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OBSERVERS DESIGN FOR DISCRETE-TIME NONLINEAR SYSTEMS

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Abstract: In this paper, it is obtained that detectability is a necessary condition for the existence of asymptotic observers for discrete-time nonlinear systems with a disturbance dynamics. Using this necessary condition, we show that there does not exist any general observer for a special class of the systems. And necessary and sufficient conditions for this class of systems that change with the disturbance under some stability assumptions are derived by using center manifold theory.

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1. Introduction

The problem of designing observers for nonlinear systems was introduced in [9]. Over the past three decades, there has been paid significant attention to the construction of observers for nonlinear systems in control systems literature, such as [4, 12, 13, 10, 11]. Recently new results on general observers for continuous-time and discrete-time nonlinear systems have been obtained in [7, 8]. In particular, the definition of observers for linear systems [5] is ex-

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tended to define general observers, and necessary and sufficient condition for the existence of general exponential observers is derived by using center manifold theory for Lyapunov stable continuous-time and discrete-time nonlinear autonomous systems in [7, 8].

In this paper, we consider the problem of finding observers for discrete-time nonlinear systems with structured uncertainty. The observer design for discrete-time nonlinear systems is very useful in applications. We establish that a necessary condition for the existence of asymptotic observers for a class of discrete-time nonlinear systems with a disturbance dynamics. As a corollary of this result, we showed that for the classical case when the state equilibrium at the origin does not change with the disturbance, and when the plant output is purely a function of the state, there is no local asymptotic observer for the plant. At last, under some stability assumptions, we use the center manifold theory [2] to derive necessary and sufficient conditions for a local exponential observer.

Firstly, we give some definitions to be used in our discussion as follows.

For a nonlinear plant of the form

$$x(k+1) = f(x(k)), \quad y(k) = h(x(k)), \tag{1.1}$$

where $x \in R^n$ is the state, $y \in R^p$ is the output of the plant (1.1). The state x belongs to an open neighborhood X of the origin of R^n. We assume that $f : X \to R^n$ is a C^1 map and $f(0) = 0$, and also that $h : X \to R^p$ is a C^1 map and $h(0) = 0$. Let $Y = h(X)$.

Definition 1. (see [8]) A discrete-time nonlinear system is described as follows:

$$z(k+1) = g(z(k), y(k))\,, \tag{1.2}$$

where z is defined in a neighborhood Z of the origin of R^m and $g : Z \times Y \to R^m$ is a C^1 map with $g(0,0) = 0$. Given that

$$\omega = q(z), \tag{1.3}$$

where $q : Z \to R^n$ is a C^1 map with $q(0) = 0$. Then, system (1.2) is called a general asymptotic (respectively, general exponential) observer for plant (1.1) corresponding to (1.3), if the following two requirements hold.

(O1) If $\omega(0) = x(0)$, then $\omega(k) = x(k), \forall k \in Z_+$, where Z_+ is the set of all positive integers.

(O2) There exists a neighborhood V of the origin of R^n, such that, for all $\omega(0) - x(0) \in V$, the estimation error $\omega(k) - x(k)$ tends to zero asymptotically (respectively, exponentially) as $t \to \infty$.

The following lemma provides a geometric characterization of condition (O1) of the above definition.

Lemma. (see [6]) *The following statements are equivalent.*

(a) Condition (O1) of the above definition holds for systems (1.1) and (1.2).

(b) The submanifold defined via $q(z) = x$ is invariant under the flow of the following system:

$$x(k+1) = f(x(k)), \quad z(k+1) = g(z(k), h(x(k)).$$

2. A Necessary Condition for a Local Asymptotic Observer

In this paper, we consider a discrete-time nonlinear plants of the form

$$x(k+1) = f(x(k), \omega(k)),\ \omega(k+1) = s(\omega(k)),\ y(k) = h(x(k), \omega(k)), \qquad (2.4)$$

where $x \in R^n$ is the state, $y \in R^p$ is the output of the plant (2.4) and $\omega \in R^v$ is the disturbance. The state x belongs to an open neighborhood X of the origin of R^n, and the disturbance ω belongs to an open neighborhood Ω of the origin of R^v. We assume that $f : X \times \Omega \to R^n$ is a C^1 map and $f(0,0) = 0$, $s : \Omega \to R^v$ is a C^1 map and $h : X \times \Omega \to R^p$ is a C^1 map and $h(0,0) = 0$. The disturbance dynamics

$$\omega(k+1) = s(\omega(k)) \qquad (2.5)$$

is called the *exosystem* for the plant (2.4).

