1. Introduction

For the standard definitions of real algebraic varieties, regular maps between two real algebraic varieties and Nash maps, see [1]. Here we just recall the following definition heavily used in [4] and [5] ([5], Definition 1.6). Let X and $\tilde{X}$ be real algebraic varieties and $\phi : \tilde{X} \to X$ a regular map. The map ϕ is a weak change of the algebraic structure of X in the sense of [4] and [5] if it is bijective and ϕ^{-1} is a Nash map.

Hence a regular map $\phi : \tilde{X} \to X$ between real algebraic varieties is a weak change of the algebraic structure of X (in the sense of [4] and [5]) if and only if it is a Nash isomorphism. In particular, it must be a homeomorphism for the euclidean topology. Notice that, if ϕ is a weak change of the algebraic structure of X, then ϕ^{-1} maps nonsingular points to nonsingular points. Hence if X is nonsingular, then $\tilde{X}$ is nonsingular also.

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However, we want to discuss properties of a real algebraic variety which heavyly depends from the choice of a fixed projective model of it. Hence we will use the following notations and definitions.

Definition 1. Let X, Y, Z be integral projective schemes defined over $\mathbb{R}$. Let $X(\mathbb{C})$ denote the set of the complex points of X. Let $X_{\mathbb{R}}$ denote the set of all equivalence classes of morphisms $\mathrm{Spec}(\mathbb{R}) \to X$ whose associated residual field is $\mathbb{R}$. Hence in $X_{\mathbb{R}}$ we do not take the pairs of complex conjugate points of $X(\mathbb{C})$; we will call $X_{\mathbb{R}}$ the set of all real points of X and we may see it as a real algebraic variety in the sense of [4] and [5]. We will always assume $X_{reg,\mathbb{R}} := X_{\mathbb{R}} \cap X_{reg}(\mathbb{C}) \neq \emptyset$ (i.e. that the set $X_{\mathbb{R}}$ is Zariski dense in $X(\mathbb{C})$) and that the same is true for Y and Z. Let $f : Y \to X$ be a morphism defined over $\mathbb{R}$. We will say that f is a weak change (resp. an étale change) of the real algebraic structure of X if it is finite, it is étale over each point of $f^{-1}(X_{\mathbb{R}})$ (resp. at each point of Y) and it induces a bijection between $Y_{\mathbb{R}}$ and $X_{\mathbb{R}}$. A weak change $f : Y \to X$ of the algebraic structure of X will be said to be *lite* (resp. *vlite*) if $f^{-1}(\mathrm{Sing}(X)) \subseteq \mathrm{Sing}(Y)$ (resp. $f^{-1}(\mathrm{Sing}(X)) \subseteq \mathrm{Sing}(Y)$). We will say that Z is a real birational model of X if there is a morphism $u : Z \to X$ which is defined over $\mathbb{R}$, $u^{-1}(X_{\mathbb{R}}) = Z_{\mathbb{R}}$) and u is a local isomorphism at each point of $Z_{\mathbb{R}}$.

Notice that any weak change f of a real algebraic structure has odd degree. If X is smooth, then f is lite if and only if it is vlite if and only if Y is smooth.

We discuss effective bounds for the degree of certain ϕ (as in Definition 1) may be obtained using the scheme-theoretic set-up of Definition 1, while no such ϕ exists if we take other definitions. Indeed, we will see that étale change of the real algebraic structure is too strong for the set-up of [4] and [5] (see Remark 1 and Example 1).

Remark 1. Let X, Z be integral projective schemes defined over $\mathbb{R}$ and $f : Z \to X$ an étale change of the real algebraic structure of X. If X is algebraically simply connected, then f is an isomorphism. Recall that X is algebraically simply connected if it either normal and unirational or a smooth complete intersection in a projective space and $\dim(X) \geq 2$.

Example 1. Take $X := \mathbb{P}^n$, $n \geq 3$, with its usual real stucture and hence with $\mathbb{R}\mathbb{P}^n = \mathbb{P}^n{}_{\mathbb{R}}$. Let $f : Y \to X$ be a weak change of the real structure of X such that $\deg(f) \leq n$. Assume that Y is normal. Then Y is algebraically simply connected ([3], Theorem 2).

Now we can state our result.

Theorem 1. *Fix an odd integer $d \geq 3$. Let X be a smooth projective* *scheme defined over* $\mathbb{R}$ *such that* $X_{reg,\mathbb{R}} \neq \emptyset$*,* $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$*,* $H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ *and* $n := \dim(X) \geq 3$*. Then there exists a degree* d *vlite weak change of the real structure* $f : Y \to X$ *of* X *such that* Y *has general type,* ω_Y *is very ample,* $\mathrm{Pic}(Y) \cong \mathbb{Z}$*,* $H^1(Y, \mathcal{O}_X) = 0$*,* $H_1(Y(\mathbb{C}), \mathbb{Z}) = 0$*,* $H^2(Y(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$*) and* $\mathrm{Aut}(Y) = \{\mathrm{Id}_Y\}$.

