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ON THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

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Abstract: In this paper, the authors investigate the Hyers-Ulam-Rassias stability of a quadratic functional equation

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)=4f(x)+4f(y)+4f(z)$$

and prove its stability on bounded domain.

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1. Introduction

S.M. Ulam [21] in 1940 raised the following stability problem: Let E_1 be a group and let E_2 be a metric group with a metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : E_1 \to E_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in E_1$, then there exists a homomorphism $H : E_1 \to E_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in E_1$?

In 1941, Hyers [8] proved that if $f : E_1 \to E_2$ is a mapping satisfying

$$||f(x+y) - f(x) - f(y)|| \leq \delta\,,$$

for all $x, y \in E_1$, where E_1 and E_2 are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

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$$||f(x) - T(x)|| \le \delta\,,$$

for all $x \in E_1$. In addition, he also proved that if $f(tx)$ is continuous in the real variable t for any fixed $x \in E_1$, then T is linear. Th.M. Rassias [18] and Gajda [7] showed some generalizations of the result of Hyers in the following ways. Let $f : E_1 \to E_2$ is a mapping satisfying

$$||f(x+y) - f(x) - f(y)|| \le \theta\left(||x||^p + ||y||^p\right)\,,$$

for all $x, y \in E_1$ where $\theta \ge 0$ and $p \ne 1$. Suppose $f(tx)$ is continuous in t for each fixed $x \in E_1$, then there is a unique linear mapping $T : E_1 \to E_2$ such that

$$||f(x) - T(x)|| \le \frac{k\theta}{||k^p - k||}||x||^p$$

for all $x \in E_1$, and any given integer $k > 1$.

Further more, in 1993 Rassias and Smerl [17] proved a more generalized form of the above result. Since then, the stability problems of functional equations have been extensively investigated by a number of mathematicians, one can refer to [2], [3], [4], [5], [13], [14], [18], [19].

The stability of the quadratic functional equations is discussed by few authors [10,11]. The equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,, \tag{1}$$

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z)\,, \tag{2}$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x) \tag{3}$$

are called quadratic functional equations because they satisfies the quadratic function $f(x) = cx^2 (x \in R)$. Recently, J.H. Bae investigated the stability of the $N-$dimensional quadratic functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i), \ \ (n \ge 2).$$

For further information on the stability of the quadratic functional equation, one may refer to [19]. In this paper, the authors investigated the generalized Hyers-Ulam-Rassias stability of a quadratic functional equation

$$\begin{aligned} f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z) \\ = 4f(x) + 4f(y) + 4f(z)\,, \end{aligned} \tag{4}$$

which is some what different from the equations appeared in [3].

2. Main Results

In this section, we denote N be the set of positive integers and R be the set of real numbers and let B_1 be a normed space and B_2 be a Banach space.

We define a function

$$\psi : B_1 \times B_1 \times B_1 \to [0, \infty)$$

be such that either

$$\phi_i(x,y,z) = \sum_{k=0}^{\infty} \frac{i}{i^{2k}} \psi(i^k x, i^k y, i^k z) < \infty, \quad \forall\ x,y,z \in B_1, \tag{5}$$

or

$$\widetilde{\phi}_i(x,y,z) = \sum_{k=0}^{\infty} i^{2(k+1)} \psi(\frac{i}{i^{k+1}}x, \frac{i}{i^{k+1}}y, \frac{i}{i^{k+1}}z) < \infty, \quad \forall\ x,y,z \in B_1. \tag{6}$$

We denote

$$\begin{aligned} Df(x,y,z) = f(x+y+z) + f(x-y+z) + f(x+y-z) \\ + f(-x+y+z) - 4f(x) - 4f(y) - 4f(z) \end{aligned}$$

be such that

$$||Df(x,y,z)|| \le \psi(x,y,z), \ \forall x,y,z \in B_1 \tag{7}$$

and $f : B_1 \to B_2$. We also define a real sequence b_k be such that $b_1 = \frac{1}{2}$, $b_2 = 1$ and $b_k = 2b_{k-1} + b_{k-2}$ $(k \ge 3)$. Now we are ready to prove the generalized Hyers-Ulam-Rassias stability of the quadratic functional equation of the form (4). Before proving our main theorem, we now prove the following lemma, which will be useful for our discussions.

Lemma 2.1. *With respect to the above assumptions:*
(i) If f is odd function then

$$\left\| \frac{1}{i^{2n}} f(i^n x) \right\| \le \frac{1}{2i^{2n}} \psi(0, i^n x, 0) \tag{8}$$

is true for $x \in B_1$.