We assume that the equilibrium $\omega = 0$ is a neutrally stable equilibrium of the exosystem (2.5), i.e. that the equilibrium $\omega = 0$ is Lyapunov stable and for some neighborhood W of $\omega = 0$, the set Ω of all positive limit points of all trajectories which are initialized in W is such that $\Omega \cap W$ is dense in W. Note that the class of exosystems with neutrally stable equilibria includes the (very important in practice) class of systems in which every solution is a periodic solution. We also assume that $f(0,0) = 0, s(0) = 0$ and $h(0,0) = 0$.

We consider the problem of finding robust (exponential) observers for the plant (2.4) with structured uncertainty. since ω is not known, it is convenient to consider ω as an additional state variable, i.e. we consider the plant (2.4) in the form

$$\begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} f(x(k), \omega(k)) \\ s(\omega(k)) \end{bmatrix}, \quad y(k) = h(x(k), \omega(k)). \qquad (2.6)$$

Corresponding to the plant (2.6), we find a candidate observer of the form

$$\begin{bmatrix} z(k+1) \\ \mu(k+1) \end{bmatrix} = \begin{bmatrix} g(z(k), \mu(k), y(k)) \\ j(z(k), \mu(k), y(k)) \end{bmatrix} \tag{2.7}$$

so that $\begin{bmatrix} z \\ \mu \end{bmatrix}$ serves as an estimate of the state $\begin{bmatrix} x \\ \omega \end{bmatrix}$ of the plant (2.6), where g is an n-column vector and j is a ν-column vector with $g(0,0,0) = 0$ and $j(0,0,0) = 0$.

We define the *error* e by

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} z \\ \mu \end{bmatrix} - \begin{bmatrix} x \\ \omega \end{bmatrix} = \begin{bmatrix} z - x \\ \mu - \omega \end{bmatrix}. \tag{2.8}$$

Now we consider the composite system

$$\begin{aligned} x(k+1) &= f(x(k), \omega(k)), \\ \omega(k+1) &= s(\omega(k)), \\ e_1(k+1) &= g(x(k) + e_1(k), \omega(k) + e_2(k), h(x(k), \omega(k))) \\ &\qquad - f(x(k), \omega(k)), \\ e_2(k+1) &= j(x(k) + e_1(k), \omega(k) + e_2(k), h(x(k), \omega(k))) - s(\omega(k)). \end{aligned} \tag{2.9}$$

We start with a necessary condition for the plant (2.6) to have local asymptotic observers.

Theorem 1. *A necessary condition for the existence of a local asymptotic observer for the plant (2.6) is that it is asymptotically detectable, i.e. any state trajectory $(x(k), \omega(k))$ of (2.6) with small initial condition (x_0, ω_0) satisfying $h(x(k), \omega(k)) \equiv 0$, must be such that*

$$\| (x(k), \omega(k)) \| \to 0 \quad \text{as} \quad k \to \infty .$$

Proof. Suppose that the plant (2.6) has a local asymptotic observer of the form (2.7). Let $(x(k), \omega(k))$ be any state trajectory of (2.6) with small initial condition (x_0, ω_0) satisfying

$$h(x(k), \omega(k)) \equiv 0.$$

Then the observer dynamics in (2.7) takes the form

$$\begin{bmatrix} z(k+1) \\ \mu(k+1) \end{bmatrix} = \begin{bmatrix} g(z(k), \mu(k), 0) \\ j(z(k), \mu(k), 0) \end{bmatrix}. \tag{2.10}$$

Hence $(z_0, \mu_0) = (0, 0)$ is an equilibrium of the dynamics (2.10).

Since (2.7) is a local asymptotic observer for the plant (2.6), it follows that for all small values of

$$\| \begin{bmatrix} z(0) - x(0) \\ \mu(0) - \omega(0) \end{bmatrix} \|,$$

we have

$$\| \begin{bmatrix} z(k) - x(k) \\ \mu(k) - \omega(k) \end{bmatrix} \| \to 0, \quad \text{as} \quad t \to \infty.$$

If we take $z(0) = z_0 = 0$ and $\mu(0) = \mu_0 = 0$, then we have

$$\| (x(k), \omega(k)) \| \to 0 \quad \text{as} \quad t \to \infty.$$

This completes the proof. □

Next, using Theorem 1, we prove that there is no local asymptotic observer for the discrete-time nonlinear plant (2.6) if the equilibrium $x = 0$ does not change with the disturbance, i.e. if $f(0, \omega) \equiv 0$ and if the output function y is purely a function of x, i.e. if the output function has the form $h(x, \omega) \equiv \psi(x)$.