Proof. Since $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$, X is algebraically simply connected (i.e. it has no degree > 1 connected étale covering) and $H^1(X(\mathbb{C}, \mathbb{C} = 0$. Hence $h^1(X, \mathcal{O}_X) = 0$ (Hodge theory). Since $H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$, the exponential sequence gives $\mathrm{Pic}(X) \cong \mathbb{Z}$. Fix any very ample line bundle H on X. Fix an integer $k > 0$ and set $L := H^{\otimes 2k}$. Hence $L^{\otimes d} \cong H^{\otimes 2kd}$. Since $2k$ is an even positive integer, there is a smooth effective divisor $D \in |L|$ defined ove $\mathbb{R}$ and with no real point. Since $\mathrm{Aut}(X)$ is an algebraic group, there is $k \gg 0$ such that there is a smooth effective divisor $D \in |L|$ defined ove $\mathbb{R}$ and with no real point. We fix any such integer $k > 0$ and any such divisor D. The pair (L, D) induces a degree d simple cyclic covering $u : M \to X$ such that D is the branch locus of u ([2], Example 1.1). Since D is smooth, M is smooth ([2], Example 1.1). Set $\Sigma := \mathbb{P}(\mathcal{O}_X \oplus L)$. Let $\mathcal{O}_\Sigma(1)$ denote the tautological degre one line bundle of the $\mathbb{P}^1$-bundle $v : \Sigma \to X$. Thus $v_*(\mathcal{O}_\Sigma(1)) \cong \mathcal{O}_X \oplus L$. The Abelian group $\mathrm{Pic}(\Sigma)$ is freely generated by $\mathcal{O}_\Sigma(1)$ and by $v^*(\mathrm{Pic}(X))$. We have $M \in |\mathcal{O}_\Sigma(d)|$. Σ, L, v and hence the linear system $|\mathcal{O}_\Sigma(d)|$ are defined over $\mathbb{R}$. Since D has no real point, $v|M$ is a vlite weak change of the algebraic structure of X. Notice that $h^0(\Sigma, \mathcal{O}_\Sigma(1)) = h^0(X, \mathcal{O}_X \oplus L)) + 1 > h^0(X, L)$ and that $\mathcal{O}_X(1)$ (and thus $\mathcal{O}_\Sigma(d)$) is spanned by its global sections. Hence a general element of $|\mathcal{O}_\Sigma(d)|$ is smooth (Bertini's Theorem). Let $Y \in |\mathcal{O}_\Sigma(d)|$ a divisor defined over $\mathbb{R}$ and sufficiently near to M. Thus Y is smooth, $e : v|Y : Y \to X$ is defined over $\mathbb{R}$ and Y has no real ramification point. Thus f is a vlite change of the algebraic structure of X. Notice that the ramification divisor (as a cycle) of $v|M$ is the Cartier divisor $(d-1)D$. Now assume $k \gg 0$ (indeed, it is sufficient to assume $k \geq n+2$). From the adjunction formula we get that ω_Y is ample and spanned and hence that Y is of general type. Thus $\mathrm{Aut}(Y)$ is finite. By [6], Theorem 2.1, (or see [3]) we have $H^1(Y(\mathbb{C}), \mathbb{C}) = H^1(X(\mathbb{C}), \mathbb{C}) = 0$. Thus $h^1(Y, \mathcal{O}_Y(1)) = 0$. By [6], Theorem 2.1, we have $H^2(Y(\mathbb{C}), \mathbb{Z}) \cong H^2(X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, Y is algebraically simply connected ([3], Theorem 2). Hence $H^2(Y(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ implies that that $H_2(Y(\mathbb{C}), \mathbb{Z})$ has rank one. It follows from the universal coefficient theorem that $H^2(Y(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$. Hence the exponential sequence gives $\mathrm{Pic}(Y) \cong \mathbb{Z}$). Thus to conclude the proof it is sufficient to prove the following claim.

Claim. *For general* Y *we have* $\mathrm{Aut}(Y) = \{\mathrm{Id}_Y\}$.

Proof of Claim. Fix $\beta \in \mathrm{Aut}(Y)$. Since $\mathrm{Pic}(Y) \cong \mathbb{Z}$, we get $\beta^*(f^*(H)) \cong$

$f^*(H)$. Notice that $f^*(H)$ is ample and spanned. By the projection formula we have $h^0(Y, f^*(H)) = h^0(X, H) + h^0(X, H \otimes L^*) = h^0(X, H)$; for the last equality we use that $1 - 2k < 0$ and H is ample. Hence the base point free linear system $|f^*(H)|$ factors through f. Since H is very ample, we get that β preserves f. Take $h \in \mathrm{Aut}(Y)$ such that $h \circ f = e$. To conclude the proof of claim it is sufficient to prove $h = \mathrm{Id}_Y$, when Y is sufficiently general. Since H is very ample, we may find Y such that the ramification divisor R is reduced and maps birationally onto its image in X. We obtain that the finite map f cannot factor through a non-trivial finite covering of X. Hence $h = \mathrm{Id}_Y$, concluding the proof of Claim and hence of Theorem 1. □

Obviously, Theorem 1 is false if $n = 1$. □

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