(ii) If f is even function then

$$\left\| \frac{1}{i^{2n}} f(i^n x) - f(x) \right\| \le \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \sum_{k=0}^{n-1} \frac{1}{i^{2k}} \psi((i-j)i^k x, i^k x, 0) \tag{9}$$

is true for $x \in B_1$ and $f(0) = 0$ holds good.

Proof. *Case (i).* Taking f is odd function and substitute $x=0, y=x, z=0$ in (7) we get

$$||-f(x)+f(-x)|| \leq \psi(0,x,0)$$

which is

$$||2f(x)|| \leq \psi(0,x,0), \quad \forall \ x \in B_1. \tag{10}$$

Now replacing x by ix in the equation (10) and divide by $2i^2$ we arrive an expression and applying induction on that we obtain

$$\left\| \frac{1}{i^{2n}} f(i^n x) \right\| \leq \frac{1}{2i^{2n}} \psi(0, i^n x, 0). \tag{11}$$

Case (ii). When f is even function, replacing x, y, z by $kx, x, 0$ in (7) we obtain

$$\|f(kx+x)+f(kx-x)-2f(kx)-2f(x)\| \leq \frac{1}{2}\psi(kx,x,0) \tag{12}$$

for all integers $k, 2 \leq k \leq i$. Now consider

$$\begin{aligned}
& \|f(ix)-i^2 f(x)\| \\
= \ & \|2f((i-1)x)-2(i-1)^2 f(x)-f((i-2)x)+(i-2)^2 f(x) \\
& + f(ix)+f((i-2)x)-2f((i-1)x)-2f(x)\| \\
\leq \ & \|2f((i-1)x)-2(i-1)^2 f(x)\| + \|f((i-2)x)-(i-2)^2 f(x)\| \\
& + \|f(ix)+f((i-2)x)-2f((i-1)x)-2f(x)\| \\
\leq \ & \sum_{k=1}^{i-2} 2b_k \psi((i-1-k)x,x,0) + \sum_{k=1}^{i-3} b_k \psi((i-k)x,ix,0) \\
& + \frac{1}{2}\psi((i-1)x,x,0) \\
\leq \ & \sum_{k=1}^{i-1} 2b_{k-1} \psi((i-1-k)x,x,0) + \sum_{k=3}^{i-1} b_{k-2} \psi((i-k)x,ix,0) \\
& + \frac{1}{2}\psi((i-1)x,x,0) \leq \sum_{j=1}^{i-1} b_j \psi((i-j)x,x,0).
\end{aligned} \tag{13}$$

Equation (13) shows that the induction for $n=1$ on equation (9) is true. Now we assume the result (9) is true for n. We will prove the result (9) for $n+1$. For this, consider

$$\left\| \frac{1}{i^{2(n+1)}} f(i^{n+1}x) - f(x) \right\|$$

$$
\begin{aligned}
&= \left\| \frac{1}{i^{2(n+1)}} f(i^{n+1}x) - \frac{1}{i^{2n}} f(i^n x) + \frac{1}{i^{2n}} f(i^n x) - f(x) \right\| \\
&\leq \frac{1}{i^{2n}} \left\| \frac{1}{i^2} f(i i^n x) - f(i^n x) \right\| + \left\| \frac{1}{i^{2n}} f(i^n x) - f(x) \right\| \\
&\leq \frac{1}{i^{2n}} \sum_{j=1}^{i-1} b_j \psi((i-j)i^n x, i^n x, 0) + \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \sum_{k=0}^{n-1} \frac{1}{i^{2k}} \psi((i-j)i^k x, i^k x, 0) \\
&= \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \sum_{k=0}^{n} \frac{1}{i^{2k}} \psi((i-j)i^k x, i^k x, 0).
\end{aligned}
$$

This proves that the equation (9) is true for $n+1$. By induction, the proof of this Lemma completes. □

Theorem 2.2. *If a function $f : B_1 \to B_2$ satisfies the equation (7) and $f(0) = 0$. Then there exists a unique quadratic function $Q : B_1 \to B_2$ satisfying*

$$
(i) \quad \|Q(x) - f(x)\| \leq \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \phi_i((i-j)x, x, 0) + \frac{1}{2} \psi(0, x, 0) \tag{14}
$$

when (5) is true, or

$$
(ii) \quad \|Q(x) - f(x)\| \leq \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \widetilde{\phi}_i((i-j)x, x, 0) + \frac{1}{2} \psi(0, x, 0) \tag{15}
$$

when (6) is true. If f is measurable or $f(tx)$ is continuous in $t \in R$ for each fixed $x \in B_1$, then Q satisfies

$$
Q(tx) = t^2 Q(x), \quad x \in B_1, \ t \in R. \tag{16}
$$

Proof. Assume (5) is true and if f is even function, then from (9)

$$
\begin{aligned}
\lim_{n \to \infty} \left\| \frac{1}{i^{2n}} f(i^n x) - f(x) \right\| &\leq \lim_{n \to \infty} \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \sum_{k=0}^{n-1} \frac{1}{i^{2k}} \psi((i-j)i^k x, i^k x, 0) \\
&\leq \frac{1}{i^2} \sum_{j=1}^{i-1} b_j \phi_i((i-j)x, x, 0).
\end{aligned} \tag{17}
$$