Theorem 2. *Suppose that the plant (2.6) satisfies the assumption*

$$f(0, \omega) \equiv 0,$$

i.e. $x = 0$ is an equilibrium of the state dynamics for all values of the disturbance ω, and also that the output function y is purely a function of x, i.e. it has the form

$$y = \psi(x).$$

Suppose also that $\omega = 0$ is a neutrally stable equilibrium of the exosystem (2.5). Then there is no local asymptotic observer for the plants (2.6).

Proof. This is an immediate consequence of Theorem 1 which gives a necessary condition for the existence of local asymptotic observer for the plant (2.6). We show that the plant (2.6) is not asymptotically detectable. This is easily seen by taking $x(0) = x_0 = 0$ and $\omega(0) = \omega_0 \neq 0$. Then we have

$$x(k) \equiv 0 \quad \text{and} \quad h(x(k), \omega(k)) = \psi(x(k)) \equiv 0\,,$$

but $\| \omega(k) \|$ does not converge to zero as $k \to \infty$ since $\omega = 0$ is a neutrally stable equilibrium of the exosystem (2.5). Hence, the plant (2.6) is not asymptotically detectable. From Theorem 1, we deduce that there is no local asymptotic observer for the plant (2.6). □

Corollary 1. *Under the same assumptions in Theorem 2, there is no local exponential observer for the plant (2.6).*

Corollary 2. *Consider the linear plant*

$$x(k+1) = A_\omega x(k),\ \omega(k+1) = \omega(k),\ y(k) = C_\omega x(k)\,.$$

If the output function y is purely a function of x, i.e. if $C_\omega \equiv C$, there is no asymptotic nonlinear observer for the plant.

Corollary 3. *Consider the linear plant*

$$x(k+1) = Ax(k) + B\omega(k),\ \omega(k+1) = S\omega(k),\ y(k) = Cx(k) + D\omega(k)\,.$$

If the output function y is purely a function of x, i.e. if $D = 0$, then there is no general asymptotic nonlinear observer for the plant.

Remark 1. Theorem 2 and its corollaries are for the case

$$f(0,\omega) \equiv 0,$$

i.e. for the case that the equilibrium $x = 0$ of the state dynamics does not change under the disturbance ω. In sharp contrast to this case, for the general case of problems, where we allow the equilibrium $x = 0$ to change with the disturbance ω, there typically exist local exponential observers, even when the output function y is purely a function of x.

3. Necessary and Sufficient Conditions for Robust Observers for Nonlinear Systems that Change with the Disturbance

In this section, using center manifold theory [2], we prove a fundamental theorem that completely characterizes the existence of local exponential observers of form (2.7) for Lyapunov stable plants of the form (2.6). Thus we define

$$C = Dh(0,0) = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial \omega} \end{bmatrix}_{(x,\omega)=(0,0)} \text{ and } A = Df(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \omega} \\ 0 & \frac{\partial s}{\partial \omega} \end{bmatrix}_{(x,\omega)=(0,0)},$$

i.e., C and A is the system linearization pair for the given nonlinear plant (2.6). Also define

$$E = \begin{bmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial \mu} \\ \frac{\partial j}{\partial z} & \frac{\partial j}{\partial \mu} \end{bmatrix}_{(z,\mu,y)=(0,0,0)}, \text{ and } K = \begin{bmatrix} \frac{\partial g}{\partial y} \\ \frac{\partial j}{\partial y} \end{bmatrix}_{(z,\mu,y)=(0,0,0)}.$$

Now, we state and prove the following result giving a characterization of the general exponential observers for Lyapunov stable discrete-time nonlinear systems.

Theorem 3. *Suppose that the plant dynamics in (2.6) is Lyapunov stable at* $(x, \omega) = (0, 0)$. *Then the system (2.7) is a local exponential observer for the plant (2.6) if, and only if:*

(a) The submanifold define via $e = 0$ *is invariant under the flow of composite system (2.9).*

(b) The dynamics

$$z(k+1) = g(z(k), \mu(k), 0), \quad \mu(k+1) = j(z(k), \mu(k), 0), \tag{3.11}$$

is local exponentially stable at $(z, \mu) = (0, 0)$.

