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AN ITERATIVE ALGORITHM FOR THE LINEAR COMPLEMENTARITY PROBLEM WITH AN *H*-MATRIX

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Abstract: It is well known that the linear complementarity problem LCP(A, q) which consists of finding a vector $z \in R^n$ such that

$$Az + q \geq 0, \quad z \geq 0, \quad z^T(Az + q) = 0,$$

where $A \in R^{n \times n}$ and $q \in R^n$ are a given real matrix and a real vector, respectively. We have proposed an $O(n^3)$ direct recursive algorithm when A is an M-matrix [8]. In [7], a block version of the algorithm was considered. Many numerical examples are showing that the block version takes fewer number of the arithmetic operations than the non-block version. In this paper, we propose a kind of splitting algorithms for solving LCP(A, q), where A is an H-matrix. The main idea of the algorithm is to devide the H-matrix into an M-matrix and a nonnegative matrix, and to solve an LCP with the M-matrix as subproblems. Some numerical examples are shown.

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1. Introduction

We know that the following linear complementarity problem (see [5]) often appears in fields of the mathematical programming.

LCP(A, q): Let $A \in R^{n \times n}$ and $q \in R^n$, finding one or all real vectors z with satisfying

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$$Az + q \geq 0, \qquad z \geq 0, \qquad z^T(Az + q) = 0. \tag{1}$$

A matrix $A \in R^{n\times n}$ is called a P-matrix if all of its principal minors are positive. It is well known that for any real vectors $q \in R^n$, LCP(A, q) has an unique solution if and only if A is a P-matrix (see [3]). For solving LCP(A, q), some traditional methods, including the principal method and the complementary algorithm have been shown (see [3]). Recently, some authors discussed the verification methods (see [1], [4]) for LCP(A, q), and the multisplitting methods (see [5], [2]) for large sparse LCP(A, q).

If A is a special matrix, it is an interesting problem to show the computational time complexity for the LCP(A, q). Y. Fathi presented an $O(2^n)$ computational time complexity for LCP(A, q) associated with positive definite matrix A by the two well known complementary pivot methods (see [6]).

We have proposed an $O(n^3)$ direct recursive algorithm [8] and a block recursive algorithm [7] for solving LCP(A, q) with A is an $n \times n$ M-matrix. In this paper, we propose a kind of splitting algorithms for solving LCP(A, q), where A is an H-matrix. The main idea of the algorithm is to devide the H-matrix into an M-matrix and a nonnegative matrix, and to solve an LCP with the M-matrix as subproblems. Some numerical examples are shown.

2. An Iterative Algorithm

Definition 2.1. Let $A = (a_{ij}) \in R^{n\times n}$, we call that A is an H-matrix if there exist positive numbers d_i, $i = 1, 2, \ldots, n$ wich satisfy the following inequalities:

$$|a_{ii}|d_i > \sum_{j\neq i} |a_{ij}|d_j, \quad i = 1 \sim n\,.$$

It is well known that M-matrix is an important special class of H-matrix. In this section, we consider to find a solution for the LCP(A,q), where $A = (a_{ij})$ is an $n \times n$ H-matrix with $a_{ii} > 0$, $i = 1 \sim n$. We know that H-matrix always can be divided to the sum of two matrices:

$$A = M + N,$$

where M is an M-matrix with including all main diagonal elements and all negative non-diagonal elements of A, N is a nonnegative matrix with including all nonnegative non-diagonal elements. For example,

$$A = \begin{pmatrix} 2 & -0.5 & 1 \\ -0.5 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -0.5 & 0 \\ -0.5 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = M + N.$$

From $A = M + N$, the LCP(A,q) can be written as

$$(M+N)z + q \geq 0, \qquad z \geq 0, \qquad z^T((M+N)z+q) = 0.$$

We consider the following iterative process

$$Mz_{k+1} + Nz_k + q \geq 0, \qquad z_{k+1} \geq 0, \qquad z_{k+1}^T(Mz_{k+1} + Nz_k + q) = 0.$$

Assume $s_k = Nz_k + q$, then the above problem becomes LCP(M,s_k), where M is an M-matrix. Take initial value $z_0 = (0, \ldots, 0)^T$, $s_0 = q$, and solve the LCP(M,s_k) in order $k = 0, 1, \ldots$, by using the $O(n^3)$ direct recursive algorithm [8] or block version algorithm [7], up to satisfy

$$||z_{k+1} - z_k|| < \varepsilon\,,$$

where $\varepsilon > 0$ is a provided permissible error. Therefore, we have the following H-LCP(A,q) algorithm.

***H*-LCP(*A*,*q*) Algorithm.**

Step 1. Input an H-matrix $A \in R^{n\times n}$ with $a_{ii} > 0$, $i = 1, 2, \ldots, n$, and a vector $q \in R^n$.

Step 2. Let $k = 0$, $z_0 = 0$, $\varepsilon > 0$, and divide H-matrix A to the sum of M-matrix M and nonnegative matrix N, $A = M + N$.

Step 3. $s_k = Nz_k + q$.

Step 4. Call the direct recursive algorithm M-LCP(M,s_k) (see [8]) or the block recursive algorithm BM-LCP(M,s_k) (see [7]) and get a solution z_{k+1}.

Step 5. If $||z_{k+1} - z_k|| < \varepsilon$ then output z_{k+1}, Stop. Else $k = k + 1$, return to Step 3.

Next, we show a proof for the convergence of the above H-LCP(A,q) algorithm.