We will now prove $\left\{ \frac{1}{i^{2n}} f(i^n x) \right\}$ is a cauchy sequence. Let m, n be positive integers such that $n > m > 0$. Then using (17) in (5) we arrive

$$\left\|\frac{1}{i^{2n}}f(i^n x)-\frac{1}{i^{2m}}f(i^m x)\right\|=\left\|\frac{1}{i^{2n-2m+2m}}f(i^{n+m-m}x)-\frac{1}{i^{2m}}f(i^m x)\right\|$$
$$\leq\frac{1}{i^{2m}}\left\|\frac{1}{i^{2(n-m)}}f(i^{(n-m)}i^m x)-f(i^m x)\right\|\leq\frac{1}{i^{2m}}\frac{1}{i^2}\sum_{j=1}^{i-1}b_j\sum_{k=0}^{n-m-1}\frac{1}{i^{2k}}$$
$$\times\,\psi((i-j)i^k i^m x, i^k i^m x, 0)\leq\frac{1}{i^2}\sum_{j=1}^{i-1}b_j\sum_{k=m}^{n-1}\frac{1}{i^{2k}}\psi((i-j)i^k x, i^k x, 0). \quad (18)$$

The R.H.S of (18) $\to 0$ as $m\to\infty$. Also B_2 is a Banach space and therefore we define

$$Q(x)=\lim_{n\to\infty}\frac{1}{i^{2n}}f(i^n x)\,,\quad \forall\ x\in B_1. \quad (19)$$

Then using the definition of $Q(x)$ and (17), we obtain

$$\|Q(x)-f(x)\|\leq\frac{1}{i^2}\sum_{j=1}^{i-1}b_j\phi_i((i-j)x,x,0). \quad (20)$$

Suppose f if an odd function, then from (18) we have

$$\lim_{n\to\infty}\frac{1}{i^{2n}}f(i^n x)=0,\quad \forall\ x\in B_1.$$

Then by (10)

$$||Q(x)-f(x)||=||f(x)||\leq\frac{1}{2}\psi(0,x,0). \quad (21)$$

Now we write $f=f_{even}+f_{odd}$ and therefore

$$||Q(x)-f(x)||=||Q(x)-f_{even}(x)||+||f_{odd}(x)||$$
$$\leq\frac{1}{i^2}\sum_{j=1}^{i-1}b_j\phi_i((i-j)x,x,0)+\frac{1}{2}\psi(0,x,0)$$

this proves (14).

Now replacing x,y,z by $i^n x, i^n y$, $i^n z$ in (4) and dividing the resultant expression by i^{2n} and using (5), we get

$$\|\frac{1}{i^{2n}}f(i^n(x+y+z))+\frac{1}{i^{2n}}f(i^n(x-y+z))+\frac{1}{i^{2n}}f(i^n(x+y-z))$$
$$+\frac{1}{i^{2n}}f(i^n(-x+y+z))-\frac{1}{i^{2n}}4f(i^n x)-\frac{1}{i^{2n}}4f(i^n y)-\frac{1}{i^{2n}}4f(i^n z)\|$$

$$\leq \frac{1}{i^{2n}}\psi(i^n x, i^n y, i^n z). \quad (22)$$

Taking lim $n \to \infty$ on both sides of (22) and using (19) we obtain

$$\|Q(x+y+z)+Q(x-y+z)+Q(x+y-z)+Q(-x+y+z) - 4Q(x)-4Q(y)-4Q(z)\| \leq 0\,,$$

which gives raise to

$$Q(x+y+z)+Q(x-y+z)+Q(x+y-z)+Q(-x+y+z) = 4Q(x)+4Q(y)+4Q(z)\,,$$

from which we can prove Q is a quadratic function.