Proof. Necessity. Suppose that system (2.7) is a local exponential observer for plant (2.6). Then, conditions (O1) and (O2) in Definition 1 are readily satisfied. By Lemma 1, condition (O1) implies condition (a). To see that condition (b) also holds, take $(x(0), \omega(0)) = (0, 0)$. Then, $x(k) \equiv 0$, $\omega(k) \equiv 0$ and $y(k) = h(x(k), \omega(k)) \equiv 0$, for all $k \in Z_+$. Now, the dynamics for (z, μ) become

$$\begin{bmatrix} z(k+1) \\ \mu(k+1) \end{bmatrix} = \begin{bmatrix} g(z(k), \mu(k), 0) \\ j(z(k), \mu(k), 0) \end{bmatrix}.$$

By condition (O2) in Definition 1, for all small initial condition (z_0, μ_0) and $e^0 = (z_0, \mu_0) - (x(0), \omega(0))$ in a neighborhood of the origin, it is immediate that

$$\| e(k) \| = \| \left[\begin{array}{c} z(k) - x(k) \\ \mu(k) - \omega(k) \end{array} \right] \| \to 0$$

exponentially, as $k \to \infty$. Form the plant dynamics in (2.6) is Lyapunov stable at $(x, \omega) = (0, 0)$, we can obtain that

$$\| (x(k), \omega(k)) \| \to 0 \, .$$

Hence, we must have

$$\| (z(k), \mu(k)) \| \to 0$$

exponentially, as $k \to \infty$, for the solution trajectory $(z(k), \mu(k))$ of the dynamics (3.11). Hence, we conclude that the dynamics (3.11) is locally exponentially stable at $(0, 0)$. Thus, we have established the necessary of conditions (a) and (b).

Sufficiency. Suppose that conditions (a) and (b) holds for plant (2.6) and (2.7). Since condition (a) implies condition (O1) in Definition 1 by Lemma 1, it suffices to show that condition (O2) in Definition 1 also holds.

By hypotheses, the dynamics (3.11) is locally exponentially stable $(z, \mu) = (0, 0)$, and the equilibrium $(x, \omega) = (0, 0)$ of the plant dynamics in (2.6) is

Lyapunov stable. Hence, matrix E must be convergent, i.e, it has all the eigenvalues inside the open unit disc of the complex plane, and matrix A must have all eigenvalues in the closed unit disc of the complex plane. We have two cases to consider.

Case 1. A is convergent.

By Hartman-Grobman Theorem, it follows that composite system (2.9) is locally topologically conjugate to the system,

$$\begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = A \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}, \quad \begin{bmatrix} z(k+1) \\ \mu(k+1) \end{bmatrix} = E \begin{bmatrix} z(k) \\ \mu(k) \end{bmatrix} + KC \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}. \tag{3.12}$$

Note that,

$$eig \begin{bmatrix} A & 0 \\ KC & E \end{bmatrix} = eig(A) \bigcup eig(E).$$

Since both A and E are convergent matrices, it is immediate that $(x(k), w(k))$ and $(z(k), \mu(k))$ tend to zero exponentially as $k \to \infty$. Then, it follows trivially that

$$\| \begin{bmatrix} z(k) - x(k) \\ \mu(k) - \omega(k) \end{bmatrix} \| \to 0$$

exponentially, as $k \to \infty$, for all small initial condition $(z(0), \mu(0))$ and $(x(0), \omega(0))$.

Case 2. A is not convergent.

First, we denote

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}, \quad \bar{z}(k) = \begin{bmatrix} z(k) \\ \mu(k) \end{bmatrix}.$$

Then, equations (3.12) become in the form:

$$\bar{x}(k+1) = A\bar{x}(k), \quad \bar{z}(k+1) = E\bar{x}(k) + KC\bar{x}(k). \tag{3.13}$$

Without loss of of generality, we can assume that the plant dynamics in (3.13) has the form

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} f_1(\bar{x}_1(k), \bar{x}_2(k)) \\ f_2(\bar{x}_1(k), \bar{x}_2(k)) \end{bmatrix} = \begin{bmatrix} A_1\bar{x}_1(k) + \phi_1(\bar{x}_1(k), \bar{x}_2(k)) \\ A_2\bar{x}_2(k) + \phi_2(\bar{x}_1(k), \bar{x}_2(k)) \end{bmatrix},$$

where $\bar{x}_1 \in R^{n_1}, \bar{x}_2 \in R^{n_2} (n_1 + n_2 = n + v)$, A_1 is an $n_1 \times n_1$ matrix having all eigenvalues with unit modulus, A_2 is an $n_2 \times n_2$ convergent matrix, and ϕ_1

and ϕ_2 are C^1 functions vanishing at $(\bar{x}_1, \bar{x}_2) = (0,0)$ together with all their first-order partial derivatives.