Let us to see a simple example. Suppose

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \varepsilon = 0.0001\,.$$

It is obvious that A is an H-matrix and it can be devided to the sum of M-matrix and nonnegative matrix:

$$A = M + N = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $z_0 = 0$ is an initial value, and $s_0 = Nz_0 + q = q$, solve LCP(M,s_0), we get

$$z_1 = (0, 0, 0.5)^T\,.$$

From $||z_1 - z_0|| = 0.5 > \varepsilon$, compute

$$s_1 = Nz_1 + q = (1, 1.5, -1)^T$$

Solve LCP(M,s_1) and get the solution $z_2 = (0, 0, 0.5)^T$. Because $||z_2 - z_1|| =$

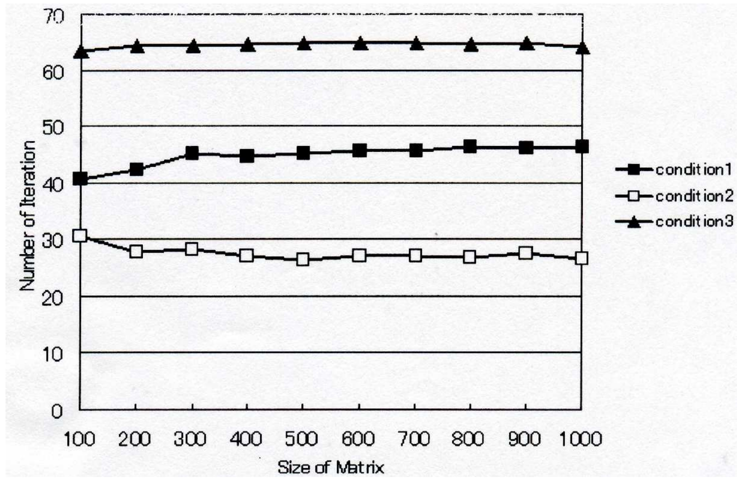

Figure 1: A comparison for number of iterations

$0 < \varepsilon$, stop the iteration and get $z = z_2$. It is obvious that

$$z_2 \geq 0, \quad w = Az_2 + q = (1, 1.5, 0)^T, \quad z_2^T w = (0, 0, 0)^T.$$

3. Numerical Examples

We computed three numerical examples based on the following computer environment. *CPU: Pentium4 2.6GHz, Memori: 768MHz, OS: Windows XP Pro., Software: Visual C++ 6.0.* We got the correct solutions in all examples with a permissible error $\varepsilon = 1.0e - 15$.

Example 1. Let A is the following $n \times n$ tridiagonal H-matrix

$$A = \text{tridiag}\,(1, 4, 1),$$

any element in q takes 1 or -1 randomly.

Example 2. Let $t_1, t_2, \ldots, t_n$ is random numbers in the interval $[n,\ 2n]$, and A is the following $n \times n$ H-matrix:

$$A = (a_{ij}), \quad a_{ii} = t_i, \quad a_{ij} = \pm\frac{t_i}{n}, \quad i \neq j$$

and $q = (q_1, q_2, \ldots, q_n)^T$, $q_i = (1 - \frac{1}{n})t_i$.

Example 3. The coefficient matrix A is the same with Example 1, and

$$q_i = -\sum_{j=1}^{n} a_{ij} z_j, \quad 1 \leq i \leq n\,,$$

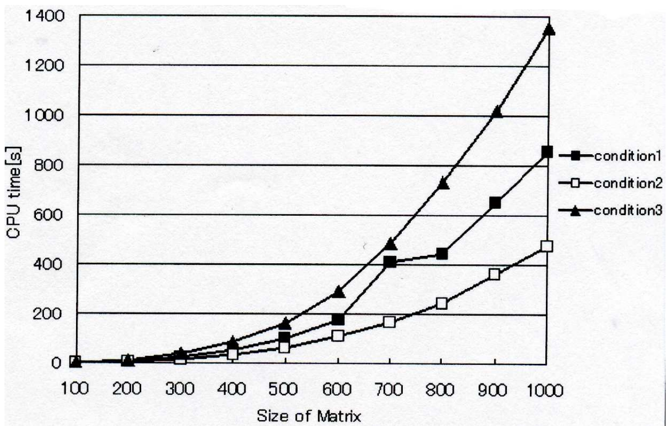

Figure 2: A comparison for CPU times

where $z_j = 1$ if j is even number, else $z_j = 0$. In this example, z is obvious solution, so we can estimate the precision of the algorithm.

Some comparisons for number of iterations, CPU time and precision with the above three examples, are shown in Figure 1, Figure 2 and Figure 3 (for $i = 1, 2, 3$, condition i means to take conditions in Example i).

4. Conclusion

In this paper, we proposed an iterative algorithm for LCP(A,q) with A is an H-matrix. This algorithm was based on the direct recursive algorithm or the block recursive algorithm for LCP(A,q) with A is an M-matrix. Many numerical examples shown the proposed algorithm is correct and takes fewer number of the iterations. But we still can not prove convergence of the proposed algorithm now.

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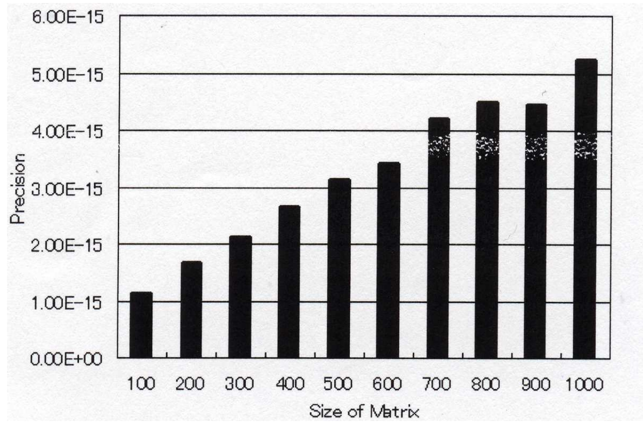

Figure 3: Precision in Example 3

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