Next we prove the uniqueness, for this, suppose $Q' : B_1 \to B_2$ be another quadratic function satisfying (14), then Q and Q' satisfying

$$Q(i^n x) = i^{2n}Q(x) \text{ and } Q'(i^n x) = i^{2n}Q'(x) \quad (23)$$

for any $n \in N$. Then using (21), (19) and (5), we obtain

$$\begin{aligned}\|Q(x)-Q'(x)\| &= \frac{1}{i^{2n}}\|Q(i^n x) - \ Q'(i^n x)\| \\ &= \frac{1}{i^{2n}}\left\{\|Q(i^n x)-f(i^n x)\| + \|f(i^n x)-Q'(i^n x)\|\right\} \\ &\leq \frac{1}{i^{2n}}\left\{\frac{1}{i^2}\sum_{j=1}^{i-1} b_j\phi_i((i-j)x,x,0)+\frac{1}{2}\psi(0,x,0)\right\} \to 0 \text{ as } n\to\infty,\end{aligned}$$

which proves $Q(x) = Q'(x)$, $\forall x \in B_1$. The last part of the theorem is true by using the result of Czerwik [6]. In the similar fashion we can prove the theorem for the function $\widetilde{\phi_i}(x,y,z)$ when (6) is true. □

Theorem 2.3. *If a function $f : B_1 \to B_2$ satisfies the inequality*

$$\|Df(x,y,z)\| \leq \epsilon, \quad \forall x,y,z \in B_1\,, \quad (24)$$

then there exists a unique quadratic function $Q : B_1 \to B_2$ satisfying

$$\|Q(x)-f(x)\| \leq \frac{2}{3}\,\epsilon, \quad \forall\ x \in B_1. \quad (25)$$

If f is measurable or $f(tx)$ is continuous in $t \in R$ for each fixed $x \in B_1$, then Q satisfies (16) for all $x \in B_1, t \in R$.

Proof. We take $\psi(x,y,z) = \epsilon$, then using equation (5) for $i = 2$. We obtain

$$\phi_2(x,y,z) = \sum_{k=0}^{\infty} \frac{i}{2^{2k}} \psi(2^k x, 2^k y, 2^k z). \tag{26}$$

Now using Theorem 2.2 then there exists a unique quadratic function $Q : B_1 \to B_2$ such that

$$\|Q(x) - f(x)\| \leq \frac{1}{2^2} b_1 \phi_2(x,x,0) + \frac{1}{2}\psi(0,x,0). \tag{27}$$

Using $b_1 = \frac{1}{2}$ and $i = 2$ in equation (14) and then applying (26) in to (27), we obtain

$$\|Q(x) - f(x)\| \leq \frac{1}{2^3} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \epsilon + \frac{1}{2}\epsilon \leq \frac{\epsilon}{6} + \frac{\epsilon}{2} = \frac{2\epsilon}{3}, \quad \forall x \in B_1. \qquad \square$$

3. Hyers-Ulam-Rassias Stability of Equation (4) on a Bounded Domain

In this we denote n be a positive integer, $r > 0$, a constant and B be a Banach space and $I^n = [-r, r]$.

Theorem 3.1. *If a function $f : I^n \to B$ satisfies (24) for all $x, y, z \in I^n$ with $x+y+z$, $x-y+z$, $x+y-z$, $-x+y+z \in I^n$ then there exists a unique quadratic function $Q : R^n \to B$ such that*

$$\|Q(x) - f(x)\| \leq \frac{7}{4}(2914n^2 + 1872n + 334)\epsilon, \quad \forall x \in I^n. \tag{28}$$

Proof. Setting $x = y = z = 0$ in (24), we obtain

$$\|f(0)\| \leq \frac{\epsilon}{8}. \tag{29}$$

Now replacing x by $-x, y$ by 0 and z by 0 in (24), we obtain

$$\|f(x) - f(-x) - 8f(0)\| \leq \epsilon. \tag{30}$$

Using (29) and (30), we obtain

$$\|f(x) - f(-x)\| \leq 2\epsilon. \tag{31}$$

Again letting $z = 0$ in (24) and dividing by 2, we arrive

$$\|f(x+y) + \frac{1}{2}f(x-y) + \frac{1}{2}f(-x+y) - 2f(x) - 2f(y) - 2f(0)\| \leq \frac{\epsilon}{2}. \quad (32)$$

Therefore, using (29), (31) and (32), we obtain

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \|f(x+y) + \frac{1}{2}f(x-y) + \frac{1}{2}f(-x+y) - 2f(x) - 2f(y) - 2f(0)\| \\ &\qquad + \frac{1}{2}\|f(x-y) - f(-x+y)\| + \|2f(0)\| \leq \frac{\epsilon}{2} + \epsilon + \frac{\epsilon}{4} = \frac{7\epsilon}{4}. \quad (33) \end{aligned}$$

Applying the result in the Theorem 3.1 [3], then there exists a unique quadratic function $Q : R^n \to B_1$ satisfying

$$\|Q(x) - f(x)\| \leq \frac{7}{4}(2914n^2 + 1872n + 334)\epsilon$$

and it holds for any $x \in I^n$. □

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