Now, $\bar{x} = (x, \omega) = (0,0)$ is a Lyapunov stable equilibrium of the plant dynamics in (2.6). Also, $\bar{z} = (z, \mu) = (0,0)$ is a locally exponential stability equilibrium of dynamics (3.11). Hence, by a total stability result, it follows that $(\bar{x}, \bar{z}) = (0,0)$ is a Lyapunov stable equilibrium of composite system (2.9).

Also, by the Center Manifold Theorem for maps, we know that composite system (2.9) has a local center manifold at $(\bar{x}, \bar{z}) = (0,0)$, the graph of a C^1 map,

$$\begin{bmatrix} \bar{x}_2 \\ \bar{z} \end{bmatrix} = \pi(\bar{x}_1) = \begin{bmatrix} \pi_1(\bar{x}_1) \\ \pi_2(\bar{x}_1) \end{bmatrix}. \tag{3.14}$$

Since composite system (2.9) is Lyapunov stable at $(\bar{x}, \bar{z}) = (0,0)$, we know that center manifold (3.14) is unique.

Since $e = 0$ is an invariant manifold for composite system (2.9), it is immediate that along the center manifold, we have

$$\pi_2(\bar{x}_1) = \begin{bmatrix} \bar{x}_1 \\ \pi_1(\bar{x}_1) \end{bmatrix}. \tag{3.15}$$

By the principle of asymptotic phase in the center manifold theory, there exists a neighborhood V of $(\bar{x}, \bar{z}) = (0,0)$, such that, for all $(\bar{x}(0), \bar{z}(0)) \in V$, we have

$$\left\| \begin{bmatrix} \bar{x}_2(k) - \pi_1(\bar{x}_1(k)) \\ \bar{z}(k) - \pi_2(\bar{x}_1(k)) \end{bmatrix} \right\| \leq M a^k \left\| \begin{bmatrix} \bar{x}_2(0) - \pi_1(\bar{x}_1(0)) \\ \bar{z}(0) - \pi_2(\bar{x}_1(0)) \end{bmatrix} \right\|, \ \forall k \in Z_+ \tag{3.16}$$

for some positive constants M and a with $0 < a < 1$.

Hence, it is immediate that

$$\bar{z}(k) \to \pi_2(\bar{x}_1(k)), \quad \text{exponentially, as} \quad k \to \infty. \tag{3.17}$$

From (3.15) and (3.17), it follows that

$$\bar{z}(k) \to \begin{bmatrix} \bar{x}_1(k) \\ \pi_1(\bar{x}_1(k)) \end{bmatrix}, \quad \text{exponentially, as} \quad k \to \infty. \tag{3.18}$$

From (3.16) and (3.18), it follows that

$$\bar{z}(k) \to \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix}, \quad \text{exponentially, as} \quad k \to \infty.$$

That is, there exists a neighborhood V of the origin, such that, for all $\bar{z}(0) - \bar{x}(0) \in V$, we have

$$\| \begin{bmatrix} z(k) - x(k) \\ \mu(k) - \omega(k) \end{bmatrix} \| \to 0$$

exponentially, as $k \to \infty$.

Thus, condition (O2) also holds. This completes the proof. □

Remark 2. There is a necessary condition for general exponential observer of form (2.7) for Lyapunov stable discrete-time plants of form (2.6) which is implicitly contained in the proof of Theorem 3. This necessary condition is that the dimension of the candidate observer cannot be lower than the number of critical eigenvalues of the linearization matrix $A = Df(0,0)$ of the plant dynamics in (2.6).

4. Conclusion

In this paper, we study the problem of finding observers for a class of discrete-time nonlinear systems with structured uncertainty. First, we show that for the classical case when the state equilibrium at the origin does not change with the disturbance, and when the plant output is purely a function of the state, there is no local asymptotic observer for the plant. At the end, we derive necessary and sufficient conditions for the given plant in the general case under some stability assumptions.

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